POLE-ZERO APPROXIMATIONS OF DIGITAL FRACTIONAL-ORDER INTEGRATORS AND DIFFERENTIATORS USING SIGNAL MODELING TECHNIQUES

Ramiro S. Barbosa*, J. A. Tenreiro Machado*, Isabel M. Ferreira+

**Department of Electrotechnical Engineering*
Institute of Engineering of Porto, Portugal
E-mail: {rbarbosa,jtm}@dee.isep.ipp.pt
+Department of Electrotechnical Engineering
Faculty of Engineering of Porto, Portugal
E-mail: imf@fe.up.pt

Abstract: A novel strategy to the development of digital pole-zero approximations to fractional-order integrators and differentiators is presented here. The scheme is based in the signal modeling techniques applied to deterministic signals, namely the Padé, the Prony and the Shanks methods. It is shown that the illustrated algorithms yield good results both in the time and the frequency domains. Moreover, they are capable to give superior approximations than other existent approaches, namely the widely used CFE method. Several examples are given that demonstrate the effectiveness of the proposed techniques.

Keywords: Fractional-Order Integrators and Differentiators, Fractional Calculus, Digital Filter Design, IIR Filters, Pole-zero Approximations.

1. INTRODUCTION

Fractional calculus (FC) deals with derivatives and integrals to an arbitrary order (*i.e.*, rational, irrational or even complex order) (Oldham and Spanier, 1974). This area of mathematics emerged at the same time as the classical differential calculus, three centuries ago. However, its inherent complexity postponed the application of the associated concepts. Nowadays, the FC theory is applied in almost all the areas of science and engineering (Oustaloup, 1995; Podlubny, 1999; Hilfer, 2000) being recognized its ability to better modeling and control many dynamical systems.

In what concerns the area of control systems the application of the FC concepts is still scarce and only in the last two decades appeared the first applications. Oustaloup (1995) introduced the fractional-order algorithms for the control of dynamic systems and demonstrated the superior performance of the CRONE (French abbreviation for *Commande Robuste d'Ordre Non Entier*) controller over the PID controller. More recently, Podlubny (1999) proposed a generalization of the PID controller, namely the $PI^{\lambda}D^{\mu}$ controller, involving an integrator of order λ and differentiator of order μ (the orders λ and μ may assume real noninteger values). He also demonstrated the better response of this type of controller, in comparison with the classical PID controller, when used for the control of fractional-order systems. The transfer function of the $PI^{\lambda}D^{\mu}$ is given by $K\,(1+1/T_i s^{-\lambda}+T_d s^{\mu})$, where λ and μ are positive real numbers; K is the proportional gain, T_i the integral time constant and T_d the derivative time constant. Clearly, taking $(\lambda, \mu) = \{(1, 1);\ (1, 0);\ (0, 1);\ (0, 0)\}$ we obtain the classical {PID, PI, PD, P} controllers, respectively. All these classical types of PID-controllers are the particular cases of the fractional $PI^{\lambda}D^{\mu}$-controller. However, the $PI^{\lambda}D^{\mu}$-controller is more flexible and gives the possibility of adjusting more carefully the dynamical properties of a fractional-order control system.

The simplest and most straightforward method to compute the fractional derivative and integral of order α of the function $f(t)$, $D^{\alpha} f(t)$ (α is a real number), is the application of the Grünwald-Letnikov (GL) definition:

$$D^{\alpha} f(t) = \lim_{h\to 0}\left\{\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k} f(t-kh)\right\} \qquad (1)$$

In control systems, we usually adopt the Laplace s-domain. The Laplace transform of $D^{\alpha} f(t)$, under null initial conditions, is given by (Podlubny, 1999):

$$L\left\{D^{\alpha}\left[f(t)\right]\right\} = s^{\alpha} F(s) \qquad (2)$$

In general, the discretization of the fractional-order operator s^{α} (α is a real number) can be expressed by the so-called generating function $s = \omega(z^{-1})$ (Vinagre, *et al.*, 2000; Chen and Moore, 2002). Table 1 lists three of the most commonly used discretization schemes, namely the trapezoidal (Tustin) rule, the backward difference (Euler) rule, and the more recently introduced Al-Alaoui operator, which is obtained by the stable inversion of the weighted sum of the Tustin integration rule and the Euler integration rule (Al-Alaoui, 1993).

Table 1 Discretization schemes

Euler Grünwald-Letnikov	$s^{\alpha} \approx \left(\frac{1}{T}\left(1-z^{-1}\right)\right)^{\alpha}$
Tustin	$s^{\alpha} \approx \left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\alpha}$
Al-Alaoui	$s^{\alpha} \approx \left(\frac{8}{7T}\frac{1-z^{-1}}{1+z^{-1}/7}\right)^{\alpha}$

There are several different ways for obtaining digital approximations from the irrational generating functions listed in Table 1. One way is to perform a power series expansion (PSE), which leads to approximations in the form of polynomials (FIR filters) (Machado, 2001). For example, by doing so, over the backward difference (Euler) rule, $\omega(z^{-1}) = (1-z^{-1})/T$, gives the discretization formula for the GL definition (1). Another possible way is to obtain rational approximations (IIR filters) by application of the continued fraction expansion (CFE) method (Vinagre, *et al.*, 2000; Chen and Moore, 2002). It is well known that rational approximations frequently converge faster than polynomial approximations and have a wider domain of convergence in the complex plane. In the work that follows, we develop rational approximations of the z variable to fractional-order integrators and differentiators, $H(z^{-1})$, of the form:

$$H^{\alpha}\left(z^{-1}\right) \approx \frac{P_m\left(z^{-1}\right)}{Q_n\left(z^{-1}\right)} = H\left(z^{-1}\right) \qquad (3)$$

where P and Q are the polynomials of degree m and n, respectively, and $H^{\alpha}(z^{-1})$ represents one of the fractional generating functions listed in Table 1.

In this paper we present a novel algorithm for obtaining digital pole-zero approximations to fractional-order integrators and differentiators of type (3). The new approach adopts techniques used in the signal modeling of deterministic signals, namely the Padé, the Prony and the Shanks methods. The process for obtaining an approximation can be synthesized in the following steps: *i*) discretize the fractional-order operator s^{α} using one of the listed generating functions, *ii*) obtain the impulse response of the discretized fractional-order operator using the PSE method (*i.e.*, Taylor series), and *iii*) apply the signal modeling techniques of Padé, Prony or Shanks between the impulse responses of the digital fractional-order operator and the desired pole-zero approximation. The proposed strategy represents an alternative form to other existent methods, namely the widely used CFE method.

Bearing these ideas in mind, the paper is organized as follows. Section 2 derives the impulse responses of the Euler, Tustin and Al-Alaoui generating functions. Section 3 gives an introduction to the problem and develops the signal modeling techniques of Padé, Prony and Shanks, in order to get pole-zero approximations to fractional-order operators. Section 4 presents some illustrative examples showing the effectiveness of the proposed strategy. Finally, section 5 draws the main conclusions.

2. IMPULSE RESPONSE OF DIGITAL FRACTIONAL-ORDER INTEGRATORS AND DIFFERENTIATORS

The impulse response of the Euler generating function, ${}^{E}h^{\alpha}(k)$, is obtained by taking the power series expansion (PSE) over ${}^{E}H^{\alpha}(z^{-1})$. By doing so, it gives:

$$\begin{aligned}
{}^{E}H^{\alpha}\left(z^{-1}\right) &= \left[\frac{1}{T}\left(1-z^{-1}\right)\right]^{\alpha} \\
&= \left(\frac{1}{T}\right)^{\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^{-k} \\
&= \sum_{k=0}^{\infty} {}^{E}h^{\alpha}(k) z^{-k} \qquad (4)
\end{aligned}$$

Then, its impulse response, ${}^{E}h^{\alpha}(k)$, is ($k \geq 0$):

$${}^{E}h^{\alpha}(k) = \left(\frac{1}{T}\right)^{\alpha} (-1)^k \binom{\alpha}{k} \qquad (5)$$

Proceeding in the same manner for the Tustin and the Al-Alaoui generating functions, ${}^{T}H^{\alpha}(z^{-1})$ and ${}^{A}H^{\alpha}(z^{-1})$, respectively, we have:

$$\begin{aligned}
{}^{T}H^{\alpha}\left(z^{-1}\right) &= \left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^{\alpha} \\
&= \left(\frac{2}{T}\right)^{\alpha} \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} (-1)^j \binom{\alpha}{j}\binom{-\alpha}{k-j}\right] z^{-k} \\
&= \sum_{k=0}^{\infty} {}^{T}h^{\alpha}(k) z^{-k} \qquad (6)
\end{aligned}$$

$$
{}^{A}H^{\alpha}\left(z^{-1}\right)=\left(\frac{8}{7T}\frac{1-z^{-1}}{1+z^{-1}/7}\right)^{\alpha}
$$

$$
=\left(\frac{8}{7T}\right)^{\alpha}\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}(-1)^{j}\left(\frac{1}{7}\right)^{k-j}\binom{\alpha}{j}\binom{-\alpha}{k-j}\right]z^{-k}
$$

$$
=\sum_{k=0}^{\infty}{}^{A}h^{\alpha}(k)z^{-k} \tag{7}
$$

Hence, their respective impulse responses, ${}^{T}h^{\alpha}(k)$ and ${}^{A}h^{\alpha}(k)$, are given as ($k \geq 0$):

$$
{}^{T}h^{\alpha}(k)=\left(\frac{2}{T}\right)^{\alpha}\sum_{j=0}^{k}(-1)^{j}\binom{\alpha}{j}\binom{-\alpha}{k-j} \tag{8}
$$

$$
{}^{A}h^{\alpha}(k)=\left(\frac{8}{7T}\right)^{\alpha}\sum_{j=0}^{k}(-1)^{j}\left(\frac{1}{7}\right)^{k-j}\binom{\alpha}{j}\binom{-\alpha}{k-j} \tag{9}
$$

Notice that the PSE method leads to impulse sequences of infinite duration. For a practically realizable form we need to truncate these sequences yielding approximations in the form of *finite impulse responses* (FIR filters).

3. SIGNAL MODELING

The pole-zero approximation $H(z^{-1})$ (IIR filter) to be designed has the form:

$$
H\left(z^{-1}\right)=\frac{B(z)}{A(z)}=\frac{b_0+b_1z^{-1}+\cdots+b_mz^{-m}}{1+a_1z^{-1}+\cdots+a_nz^{-n}} \tag{10}
$$

where $m \leq n$. The impulse response $h(k)$ is related to $H(z^{-1})$ by the Z-transform:

$$
H\left(z^{-1}\right)=\sum_{k=0}^{\infty}h(k)z^{-k} \tag{11}
$$

The pole-zero approximation (10) has $m + n + 1$ parameters, namely the coefficients a_k ($k = 1, \ldots, n$) and b_k ($k = 0, \ldots, m$), which can be selected to minimize some error criterion. Usually, we adopt the least-squares (LS) method in order to minimize the error $e_{LS}(k) = h^{\alpha}(k) - h(k)$, as shown in Fig. 1:

$$
E_{LS}=\sum_{k=0}^{N-1}\left[h^{\alpha}(k)-h(k)\right]^{2} \tag{12}
$$

where N is the number of impulse values used in the summation. However, the LS approach leads to a non-linear problem for the model parameters (a_k, b_k), which requires the solution of a set of nonlinear equations.

If we rewrite (10) as $H(z)A(z) = B(z)$, and assuming that $h^{\alpha}(k)$ is given approximately by the impulse response of $H(z^{-1})$, one can write the time-domain equation of (10) as:

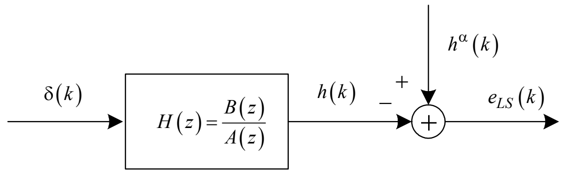

Fig. 1. Least-squares (direct) method for signal modeling.

$$
h^{\alpha}(k)+\sum_{i=1}^{n}a_ih^{\alpha}(k-i)=\begin{cases}b_k, & k=0,1,\ldots,m\\ 0, & k>m\end{cases} \tag{13}
$$

This gives a set of linear equations, which can be used in different ways to solve for the coefficients a_k and b_k. Our objective is to use simple (indirect) methods that can handle more easily the determination of the model parameters. In this perspective, this study considers three linear suboptimal solutions: the Padé approximation, the Prony's method and the Shanks' method (Hayes, 1996; Barbosa, *et al.*, 2004). In the sequel we describe these methods, for which it is assumed that $h^{\alpha}(k) = 0$ for $k < 0$ (*i.e.*, a causal system).

3.1 Padé Approximation

The Padé approximation method yields a pole-zero model that have an exactly fit to $h^{\alpha}(k)$ for the first $m + n + 1$ values of k. Then, Eq. (13) becomes:

$$
h^{\alpha}(k)+\sum_{i=1}^{n}a_ih^{\alpha}(k-i)=\begin{cases}b_k, & k=0,1,\ldots,m\\ 0, & k=m+1,\ldots,m+n\end{cases} \tag{14}
$$

Two steps are used to solve for a_k and b_k:

1) Determine a_k using the last n equations in the lower part of system (14), *i.e.*, in matrix form:

$$
\boldsymbol{H}_2\boldsymbol{a}=-\boldsymbol{h}_{21} \tag{15}
$$

where

$$
\boldsymbol{a}=\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{bmatrix}\in\mathbb{R}^n,\ \boldsymbol{h}_{21}=\begin{bmatrix}h^{\alpha}(m+1)\\ h^{\alpha}(m+2)\\ \vdots\\ h^{\alpha}(m+n)\end{bmatrix}\in\mathbb{R}^n
$$

$$
\boldsymbol{H}_2=\begin{bmatrix}h^{\alpha}(m) & h^{\alpha}(m-1) & \cdots & h^{\alpha}(m-n+1)\\ h^{\alpha}(m+1) & h^{\alpha}(m) & \cdots & h^{\alpha}(m-n+2)\\ \vdots & \vdots & \ddots & \vdots\\ h^{\alpha}(m+n-1) & h^{\alpha}(m+n-2) & \cdots & h^{\alpha}(m)\end{bmatrix}
$$

$H_2 \in \mathbb{R}^{n\times n}$ is a non-symmetric Toeplitz matrix. If $\boldsymbol{H}_2$ is nonsingular, a_k are uniquely determined by:

$$
\boldsymbol{a}=-\left(\boldsymbol{H}_2\right)^{-1}\boldsymbol{h}_{21} \tag{16}
$$

2) With a_k given, solve for b_k using the first (m+1) equations of system (14), *i.e.*, in matrix form:

$$\boldsymbol{b} = \boldsymbol{H}_1 \overline{\boldsymbol{a}} \tag{17}$$

where

$$\overline{\boldsymbol{a}} = \begin{bmatrix} 1 \\ \boldsymbol{a} \end{bmatrix} \in \mathbb{R}^{n+1}, \boldsymbol{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^{m+1}$$

$$\boldsymbol{H}_1 = \begin{bmatrix} h^\alpha(0) & 0 & \cdots & 0 \\ h^\alpha(1) & h^\alpha(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h^\alpha(m) & h^\alpha(m-1) & \cdots & h^\alpha(m-n) \end{bmatrix}$$

$\boldsymbol{H}_1 \in \mathbb{R}^{(m+1)\times(n+1)}$.

3.2 Prony's Method

Prony's method differs from the Padé approximation method in the form of finding the denominator coefficients a_k ($k = 1, 2, \ldots, n$) (Fig. 2). These are determined by LS minimization of the error $e_P(k) = a_k * h^\alpha(k) - b_k$ (where the symbol * denotes convolution), which for $k = m+1, \ldots, N-1$ becomes:

$$e_P(k) = h^\alpha(k) + \sum_{i=1}^{n} a_i h^\alpha(k-i) \tag{18}$$

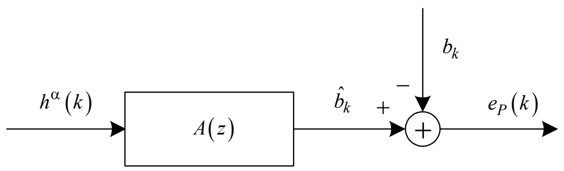

Fig. 2. Prony's method for signal modeling.

Two steps are used to solve for a_k and b_k:

1) Determine a_k by setting the error $e_P(k) = 0$ in (18) and writing these equations in matrix form:

$$\boldsymbol{H}_2 \boldsymbol{a} = -\boldsymbol{h}_{21} \tag{19}$$

where

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n, \ \boldsymbol{h}_{21} = \begin{bmatrix} h^\alpha(m+1) \\ h^\alpha(m+2) \\ \vdots \\ h^\alpha(N-1) \end{bmatrix} \in \mathbb{R}^{N-m-1}$$

$$\boldsymbol{H}_2 = \begin{bmatrix} h^\alpha(m) & h^\alpha(m-1) & \cdots & h^\alpha(m-n+1) \\ h^\alpha(m+1) & h^\alpha(m) & \cdots & h^\alpha(m-n+2) \\ \vdots & \vdots & \ddots & \vdots \\ h^\alpha(N-2) & h^\alpha(N-3) & \cdots & h^\alpha(N-n-1) \end{bmatrix}$$

$\boldsymbol{H}_2 \in \mathbb{R}^{(N-m-1)\times n}$. It is obvious that in this case (19) cannot be solved exactly. Therefore, we find the LS solution by solving the *normal equations*:

$$\left(\boldsymbol{H}_2^T \boldsymbol{H}_2\right)\boldsymbol{a} = -\boldsymbol{H}_2^T \boldsymbol{h}_{21} \tag{20}$$

If $\left(\boldsymbol{H}_2^T \boldsymbol{H}_2\right) \in \mathbb{R}^{n\times n}$ is nonsingular then the optimum coefficients a_k are given by:

$$\begin{aligned} \boldsymbol{a} &= -\left(\boldsymbol{H}_2^T \boldsymbol{H}_2\right)^{-1} \boldsymbol{H}_2^T \boldsymbol{h}_{21} \\ &= -\boldsymbol{H}_2^+ \boldsymbol{h}_{21} \end{aligned} \tag{21}$$

where $\boldsymbol{H}_2^+ = (\boldsymbol{H}_2^T\boldsymbol{H}_2)^{-1}\boldsymbol{H}_2^T$ is the pseudoinverse of $\boldsymbol{H}_2$.

2) With a_k given, determine b_k using the same way as in the Padé approximation method (*step* 2), *i.e.* by an exact fit over the interval $[0, m]$.

3.3 Shanks' Method

Shank's method provides an alternative to Prony's method of finding the numerator coefficients b_k ($k = 0, 1, \ldots, m$) (Fig. 3). Instead of forcing an exact fit for the first m+1 values of the impulse response, it performs a least squares minimization of the error $e_S(k) = h^\alpha(k) - \hat{h}(k)$ over the interval $[0, N-1]$:

$$e_S(k) = h^\alpha(k) - \sum_{i=0}^{m} b_i g(k-i), \ \ k = 0, 1, \ldots, N-1 \tag{22}$$

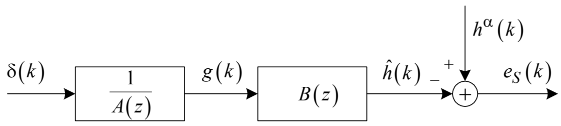

Fig. 3. Shanks' method for signal modeling.

Two steps are used to solve for a_k and b_k:

1) Determine a_k in the same way as in Prony's method (*step* 1), *i.e.*, by LS fit over the interval $[m+1, N-1]$.

2) With a_k given, determine b_k following the sequence illustrated in Fig. 3:

a) Compute the impulse response $g(k)$ of the filter $1/A(z)$ using, for example, the recursion:

$$g(k) = \delta(k) - \sum_{i=1}^{n} a_i g(k-i), \quad k = 0, 1, \ldots, N-1 \tag{23}$$

with $g(k) = 0$ for $k < 0$.

b) Solve for b_k by setting the error $e_S(k) = 0$ in (22) and writing these equations in matrix form:

$$\boldsymbol{G}\boldsymbol{b} = \boldsymbol{h} \tag{24}$$

where

$$\boldsymbol{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^{m+1}, \ \boldsymbol{h} = \begin{bmatrix} h^\alpha(0) \\ h^\alpha(1) \\ \vdots \\ h^\alpha(N-1) \end{bmatrix} \in \mathbb{R}^{N}$$

$$\boldsymbol{G} = \begin{bmatrix} g(0) & 0 & \cdots & 0 \\ g(1) & g(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g(N-1) & g(N-2) & \cdots & g(N-m-1) \end{bmatrix}$$

$\boldsymbol{G} \in \mathbb{R}^{N\times(m+1)}$. The LS solution is found by solving the linear equations:

$$\left(\boldsymbol{G}^T\boldsymbol{G}\right)\boldsymbol{b} = \boldsymbol{G}^T\boldsymbol{h} \qquad (25)$$

If $\left(\boldsymbol{G}^T\boldsymbol{G}\right) \in \mathbb{R}^{(m+1)\times(m+1)}$ is nonsingular then the optimum coefficients b_k are given by:

$$\begin{aligned} \boldsymbol{b} &= \left(\boldsymbol{G}^T\boldsymbol{G}\right)^{-1}\boldsymbol{G}^T\boldsymbol{h} \\ &= \boldsymbol{G}^{+}\boldsymbol{h} \end{aligned} \qquad (26)$$

where $\boldsymbol{G}^{+} = (\boldsymbol{G}^T\mathbf{G})^{-1}\boldsymbol{G}^T$ is the pseudoinverse of $\boldsymbol{G}$.

4. ILLUSTRATIVE EXAMPLES

In this section we use the signal modeling techniques described in the previous section to develop digital pole-zero approximations to s^{α}, for $\alpha = \pm 1/2$, and sampled at $T = 0.01$ s. It is adopted an impulse sequence length of $N = 1000$. Moreover, in practice, we normally set $m = n$ (Vinagre, *et al.*, 2000; Chen and Moore, 2002; Barbosa, *et al.*, 2004) because the case of $m < n$ leads to inferior results.

Fig. 4 depicts the Bode plots and the unit step responses of the approximations to Tustin operator for $\alpha = -1/2$, $N = 1000$ and $m = n = 1, 3, \ldots, 9$. For comparison purposes, we also plot the curves of the pole-zero approximation obtained by the Padé (or the CFE) method for $m = n = 5$, $G_5(z^{-1})$. As can be observed, the approximations are well fitted in the ideal responses (dotted lines) both in the frequency and the time responses. We also verify that Prony's approximation performs a better fitting in the low frequency range (steady-state time response) than the Padé (or the CFE) approximation. This may be justified by the fact that Prony's method performs a LS fitting over a wide range of impulse samples (*e.g.*, for $[m+1, N-1]$), while the Padé method produces an exact fit for the first $m+n+1$ samples of the impulse response, with any guarantee about the accuracy of the approximation for $k > m+n$.

Fig. 5 shows the distribution of zeros and poles of the approximations to Tustin operator for $\alpha = -1/2$, $N = 1000$ and $m = n = 1, 2, \ldots, 9$. We observe that the approximations satisfy two desired properties: (*i*) all the poles and zeros lie inside the unit circle, and (*ii*) the poles and zeros are interlaced along the segment of the real axis corresponding to $z \in (-1, 1)$.

To further illustrate the effectiveness of the proposed techniques, the approximations are used to calculate the *differintegral* of the unit step function that occurs at $t = t_0$, $u(t-t_0)$, and the sine function $s(t)$:

$$u(t-t_0) = \begin{cases} 1, & t \geq t_0 \\ 0, & t < t_0 \end{cases} \qquad (27)$$

$$s(t) = \sin(t)u(t) \qquad (28)$$

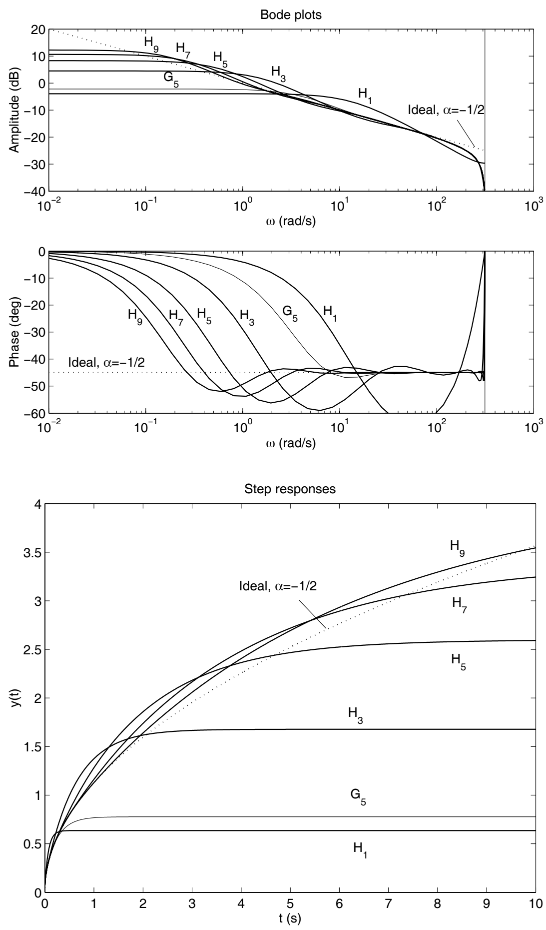

Fig. 4. Bode plots and unit step responses of Prony's approximation to Tustin operator for $\alpha = -1/2$, $N = 1000$, $T = 0.01$ s and $m = n = 1, 3, \ldots, 9$. $G_5(z)$ is the CFE (or the Padé) approximation.

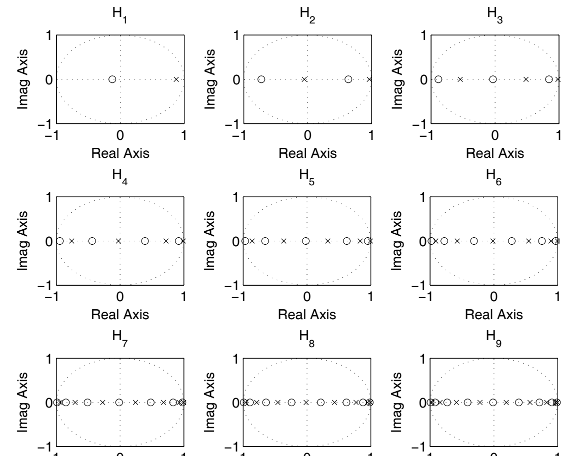

Fig. 5. Pole-zero maps of the Prony's approximation to Tustin operator for $\alpha = -1/2$, $N = 1000$, $T = 0.01$ s and $m = n = 1, 2, \ldots, 9$.

The *differintegral* of the unit step function $u(t-t_0)$ is given by (Podlubny, 1999):

$$D^{\alpha}\left[u\left(t-t_0\right)\right]=\begin{cases}\dfrac{\left(t-t_0\right)^{\alpha}}{\Gamma\left(1-\alpha\right)}, & t \geq t_0 \\ 0, & t < t_0\end{cases} \tag{29}$$

The sine function $s(t)$ is calculated for the semiderivative ($\alpha = 1/2$) and semiintegral ($\alpha = -1/2$) (Oldham and Spanier, 1974):

$$D^{1/2}\left[\sin\left(t\right)\right]=\sin\left(t+\pi/4\right)-\sqrt{2}g\left(\sqrt{2t/\pi}\right) \tag{30}$$

$$D^{-1/2}\left[\sin\left(t\right)\right]=\sin\left(t-\pi/4\right)+\sqrt{2}f\left(\sqrt{2t/\pi}\right) \tag{31}$$

where $f(\cdot)$ and $g(\cdot)$ are the auxiliary Fresnel integrals (Abramowitz and Stegun, 1974). In (30) and (31), the first and second terms represent the *steady-state* and the *transient* responses, respectively.

Figs. 6 and 7 show the semiintegral ($\alpha = -1/2$) and semiderivative ($\alpha = 1/2$) of the functions $u(t-1)$ and $s(t)$ calculated with the Shanks' and the Prony's approximations, respectively. Once more, we can see the effectiveness of the approximations fitting the ideal curves (dotted lines). Obviously, we may tune the order $m = n$ of the approximation along with the sampling period T to get better agreement between the two curves (the ideal and the calculated).

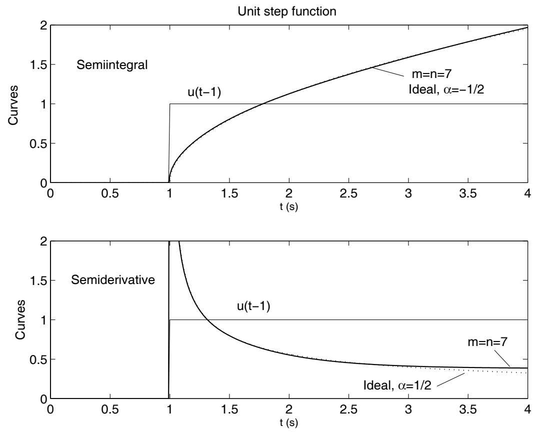

Fig. 6. The semiintegral ($\alpha = -1/2$) and semiderivative ($\alpha = 1/2$) of unit step function $u(t-1)$ calculated with Shanks' approximation to Al-Alaoui operator for $N = 1000$, $T = 0.01$ s and $m = n = 7$.

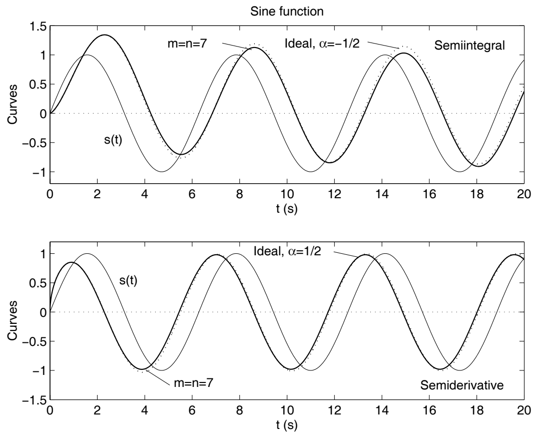

Fig. 7. The semiintegral ($\alpha = -1/2$) and semiderivative ($\alpha = 1/2$) of sine function $s(t)$ calculated with Prony's approximation to Euler operator for $N = 1000$, $T = 0.01$ s and $m = n = 7$.

5. CONCLUSIONS

We have described the application of the signal modeling techniques for deterministic signals in the design of digital pole-zero approximations (IIR filters) to fractional-order integrators and differentiators. The resulting approximations are causal, stable and minimum-phase suitable for a real-time implementation. The illustrated techniques of Padé, Prony and Shanks yield good approximations both in the time and the frequency domains. Moreover, it can produce superior approximations than other existent methods, namely the widely used CFE method. The Padé and the CFE methods produce the same pole-zero approximation ($m = n$). Some examples are given that shows the effectiveness of the proposed techniques.

REFERENCES

Abramowitz M. and I. A. Stegun (1974). *Handbook of Mathemathical Functions*. Dover Publications, New York.

Al-Alaoui, M. A. (1993). Novel Digital Integrator and Diferentiator. *Electronics Letters*, **29**(4), 376–378.

Barbosa, Ramiro S., J. A. Tenreiro Machado and Isabel M. Ferreira (2004). Least-Squares Design of Digital Fractional-Order Operators. In: *Proc. of the 1st IFAC Workshop on Fractional Differentiation and its Applications (FDA'04)*, pp. 434–439. ENSEIRB, Bordeaux, France.

Chen, Y. Q. and K. L. Moore (2002). Discretization Schemes for Fractional-Order Differentiators and Integrators. *IEEE Transactions on Circuits and Systems–I: Fundamental Theory and Applications*, **49**(3), 363–367.

Hayes, Monson H. (1996). *Statistical Digital Signal Processing and Modeling*. John Wiley & Sons, New York.

Hilfer, R. (2000). *Applications of Fractional Calculus in Physics*. World Scientific, Singapore.

Machado, J. A. Tenreiro (2001). Discrete-Time Fractional-Order Controllers. *Fractional Calculus & Applied Analysis*, **4**(1), 47–66.

Oldham, K. B. and J. Spanier (1974). *The Fractional Calculus*. Academic Press, New York.

Oustaloup, A. (1995). *The Dérivation Non Entière*. Éditions Hermès, Paris.

Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press, San Diego.

Vinagre, B., I. Podlubny, A. Hernandez, and V. Feliu (2000). Some Approximations of Fractional Order Operators Used in Control Theory and Applications. *Fractional Calculus & Applied Analysis*, **3**(3), 231–248.