UNIFORM SEMIGLOBAL ASYMPTOTIC STABILITY FOR TIME-VARYING NONLINEAR CASCADED SYSTEMS

Antoine Chaillet* **Antonio Loría***

∗CNRS–LSS, Supélec, 3 rue Joliot Curie, 91192 Gif s/Yvette, France, `chaillet@lss.supelec.fr`, `loria@lss.supelec.fr`

Abstract: In this paper we address the stability analysis problem for cascades of systems that are semiglobally uniformly asymptotically stable (USGPAS). That is, we establish that, under a uniform boundedness condition, the cascade of two USGPAS systems remains USGPAS. Our results generalize fundamental well known results for uniform global asymptotic stability for time-varying, and hence for autonomous, cascades.
Copyright© 2005 IFAC

Key Words— Semiglobal Stability, Practical Stability, Nonlinear time-varying systems, cascaded systems.

1 Introduction

Cascaded dynamical systems appear in many applications whether naturally or intentionally provoked by the control design. For instance, the cascades-based control approach consists in designing the control law so that the closed loop system has a cascaded structure. Such strategy has the advantage of, often, reducing the complexity of the controller and the difficulty of the stability analysis. This is considerably attractive when dealing with non-autonomous systems (e.g. in tracking control and or time-varying stabilization problems) since uniform forms of stability may be concluded without Lyapunov functions satisfying the usual (restrictive) conditions of sign-definiteness of the function itself and its derivative. Furthermore, in problems such as output feedback control, the cascades-based approach may lead, under appropriate conditions, to nonlinear separation principles.

Hence, on one hand the advantages that offer the analysis of nonlinear cascaded systems in control applications (see e.g. (Loría and Panteley 2005, Lefeber 2000)) as well as the complexity of the problem itself (see the seminal paper (Sussman and Kokotović 1991) or (Sepulchre *et al.* 1997) and references therein) has motivated researchers to study cascaded systems from different viewpoints and under a diversity of conditions. In general terms, for autonomous as for non-autonomous systems, one may retain that cascades of uniformly globally asymptotically stable systems (UGAS) remain UGAS if and only if the solutions are uniformly globally bounded (UGB) (cf. (Sontag 1989, Seibert and Suárez 1990) for the autonomous case and (Panteley and Loría 2001) for time-varying systems). The remaining fundamental question is how to guarantee UGB. One way, is by ensuring the stronger property of Input to State Stability (ISS); other conditions are formulated in terms of growth-rate conditions. The literature on this subject is very reach, specially in the domain of time-invariant systems. See the references in (Loría and Panteley 2005, Sepulchre *et al.* 1997, Sontag 2003) as well as (Arcak *et al.* 2002) and recent works on integrator forwarding. See also (Angeli *et al.* 2000) for a recent reference on semiglobal versions of ISS.

A considerable drawback, from a control-practice viewpoint, of most results on stability of cascades is that they address the problem of guaranteeing *global* properties. However, it is often the case, for instance in output-feedback control problems (see e.g. (Marino and Tomei 1993)), that only *semiglobal* properties can be concluded, either because of technical obstacles in the control design (due to high nonlinearities in the model as for instance in mechanical systems) or due to the physical nature of the plant (e.g. multiple equilibria). In this case the cascades approach based on global results, which has proved to be so useful, fails both in the control design and the stability analysis. Another important situation where classical results fail is when one cannot ensure asymptotic stability of the cascaded subsystems, taken separately. That is, when only some type of robust stability, such as convergence to "balls" may be ensured.

In this paper we address the stability analysis problem for cascades of systems that are uniformly semiglobally practically asymptotically stable (USGPAS, cf. Definition 3). That is, we establish that, under a uniform boundedness condition, the cascade of two USGPAS systems remains USGPAS. Our main result extends in this direction, (Panteley and Loría 2001, Lemma 1) and the main (and fundamental) results of (Sontag 1989, Seibert and Suárez 1990).

The rest of the paper is organized as follows. In next section we present some definitions of stability and two auxiliary propositions on local asymptotic stability with respect to balls and on semiglobal practical asymptotic stability. Our main result is presented in section 3 and its proof is given in section 4. We finally conclude with some remarks.

2 Preliminaries

Notation. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is strictly increasing and $\alpha(0) = 0$; $\alpha \in \mathcal{K}_\infty$ if, in addition, $\alpha(s) \to \infty$ as $s \to \infty$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{L}$ ($\sigma \in \mathcal{L}$) if it is non-increasing and tends to zero as its argument tends to infinity. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ is said to be a *class* $\mathcal{KL}$ *function* if, $\beta(\cdot, t) \in \mathcal{K}$ for any $t \geq 0$, and $\beta(s, \cdot) \in \mathcal{L}$ for any $s \geq 0$. We denote by $x(\cdot, t_0, x_0)$ the solutions of the differential equation $\dot{x} = f(t, x)$ with initial conditions (t_0, x_0). We denote by $\mathcal{B}_\delta$ the closed ball in $\mathbb{R}^n$ of radius δ. We use $\|\cdot\|$ for the Euclidean norm of vectors and the induced L_2 norm of matrices. We define $\|x\|_\delta := \inf_{z \in \mathcal{B}_\delta} \|x - z\|$. We designate by $\mathbb{N}_{\leq N}$ the set of all nonnegative integers less than or equal to N. When the context is sufficiently explicit, we may omit to write the arguments of a function by commodity.

2.1 Local asymptotic stability w.r.t. balls

We start by presenting some definitions concerning Uniform Local (or Global) Asymptotic Stability with respect to a set for nonlinear time-varying (NTLV) systems:

$$\dot{x} = f(t, x) , \qquad (1)$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$ and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz in x and piecewise continuous in t . For our present purpose, we define these notions in the particular case when the set in an open ball of $\mathbb{R}^n$.

Definition 1 (ULAS / UGAS w.r.t. a ball) Let δ and Δ be positive numbers such that $\Delta > \delta$. A system $\dot{x} = f(t, x)$ is said to be *Uniformly Locally Asymptotically Stable on* $\mathcal{B}_\Delta$ *with respect to* $\mathcal{B}_\delta$ if there exists a class $\mathcal{KL}$ function β such that its solutions starting from any initial state x_0 in $\mathcal{B}_\Delta$ at any initial time $t_0 \geq 0$ satisfy

$$\|x(t, t_0, x_0)\|_\delta \leq \beta(\|x_0\|, t - t_0) , \quad \forall t \geq t_0 .$$

The system is said to be *Uniformly Globally Asymptotically Stable with respect to* $\mathcal{B}_\delta$ if this property holds for any $x_0 \in \mathbb{R}^n$.

Remark 1 Note that the "ULAS with respect to a ball" as defined here is less restrictive than the time-varying adaptation of "Asymptotic Stability with respect to a set" given in (Lin *et al.* 1996) for the case when the set is a ball. Indeed, in the latter reference, it is imposed that the ball $\mathcal{B}_\delta$ be positively invariant.

Remark 2 If a system is ULAS on $\mathcal{B}_\Delta$ with respect to $\mathcal{B}_\delta$, then it is also ULAS on $\mathcal{B}_{\Delta'}$ with respect to $\mathcal{B}_{\delta'}$, for any $\delta' \geq \delta$ and $\Delta' \leq \Delta$ such that $\Delta' > \delta'$.

Definition 2 (ULB) The solutions of (1) are said to be *Uniformly Locally Bounded on the compact set* $\mathcal{A} \subset \mathbb{R}^n$ if there exist a class $\mathcal{K}$ function γ and a nonnegative constant μ such that, for any initial time $t_0 \in \mathbb{R}_{\geq 0}$ and any initial state $x_0 \in \mathcal{A}$, it holds that

$$\|x(t, t_0, x_0)\| \leq \gamma(\|x_0\|) + \mu , \quad \forall t \geq t_0 .$$

2.2 Semiglobal practical properties

Consider a parameterized nonlinear time-varying system of the form

$$\dot{x} = f(t, x, \theta) , \qquad (2)$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_{\geq 0}$, $\theta \in \mathbb{R}^m$ is a constant parameter and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in x and piecewise continuous in (t, θ).

Definition 3 (USGPAS) The system (2) is said to be *Uniformly Semiglobally Practically Asymptotically Stable on the parameter set* $\Theta \subset \mathbb{R}^m$ if, given any $\Delta > \delta > 0$, there exists a parameter $\theta^\star \in \Theta$ such that $\dot{x} = f(t, x, \theta^\star)$ is ULAS on $\mathcal{B}_\Delta$ with respect to $\mathcal{B}_\delta$.

Proposition 1 (Lyapunov condition for USGPAS) Suppose that, given any positive numbers Δ and δ such that $\Delta > \delta$, there exist a parameter $\theta^\star \in \Theta$, a continuously differentiable Lyapunov function V and class $\mathcal{K}$ functions α_1, α_2, α_3 such that, for any $t \geq 0$ and any $x \in \mathcal{B}_\Delta$,

$$\alpha_1(\|x\|_\delta) \leq V(t, x) \leq \alpha_2(\|x\|) , \qquad (3)$$

$$\|x\| > \delta \Rightarrow \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, \theta^\star) \leq -\alpha_3(V) . \qquad (4)$$

Then (2) is USGPAS on the set Θ.

The proof of this result is omitted here by lack of space. It can however be found in (Chaillet A. and Loría A. 2005).

We introduce now the following notation in order to simplify the statement of our main results.

Definition 4 ($\mathcal{D}$-set) For any $\Delta > \delta > 0$, the $\mathcal{D}$*-set* of (2) is defined as

$$\mathcal{D}_f(\delta, \Delta) := \{\theta \in \mathbb{R}^m | (2) \text{ is ULAS on } \mathcal{B}_\Delta \text{ w.r.t } \mathcal{B}_\delta\}.$$

3 Main results

We consider cascaded systems of the form

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, \theta_1) + g(t, x_1, x_2)x_2 \\ \dot{x}_2 = f_2(t, x_2, \theta_2) \end{cases}, \tag{5}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $t \in \mathbb{R}_{\geq 0}$, $\theta_1 \in \mathbb{R}^{m_1}$, $\theta_2 \in \mathbb{R}^{m_2}$, f_1, f_2 and g are locally Lipschitz in the state and piecewise continuous in the time, and f_1 and f_2 are piecewise continuous in the parameter.

We shall consider the stability of (5) under the following assumptions.

Assumption 1 The interconnection term g is uniformly bounded in time, i.e. there exists a non-decreasing function G such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and all $t \geq 0$,

$$\|g(t, x_1, x_2)\| \leq G(\|(x_1, x_2)\|). \tag{6}$$

Assumption 2 Given any $\Delta_1 > \delta_1 > 0$, there exists a parameter $\theta_1^\star(\delta_1, \Delta_1) \in \Theta_1$, a smooth Lyapunov function V_1, class $\mathcal{K}_\infty$ functions α_1 and α_2, a class $\mathcal{K}$ functions α_4, a continuous positive non-decreasing function c_1, and a positive constant ϵ such that, for any $x_1 \in \mathbb{R}^{n_1}$ and any $t \geq 0$,

$$\alpha_1(\|x_1\|_{\delta_1}) \leq V_1(t, x_1) \leq \alpha_2(\|x_1\|)$$

$$\|x_1\| \geq \delta_1 \Rightarrow \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t, x_1, \theta_1^\star) \leq -\epsilon V_1 + \alpha_4(\|x_1\|_{\Delta_1})$$

$$\left\| \frac{\partial V_1}{\partial x_1}(t, x_1) \right\| \leq c_1(\|x_1\|).$$

Assumption 3 The system $\dot{x}_2 = f_2(t, x_2, \theta_2)$ is USGPAS on Θ_2.

Assumption 4 Given any positive numbers δ_1, Δ_1, δ_2, Δ_2, such that $\Delta_1 > \delta_1$ and $\Delta_2 > \delta_2$, and for the parameter $\theta_1^\star(\delta_1, \Delta_1)$ as defined in Assumption 2, there exists a parameter $\theta_2^\star \in \mathcal{D}_{f_2}(\delta_2, \Delta_2) \cap \Theta_2$ (see Definition 4) such that the trajectories of

$$\dot{x}_1 = f_1(t, x_1, \theta_1^\star) + g(t, x_1, x_2)x_2 \tag{7a}$$

$$\dot{x}_2 = f_2(t, x_2, \theta_2^\star) \tag{7b}$$

are ULB on $\mathcal{B}_{\Delta_1} \times \mathcal{B}_{\Delta_2}$.

Theorem 1 Under Assumptions 1, 2, 3 and 4, the cascaded system (5) is USGPAS on $\Theta_1 \times \Theta_2$.

Proof. See section 4. ■

Remark 3 In view of Proposition 1, Assumption 2 implies that the subsystem $\dot{x}_1 = f_1(t, x_1, \theta_1)$ is USGPAS on Θ_1. Hence, roughly speaking, Theorem 1 states that, under a condition of boundedness of solutions and provided the knowledge of a Lyapunov function, the cascade composed of two USGPAS systems remains USGPAS. The requirement on the gradient of V_1 in Assumption 2 is little restrictive, and is satisfied in many concrete applications. See (Chaillet A. and Loría A. 2005) for an example in robot control.

4 Proof of Theorem 1

For any positive numbers δ_1, Δ_1, δ_2 and Δ_2 such that $\delta_1 < \Delta_1$ and $\delta_2 < \Delta_2$, choose a $\theta_1^\star \in \Theta_1$ satisfying Assumption 2 and any $\theta_2^\star \in \mathcal{D}_{f_2}(\delta_2, \Delta_2) \cap \Theta_2$ given by Assumption 4. We first show that there exist $\delta > 0$ and $\Delta > 0$ such that (7) is ULAS on $\mathcal{B}_\Delta$ with respect to $\mathcal{B}_\delta$. To that end, we first show that the system is "stable w.r.t. a ball", more precisely, we construct $\underline{\alpha} \in \mathcal{K}_\infty$ and $\underline{\delta}_3 > 0$ such that

$$\|x_1(t, t_0, x_{10})\|_{\underline{\delta}_3} \leq \underline{\alpha}(\|x_0\|). \tag{8}$$

We then use this property to prove that a ball, larger than $\mathcal{B}_{\delta_3}$, is ULA and we construct a $\mathcal{KL}$ estimate for the solutions. Finally, we show that the estimates of the domain of attraction and of the ball to which solutions converge can be arbitrarily enlarged and diminished respectively.

4.1 Proof of "stability w.r.t. a ball"

The time derivative of V_1 along the trajectories of (7) yields

$$\dot{V}_1 = \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1}\big(f_1(t, x_1, \theta_1^\star) + g(t, x_1, x_2)x_2\big).$$

Therefore, according to Assumption 2, for any $x_1 \in \mathbb{R}^{n_1} \setminus \mathcal{B}_{\delta_1}$,

$$\begin{aligned} \dot{V}_1 &\leq -\epsilon V_1 + \left\| \frac{\partial V_1}{\partial x_1} \right\| \|g(t, x_1, x_2)\| \, \|x_2\| + \alpha_4(\|x_1\|_{\Delta_1}) \\ &\leq -\epsilon V_1 + c_1(\|x_1\|) G(\|x\|) \, \|x_2\| + \alpha_4(\|x_1\|_{\Delta_1}), \end{aligned}$$

where $x := (x_1, x_2)$. Defining

$$\Gamma := \{t \geq t_0 \mid \|x_1(t, t_0, x_{10})\| \geq \delta_1\}, \tag{9}$$

and using the shorthand notation $x_1(t)$ for $x_1(t, t_0, x_0)$ and $v_1(t) := V_1(t, x_1(t))$ we get that, for any $x_0 \in \mathbb{R}^{n_1}$ and any $t \in \Gamma$,

$$\begin{aligned} \dot{v}_1(t) \leq\; & -\epsilon v_1(t) + c_1(\|x_1(t)\|) G(\|x(t)\|) \, \|x_2(t)\| \\ & + \alpha_4(\|x_1(t)\|_{\Delta_1}). \end{aligned}$$

Using Assumption 4, for all $x_0 \in \mathcal{B}_{\Delta_1} \times \mathcal{B}_{\Delta_2}$ and all $t \in \Gamma$,

$$\dot{v}_1(t) \leq -\epsilon v_1(t) + c_2(\|x_0\|)\|x_2(t)\| + c_3(\|x_0\|) \quad (10)$$

where

$$c_2(\|x_0\|) := c_1(\gamma(\|x_0\|) + \mu)G(\gamma(\|x_0\|) + \mu) \quad (11)$$
$$c_3(\|x_0\|) := \alpha_4(\|\gamma(\|x_0\|) + \mu\|_{\Delta_1}) . \quad (12)$$

In addition, Assumption 3 ensures the existence of a class $\mathcal{KL}$ function β_2 such that[1] for any $x_{20} \in \mathcal{B}_{\Delta_2}$ and any $t \geq t_0$,

$$\|x_2(t, t_0, x_{20})\| \leq \beta_2(\|x_{20}\|, t - t_0) + \delta_2 . \quad (13)$$

From this and inequality (10), it follows that for all $x_0 \in \mathcal{B}_{\Delta_1} \times \mathcal{B}_{\Delta_2}$ and all $t \in \Gamma$,

$$\dot{v}_1(t) \leq -\epsilon v_1(t) + c_2(\|x_0\|)(\beta_2(\|x_{20}\|, t - t_0) + \delta_2) + c_3(\|x_0\|) \quad (14)$$

Since β_2 is a $\mathcal{KL}$ function, we have that

$$\dot{v}_1(t) \leq -\epsilon v_1(t) + c_4(\|x_0\|) \quad (15)$$

where

$$c_4(s) := c_2(s)(\beta_2(s, 0) + \delta_2) + c_3(s) , \ \forall s \geq 0 . \quad (16)$$

Now, notice that the interior of Γ can be divided into open intervals in the following way:

$$\overset{\circ}{\Gamma} = \bigcup_{i=0}^{N} \,]T_{2i} \ ; \ T_{2i+1}[\ , \quad (17)$$

where the sequence $\{T_i\}_{i \in \mathbb{N}_{\leq N}}$ is nondecreasing, $T_0 \geq t_0$, $N \in \mathbb{N}$ is potentially infinite, $T_{2i} < T_{2i+1}$ for all $i \in \mathbb{N}_{\leq N}$ and $\|x_1(T_i, t_0, x_{10})\| = \delta_1$ for all $i \geq 1$ by continuity of the solutions. We consider two cases: whether the trajectories start from outside (i.e. $T_0 = t_0$) or inside[2] $\mathcal{B}_{\delta_1}$.

<u>Case 1</u>: $T_0 = t_0$. In this case $[t_0; T_1] \subset \Gamma$. Hence integrating (15) and using a comparison lemma (see e.g. (Khalil 1996, Lemma 2.5)), we have that, for any $t \in [t_0; T_1]$,

$$\begin{aligned} \dot{v}_1(t) &\leq \big(v_1(t_0) - c_4(\|x_0\|)\big)e^{-\epsilon(t-t_0)} + c_4(\|x_0\|) \\ &\leq v_1(t_0) + c_4(\|x_0\|) . \end{aligned}$$

But, by Assumption 2,

$$\|x_1(t)\|_{\delta_1} \leq \alpha_1^{-1}(v_1(t)) , \quad v_1(t_0) \leq \alpha_2(\|x_0\|) ,$$

hence

$$\|x_1(t)\|_{\delta_1} \leq \alpha_1^{-1}\big(\alpha_2(\|x_0\|) + c_4(\|x_0\|)\big) .$$

Define next the function $\tilde{\alpha} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as

$$\tilde{\alpha}(s) := \alpha_1^{-1}\big(\alpha_2(s) + c_4(s)\big) - \alpha_1^{-1}\big(c_4(0)\big) . \quad (18)$$

In view of (16) and noticing that c_2 and c_3 are nondecreasing functions, we see that $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function, and we have

$$\|x_1(t)\|_{\delta_1} \leq \tilde{\alpha}(\|x_0\|) + \alpha_1^{-1}\big(c_4(0)\big) . \quad (19)$$

This in turn implies that, for any $x_0 \in \mathcal{B}_{\Delta_1} \times \mathcal{B}_{\Delta_2}$,

$$\|x_1(t)\|_{\tilde{\delta}_3} \leq \tilde{\alpha}(\|x_0\|) , \quad \forall t \in [t_0; T_1] , \quad (20)$$

where $\tilde{\delta}_3 := \delta_1 + \alpha_1^{-1}\big(c_4(0)\big)$, i.e.

$$\tilde{\delta}_3 = \delta_1 + \alpha_1^{-1}\big(c_1(\mu)G(\mu)\delta_2 + \alpha_4(\|\mu\|_{\Delta_1})\big) . \quad (21)$$

Furthermore, for any $t \in [T_{2i} \ ; \ T_{2i+1}]$, $i \geq 1$, we develop a similar reasoning, observing first that, by Assumption 2 and the definition of the sequence $\{T_i\}_{i \in \mathbb{N}_{\leq N}}$,

$$v_1(T_{2i}) \leq \alpha_2(\|x_1(T_{2i}, t_0, x_0)\|) = \alpha_2(\delta_1) ,$$

and, consequently, we get that

$$\|x_1(t)\|_{\delta_1} \leq \alpha_1^{-1}\big(\alpha_2(\delta_1) + c_4(\|x_0\|)\big) .$$

In other words, defining

$$\overline{\alpha}(s) := \alpha_1^{-1}\big(\alpha_2(\delta_1) + c_4(s)\big) - \alpha_1^{-1}\big(\alpha_2(\delta_1) + c_4(0)\big)$$

$$\delta_3 := \delta_1 + \alpha_1^{-1}\big(\alpha_2(\delta_1) + c_4(0)\big) ,$$

we see that $\overline{\alpha}$ is also a $\mathcal{K}_\infty$ function, and we obtain that, for all $x_0 \in \mathcal{B}_{\Delta_1} \times \mathcal{B}_{\Delta_2}$ and all $\forall t \in [T_{2i}; T_{2i+1}]$, $i \geq 1$,

$$\|x_1(t, t_0, x_{10})\|_{\delta_3} \leq \overline{\alpha}(\|x_0\|). \quad (22)$$

Thus, noticing, in view of (11), (12) and (16) that $\delta_3 \geq \tilde{\delta}_3$, and defining

$$\alpha(s) := \max\{\tilde{\alpha}(s) \ ; \ \overline{\alpha}(s)\} \quad (23)$$

(which is also a $\mathcal{K}_\infty$ function), inequalities (20) and (22) imply that, for all $x_0 \in \mathcal{B}_{\Delta_1} \times \mathcal{B}_{\Delta_2}$,

$$\|x_1(t, t_0, x_{10})\|_{\delta_3} \leq \alpha(\|x_0\|) , \ \forall t \in \Gamma . \quad (24)$$

<u>Case 2</u>: $T_0 > t_0$. In this case, by Assumption 2 and the definition of Γ we have that, for any $i \in \mathbb{N}_{\leq N}$,

$$v_1(T_{2i}) \leq \alpha_2(\|x_1(T_{2i}, t_0, x_0)\| = \alpha_2(\delta_1) .$$

Hence, following the same reasoning as before, we obtain again that, for all $x_0 \in \mathcal{B}_{\Delta_1} \times \mathcal{B}_{\Delta_2}$,

$$\|x_1(t, t_0, x_{10})\|_{\delta_3} \leq \alpha(\|x_0\|) , \ \forall t \in \Gamma . \quad (25)$$

Notice finally that, for any $t \in \mathbb{R}_{\geq t_0} \setminus \Gamma$, we have that $\|x_1(t, t_0, x_0)\| < \delta_1 \leq \delta_3$, hence

[1] Notice that $\|s\|_a \leq b \Leftrightarrow \|s\| \leq a + b$, $\forall s \in \mathbb{R}^n$, $a > 0$, $b \geq 0$.

[2] In the case when $x_1(t, t_0, x_{10})$ never enters the ball $\mathcal{B}_{\delta_1}$, we consider, by an abuse of notation, that $T_0 = +\infty$.

$\|x_1(t,t_0,x_0)\|_{\delta_3}=0$. Thus, we conclude from (24) and (25) that for all $x_0\in\mathcal{B}_{\Delta_1}\times\mathcal{B}_{\Delta_2}$ and all $t\ge t_0$,

$$\|x_1(t,t_0,x_{10})\|_{\delta_3}\le\alpha(\|x_0\|) \tag{26}$$

where $\alpha\in\mathcal{K}_\infty$. From the bound (26) we now construct a ball which is such that, any solution $x_1(\cdot,t_0,x_{10})$ starting in it remains forever after in the ball $\mathcal{B}_{\Delta_1}$. To that end, let $\tilde\Delta$ denote the radius of such a ball. Then, as long as $\Delta_1>\delta_3$, the following choice is convenient:

$$\tilde\Delta:=\min\{\alpha^{-1}(\Delta_1-\delta_3)\ ;\ \Delta_1\ ;\ \Delta_2\}\,. \tag{27}$$

Indeed, if $\|x_0\|<\tilde\Delta$, then (26) implies that

$$\|x_1(t)\|\le\delta_3+\alpha(\alpha^{-1}(\Delta_1-\delta_3))=\Delta_1\,,\quad\forall t\ge t_0\,.$$

Note that the previous reasoning can be repeated with initial states $x_0=(x_{10},x_{20})$ in the ball $\mathcal{B}_{\tilde\Delta}$ instead of $\mathcal{B}_{\Delta_1}\times\mathcal{B}_{\Delta_2}$. Then, we can get rid of the terms in α_4 and consequently those in c_3. Thus, even though the explicit demonstration is voluntarily omitted due to lack of space, it can be shown that for any $x_0\in\mathcal{B}_\Delta$ and all $t\ge t_0$, (8) holds with

$$\underline{\delta}_3:=\delta_1+\alpha_1^{-1}\big(\alpha_2(\delta_1)+c_1(\mu)G(\mu)\delta_2\big) \tag{28}$$
$$\underline{\alpha}(s):=\max\{\underline{\tilde\alpha}(s)\ ;\ \overline{\alpha}(s)\} \tag{29}$$
$$\underline{\tilde\alpha}(s):=\alpha_1^{-1}\big(\alpha_2(s)+c_2(s)(\beta_2(s,0)+\delta_2)\big)-\alpha_1^{-1}\big(c_1(\mu)G(\mu)\delta_2\big)$$
$$\overline{\alpha}(s):=\alpha_1^{-1}\big(\alpha_2(\delta_1)+c_2(s)(\beta_2(s,0)+\delta_2)\big)-\alpha_1^{-1}\big(\alpha_2(\delta_1)+c_1(\mu)G(\mu)\delta_2)\big)$$
$$\Delta:=\min\big\{\Delta_1\ ;\ \Delta_2\ ;\ \underline{\alpha}^{-1}(\Delta_1-\underline{\delta}_3)\big\} \tag{30}$$

as long as
$$\Delta_1>\underline{\delta}_3\,. \tag{31}$$

Notice that $\underline{\alpha}$, $\underline{\tilde\alpha}$, $\overline{\alpha}$ are also class $\mathcal{K}_\infty$ functions.

4.2 Proof of "attractivity w.r.t. a ball"

Consider again (14). Since β_2 is a $\mathcal{KL}$ function, given any $\varepsilon_1>0$ and any initial condition x_0, there is a time $t_1\ge0$ such that, for any $t_0\ge0$,

$$\beta_2(\|x_0\|,t-t_0)\le\varepsilon_1\,,\quad\forall\,t\ge t_0+t_1\,.$$

In view of (12), for any initial state $x_0\in\mathcal{B}_\Delta$, (14) implies that, for all $t\in\Gamma\cap\mathbb{R}_{\ge t_0+t_1}$,

$$\dot v_1(t)\le-\epsilon v_1(t)+c_2(\|x_0\|)(\varepsilon_1+\delta_2)\,. \tag{32}$$

Consider an initial state $x_0\in\mathcal{B}_\Delta$ and assume, for the time being, that $\|x(t_0+t_1,t_0,x_{10})\|>\delta_1$, i.e. $t_0+t_1\in\Gamma$. Then, there exists a time $t^\star>t_1$, potentially infinite and depending on t_0, such that

$$\|x_1(t,t_0,x_{10})\|>\delta_1\,,\ \forall t\in[t_0+t_1;t_0+t^\star) \tag{33a}$$
$$\|x_1(t^\star,t_0,x_{10})\|=\delta_1 \tag{33b}$$

From this observation, (32) holds on the time interval $[t_0+t_1;t_0+t^\star)$. Two cases are then possible: either $t^\star$ is finite or not.

Case 1: $\|x_1(t,t_0,x_{10})\|>\delta_1$ for all $t\ge t_0+t_1$. In this case, $t^\star=\infty$ and the integration of (32) from t_0+t_1 to any $t\ge t_0+t_1$ yields

$$v_1(t)\ \le\ \big(v_1(t_0+t_1)-c_2(\|x_0\|)(\varepsilon_1+\delta_2)\big)\times e^{-\epsilon(t-t_0-t_1)}+c_2(\|x_0\|)(\varepsilon_1+\delta_2)$$

In view of Assumptions 2 and 4, we get that

$$v_1(t)\le\alpha_2(\gamma(\|x_0\|+\mu)e^{-\epsilon(t-t_0-t_1)}+c_2(\|x_0\|)(\varepsilon_1+\delta_2)\,.$$

Therefore, for any $\varepsilon_2>0$, we have that

$$t\ge t_0+t_2\ \Rightarrow\ v_1(t)\le\varepsilon_2+c_2(\|x_0\|)(\varepsilon_1+\delta_2)\,.$$

where $t_2:=t_1+\frac{1}{\epsilon}\ln\left(\frac{\alpha_2(\gamma(\Delta)+\mu)}{\varepsilon_2}\right)$. Notably, by picking $\varepsilon_2\le\alpha_2(\delta_1)$, we ensure the existence of a finite time t_2, independent on the initial conditions t_0 and x_0, such that, for all time $t\ge t_0+t_2$,

$$\|x_1(t)\|\le\delta_1+\alpha_1^{-1}\big(\alpha_2(\delta_1)+c_2(\|x_0\|)(\varepsilon_1+\delta_2)\big)\,.$$

Since ε_1 is arbitrary and in view of (28) we conclude that, for all time $t\ge t_0+t_2$ and any $x_0\in\mathcal{B}_\Delta$,

$$\|x_1(t,t_0,x_0)\|\le\delta_4\,, \tag{34}$$

where δ_4 is any positive number such that

$$\delta_4>\underline{\delta}_3\,. \tag{35}$$

Case 2: $\|x_1(t^\star)\|\le\delta_1$ for some finite time $t^\star\ge t_1$. In this case, since $2\delta_4\ge\delta_1$ and invoking the continuity of the solutions of (7), there exists a time $t_3\le t^\star$, potentially dependent on the initial time t_0 such that $\|x_1(t_0+t_3,t_0,x_{10})\|\le2\delta_4$. Then, using the bound (8) by picking t_3 as the "initial time", we get that, for all $x_0\in\mathcal{B}_\Delta$ and all $t\ge t_0+t_3$,

$$\|x_1(t)\|_{\underline{\delta}_3}\le\underline{\alpha}(2\delta_4) \tag{36}$$

Said differently, for all $\forall x_0\in\mathcal{B}_\Delta$,

$$\|x_1(t,t_0,x_{10})\|_{\delta_5}=0\,,\quad\forall t\ge t_0+t_3\,, \tag{37}$$

where
$$\delta_5:=\underline{\delta}_3+\underline{\alpha}(2\delta_4)\,. \tag{38}$$

But, from (32), we know that such a time t_3 should satisfy

$$t_3\le t_4:=\frac{1}{\epsilon}\ln\left(\frac{\alpha_2(\Delta)}{\delta_4}\right)\,.$$

Hence, (37) implies that, for all $x_0\in\mathcal{B}_\Delta$,

$$\|x_1(t,t_0,x_{10})\|_{\delta_5}=0\,,\quad\forall t\ge t_0+t_4\,, \tag{39}$$

where, this time, t_4 is *independent* of t_0.

Finally, let us examine the case when $\|x_1(t_0+t_1,t_0,x_{10})\|\le\delta_1$. By (8), we get that, for all $t\ge t_0+t_1$, $\|x_1(t)\|_{\underline{\delta}_3}\le\underline{\alpha}(\delta_1)$. Since $\alpha\in\mathcal{K}_\infty$ and $\delta_1\le2\delta_4$, we can establish that (39) holds for all $t\ge t_0+t_1$.

Thus, we have shown that defining,

$$t_5 := \begin{cases} t_2 & \text{if } t^\star = \infty \\ t_4 & \text{otherwise,} \end{cases}$$

and

$$\delta_6 := \max\{\delta_4; \delta_5\}, \quad (40)$$

we have that[3], for all time $t \geq t_0 + t_5$,

$$\|x_1(t, t_0, x_{10})\|_{\delta_6} = 0 .$$

Using finally the uniform bound on the solutions of (7) on the interval $[t_0; \max\{t_2; t_3\}]$, we conclude that there exists a class $\mathcal{L}$ function σ such that, for any $x_0 \in \mathcal{B}_\Delta$, and any $t_0 \geq 0$,

$$\|x_1(t, t_0, x_{10})\|_{\delta_6} \leq \sigma(t - t_0) \quad \forall t \geq t_0 . \quad (41)$$

For example, one can take $sigma(t) = (\gamma(\Delta) + \mu)e^{-(t-t_6)}$. From (8) and (41), we are now ready to exhibit a $\mathcal{KL}$ bound on the trajectories. Indeed, first notice that $\delta_6 \geq \underline{\delta}_3$. Hence, (8) implies that

$$\|x_1(t, t_0, x_{10})\|_{\delta_6} \leq \alpha(\|x_0\|) , \quad \forall x_0 \in \mathcal{B}_\Delta . \quad (42)$$

Multiplying (41) and (42) gives

$$\|x_1(t, t_0, x_{10})\|_{\delta_6} \leq \sqrt{\alpha(\|x_0\|)\sigma(t - t_0)} .$$

Hence, using the equivalent formulation for (13):

$$\|x_2(t, t_0, x_{20})\|_{\delta_2} \leq \beta_2(\|x_{20}\| , t - t_0) , \quad (43)$$

we obtain that, for all $x_0 \in \mathcal{B}_\Delta$ and all $t \geq t_0$,

$$\|x(t, t_0, x_0)\|_\delta \leq \beta(\|x_0\| , t - t_0) \quad \forall x_0 \in \mathcal{B}_\Delta , \quad (44)$$

where Δ is given in (30), $\delta := \max\{\delta_2; \delta_6\}$, and

$$\beta(s, t) := \sqrt{\alpha(s)\sigma(t)} + \beta_2(s, t) .$$

Since α, σ and β_2 are respectively of class $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{KL}$, β is clearly a class $\mathcal{KL}$ function.

It is only left to show that δ and Δ can be arbitrarily diminished and enlarged respectively. To that end, first notice that Δ can be taken arbitrarily by a convenient choice of Δ_1 and Δ_2. Indeed, if one choose for example

$$\Delta_1 = \max\left\{\Delta_2 ; \alpha_1^{-1}\big(\alpha_2(\delta_1) + c_1(\mu)G(\mu)\delta_2\big) + \underline{\alpha}(\Delta_2) + \delta_1\right\} ,$$

then, according to (28) and (30), we get that[4] $\Delta = \Delta_2$. In addition, for these convenient Δ_1 and Δ_2, δ_4 can be taken as small as wanted by picking δ_1 and δ_2 sufficiently small. Hence, in view of (38) and (40), it is also the case for δ_5 and δ_6. Therefore, δ can be arbitrarily diminished by a convenient choice of δ_1 and δ_2. Notably, the condition $\Delta > \delta$ can be fulfilled.

Hence, it suffices to pick the parameters $\theta_1^\star(\delta_1, \Delta_1)$ and $\theta_2^\star$ in the set $\mathcal{D}_{f_2}(\delta_2, \Delta_2) \cap \Theta_2$ generated by the chosen δ_1, Δ_1, δ_2 and Δ_2, to conclude that, for any $\Delta > \delta > 0$, there exists some parameters $\theta_1^\star \in \Theta_1$ and $\theta_2^\star \in \Theta_2$ such that (7) is ULAS on $\mathcal{B}_\Delta$ with respect $\mathcal{B}_\delta$, which establishes the result.

[3] This is done by noticing that $t_4 \geq t_1$.

[4] Notice that we also satisfy the requirement (31).

5 Conclusion

Our main result concerns the cascade of two nonlinear time-varying subsystems which are assumed to be USGPAS. It was shown that, under a condition of boundedness of its solutions and provided the knowledge of a Lyapunov function for the subsystem which is "perturbed" by the other one, the resulting cascaded system remains USGPAS. As a perspective, we want first to relax the assumptions by imposing them only on the "doughnut" $\mathcal{B}_\Delta \setminus \mathcal{B}_\delta$, which will make the use of α_4 obsolete (see (Chaillet A. and Loría A. 2005)). In a second time, we plan to get rid of the knowledge of the Lyapunov function, by adapting and using some converse theorems for USGPAS.

References

Angeli, D., E. D. Sontag and Y. Wang (2000). Further equivalences and semiglobal versions of integral input to state stability. *Dynamics and Control Journal* **10**, 127–149.

Arcak, M., D. Angeli and E. Sontag (2002). A unifying integral ISS framework for stability of nonlinear cascades. *SIAM J. on Contr. and Opt.* **40**, 888–1904.

Chaillet A. and Loría A. (2005). Uniform semiglobal practical asymptotic stability for non-autonomous cascaded systems and applications. eprint no. arXiv:math.DS/0503039. Available from `http://front.math.ucdavis.edu/math.OC/0503039`.

Khalil, H. (1996). *Nonlinear systems*. Macmillan Publishing Co., 2nd ed.. New York.

Lefeber, A. A. J. (2000). Tracking control of nonlinear mechanical systems. PhD thesis. University of Twente. Enschede, The Netherlands.

Lin, Y., E. D. Sontag and Y. Wang (1996). A smooth converse Lyapunov theorem for robust stability. *SIAM J. on Contr. and Opt.* **34**, 124–160.

Loría A. and E. Panteley (2005). Cascaded nonlinear time-varying systems: analysis and design. Chap. 2 in *New directions in nonlinear observer design*. Vol. 311 of *Lecture Notes in Control and Information Sciences*. Springer Verlag: London. F. Lamnabhi-Lagarrigue, A. Loría, E. Panteley, eds.

Marino, R. and P. Tomei (1993). Global adaptive output feedback control of nonlinear systems. Part I : Linear parameterization. *IEEE Trans. on Automat. Contr.* **38**, 17–32.

Panteley, E. and A. Loría (2001). Growth rate conditions for stability of cascaded time-varying systems. *Automatica* **37**(3), 453–460.

Seibert, P. and R. Suárez (1990). Global stabilization of nonlinear cascaded systems. *Syst. & Contr. Letters* **14**, 347–352.

Sepulchre, R., M. Janković and P. Kokotović (1997). *Constructive nonlinear control*. Springer Verlag: London.

Sontag, E. D. (1989). Remarks on stabilization and Input-to-State stability. In: *Proc. 28th. IEEE Conf. Decision Contr.*. Tampa, Fl. pp. 1376–1378.

Sontag, E. D. (2003). A remark on the converging-input converging-state property. *IEEE Trans. on Automat. Contr.*

Sussman, H. J. and P. V. Kokotović (1991). The peaking phenomenon and the global stabilization of nonlinear systems. *IEEE Trans. on Automat. Contr.* **36**(4), 424–439.