

STRICT LYAPUNOV FUNCTIONS FOR GENERATING ROBUST OSCILLATIONS IN NONLINEAR SYSTEMS

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Abstract: This paper deals with the problem of generating stable and robust oscillations in triangular nonlinear systems. The proposed method consists of two steps. First, a globally attractive oscillation is generated in a nominal second-order subsystem. Based on a partition of the state space and solving the Lyapunov equation on each part, a strict Lyapunov function is obtained that ensures exponential convergence to a ring-shaped region containing the target limit cycle. Then, the nominal stabilizing controller and the strict Lyapunov function are extended to arbitrary order systems, via a method in the essence of backstepping. Thanks to the strict Lyapunov functions, a new ability to deal with unmodeled dynamics is acquired, and the original controller is easily extended to quasi-triangular structures. *Copyright*©2005 IFAC.

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1. INTRODUCTION

The present work stems from a control design presented in (Gordillo *et al.*, 2002; Gómez-Estern *et al.*, 2002), see also (Aracil *et al.*, 2004) for stabilizing a class of cascaded nonlinear systems of arbitrary order. In a subsequent work (Barreiro *et al.*, 2004), the domains of attraction (DOA) for this scheme with saturated control were studied. However the DOAs and robustness problems due to unmodelled dynamics and external disturbances in this framework were left aside.

The main drawback of the aforementioned method lies in the fact that it uses a *non-strict* Lyapunov function, i.e., its derivative is only a semidefinite function of the state. Strict Lyapunov functions are the key for robustness analysis, sensitivity to

disturbances, DOA analysis, adaptive control and singular perturbation theory (Khalil, 2002).

Here we will obtain the desired closed-loop dynamics with strict Lyapunov functions via a twofold method. First, a globally attractive oscillation is generated in a nominal second-order subsystem. Then we introduce a partition of the state space based on the generated limit cycle and the Lyapunov equation is solved separately on each part. By merging the results we obtain a globally defined smooth *strict* Lyapunov function that guarantees that even in the presence of disturbances, a ring-shaped region containing the target limit cycle is made attractive. In a second step, *backstepping* (Khalil, 2002; Kritic *et al.*, 1995; Sepulchre *et al.*, 1997) is applied to

extend the oscillating behavior to arbitrary order systems, still obtaining a *strict* Lyapunov function in the augmented state space.

The *strict* Lyapunov function obtained with this method is to be exploited for robustness analysis in systems with static uncertainties. Hence, we provide the tools for estimating the maximum amplitude error and the pseudo-period of the perturbed oscillation, which depend on the estimated upper bounds of the unknown terms.

2. TARGET DYNAMICS

In this section we define a simple two-dimensional system $\dot{x} = f(x)$, $x = [x_1, x_2]^\top$, that presents an attractive limit cycle. This system will be used as the target dynamics in a subspace of dimension 2 of the controlled system in subsequent sections. Given a pair of *positive* design parameters $\omega_c \in \mathbb{R}$ and $\mu \in \mathbb{R}$, we will define the target set as the ellipse where the function

$$\Gamma(x_1, x_2) \triangleq \omega_c^2 x_1^2 + x_2^2 - \mu$$

is equal to zero. Now consider the Lyapunov function candidate $V_0 = \frac{1}{4}\Gamma^2$. Obviously, the minima of V_0 are reached for $\Gamma = 0$, as depicted in Fig. 1

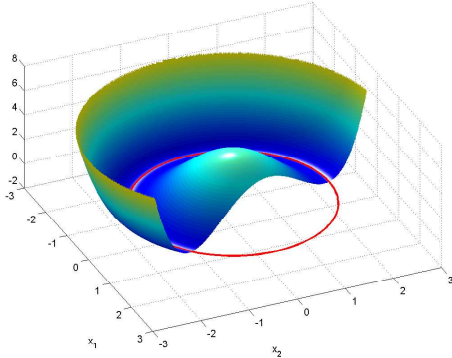


Fig. 1. Minimal set of V_0 : the closed curve $\Gamma(x_1, x_2) = 0$.

The shape of $V_0(x_1, x_2)$ invites us to consider systems for which V_0 is a Lyapunov function (Mees and Chua, 1979). The limit sets are expected to be limit cycles. One way to get a dynamical system with V_0 as a Lyapunov function is to define the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\omega_c^2 x_1 - k_0 \Gamma x_2, \quad (1)$$

Proposition 1. (Aracil *et al.*, 2004) Consider system (1). If $\mu < 0$ the origin is globally asymptotically stable. If $\mu > 0$, for any initial condition except the origin, the trajectories tend to the limit cycle $\Gamma = 0$ with period $2\pi/\omega_c$.

This is based on the fact that along the trajectories of (1),

$$\dot{V}_0 = -k_0 \Gamma^2 x_2^2,$$

while the details of the proof can be read in the cited paper.

Remark 1. For $\mu > 0$, the change of variables $z_1 \triangleq \omega_c x_1 / \sqrt{\mu}$ and $z_2 \triangleq x_2 / \sqrt{\mu}$ and the time scaling $\tau = \omega_c t$, transform (1) into the canonical form

$$z_1' = z_2 \quad z_2' = -z_1 - \tilde{k}_0 \tilde{\Gamma} z_2 \quad (2)$$

where $\tilde{k}_0 = k_0 \mu / \omega_c$, $\tilde{\Gamma} = z_1^2 + z_2^2 - 1$ and the derivatives are expressed with respect to τ . This corresponds to an oscillation of unitary period and amplitude that will be used for simplicity in subsequent sections.

3. FEEDBACK STABILIZATION OF OSCILLATIONS

3.1 Feedback law for second-order systems

We will consider second-order mechanical-like systems (where the input enters the acceleration), for which the target dynamics (1) are easily achievable. Indeed, for the system

$$\dot{x}_1 = x_2 \quad (3)$$

$$\dot{x}_2 = f(x) + g(x)h(u) \quad (4)$$

where $x = [x_1 \ x_2]^\top$, $g(x) \neq 0$ and $h^{-1}(\cdot)$ exists over the domain of interest, the feedback

$$u = h^{-1}((-x_1 - k_0 \Gamma x_2 - f(x))/g(x)). \quad (5)$$

trivially obtains the desired closed-loop behavior from Proposition 1 and V_0 is a control Lyapunov function of the system.

3.2 Strict Lyapunov functions in oscillating systems

Strict Lyapunov functions are used to solve adaptive control and robustness problems (Khalil, 2002). It is well known that in exponentially stable linear systems¹ a strict Lyapunov function can be found by choosing the candidate $V = x^\top P x$ with $P > 0$ and solving the Lyapunov equation

$$A^\top P + P A = -Q \quad (6)$$

If the (strictly negative) derivative of V dominates the effect of unmodelled dynamics at least in a neighborhood of the origin, then stability is preserved under the effect of disturbances for trajectories starting in that region.

As $\dot{V}_0 = -k_0 \Gamma^2 x_2^2$ in our oscillatory second-order system, the region where $\dot{V}_0 = 0$ is *not restricted* to the target set $\Gamma = 0$ nor its proximity. When trying to adapt the above strategy to the problem of oscillations, we find two main obstacles,

- (1) Degeneration of the target set.
- (2) Oscillations away from the origin do not allow to use linear tools like the Lyapunov equation and exponential stability.

¹ i.e. if the system matrix A is *Hurwitz*.

The first issue arises at the proximity of the orbit. If some disturbance is added to the state equations the system may not converge asymptotically to the curve $\Gamma(x_1, x_2) = 0$ but to a different manifold possibly close to the latter, but not arbitrarily close. Let us call this new set, $\Omega \in \mathbb{R}^2$. In general, the value of $\Gamma(x_1, x_2)$ at Ω is different from zero. Therefore $\Gamma^2/4$ can no longer be taken as a Lyapunov function, not even in a neighborhood of the limit set. On the other hand, the analytical search for a new function $\bar{\Gamma}$ whose value is exactly zero at the perturbed orbit (assuming particular forms of the disturbances) has resulted unsuccessful.

The second issue states that away from $\Gamma = 0$ no strict Lyapunov function can easily be found. Indeed, the dynamics (1) cannot be approximated by a linear system (not even close to the target set, as happens in set point stabilization) and hence the Lyapunov equation (6) is of no use.

Strict Lyapunov functions away from the target orbit. The aforementioned difficulties can be tackled by defining a minimal ring-shaped region $\mathcal{S} \in \mathbb{R}^2$ that contains both the desired trajectory and the perturbed one, such that for any initial state outside \mathcal{S} , the system asymptotically converges to it with *strict* Lyapunov function. This means that the convergence to this region will be guaranteed even in the presence of unmodeled dynamics. That band contains, at least, \mathcal{S}_1 (area between A and B in Fig. 2).

In order to obtain a strict Lyapunov function we will first observe the behavior of the “distance to the origin” function, $r = (x_1^2 + x_2^2)/2$ along the trajectories of the nominal system (2),

$$\dot{r} = -k_0 \Gamma x_2^2.$$

This expression has opposite sign than Γ , which means that if $\Gamma < 0$ (area encircled by the inner target orbit), r will grow and the system escapes from the origin, whilst outside the orbit the system will tend to the origin.

According to this observation, we will redefine the damping term in (5) maintaining the stability of $\Gamma = 0$, but transforming the system into a *piecewise linear* one with a step transition at $\Gamma = 0$. With this partition, a different Lyapunov equation can be solved on each linear part and a strict Lyapunov function merging both solutions will be obtained for robust convergence. Let us rewrite the control law (5) as follows

$$u = h^{-1}((-x_1 - k_0 \text{sgn}(\Gamma)x_2 - f(x))/g(x)). \quad (7)$$

The sign function introduced in the dissipation term does not affect the stability analysis, as the Lyapunov function is continuous and its partial derivatives are still \mathcal{C}^1 . The new closed-loop dynamics are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\text{sgn}(\Gamma)k_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (8)$$

Now we can obtain a strict Lyapunov function for each of the two *linear* systems separated by the closed curve $\Gamma = 0$, by solving the Lyapunov equation with a matrix $Q > 0$ of our choice,

$$\begin{bmatrix} 0 & -1 \\ 1 & -\text{sgn}(\Gamma)k_0 \end{bmatrix} R + R \begin{bmatrix} 0 & 1 \\ -1 & -\text{sgn}(\Gamma)k_0 \end{bmatrix} = -Q$$

In the outer part ($\Gamma > 0$), we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -k_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In that region we need a strict Lyapunov function of the form $V^+ = x^\top R^+ x$, $R^+ > 0$ with negative derivative for the trajectories to tend to the inside of the orbit. By choosing $Q = I$, we obtain

$$R^+ = \begin{bmatrix} \frac{1}{k_0} + \frac{k_0}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{k_0} \end{bmatrix} > 0 \quad (9)$$

With this solution we have as desired

$$\dot{V}^+ = -x^\top Q x = -\|x\|^2$$

On the other hand, at the inner region of the orbit the dynamics are unstable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & k_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

As the system must escape from the origin and move into the band containing the target orbit, a Lyapunov function will be found with a maximum at the origin and negative derivative. A possible choice is now $V^- = -x^\top R^- x$ with $R^- > 0$. The Lyapunov equation becomes

$$\begin{bmatrix} 0 & -1 \\ 1 & k_0 \end{bmatrix} (-R^-) + (-R^-) \begin{bmatrix} 0 & 1 \\ -1 & k_0 \end{bmatrix} = -Q$$

this equation is turned into form (6) by rearranging signs

$$\begin{bmatrix} 0 & 1 \\ -1 & -k_0 \end{bmatrix} R^- + R^- \begin{bmatrix} 0 & -1 \\ 1 & -k_0 \end{bmatrix} = -Q$$

and the solution is guaranteed because the expanded matrix is *Hurwitz*. Proposing again $Q = I$ yields

$$R^- = \begin{bmatrix} \frac{1}{k_0} + \frac{k_0}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{k_0} \end{bmatrix} > 0 \quad (10)$$

As V^+ drives the system to the origin its effect is only important as long as $\Gamma > 0$. Conversely, V^- is related to trajectories escaping from the origin, and should only be used to approach $\Gamma = 0$ when this last is negative. This suggests a compound Lyapunov function that takes the convenient behavior on either side of $\Gamma = 0$. Moreover, stability theorems require that the partial derivatives of the Lyapunov function are continuous. A smooth

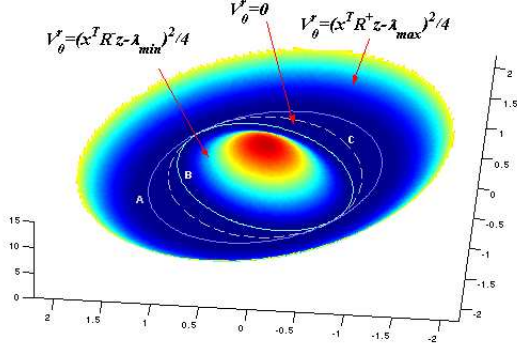


Fig. 2. Function V_0^r . The transition band \mathcal{S}_1 is limited by the ellipses A and B . C depicts the desired orbit $\Gamma = 0$.

transition between the two regions is obtained by defining, motivated by the fourth order polynomial V_0 , the Lyapunov function candidate

$$V_0^r \triangleq \begin{cases} \frac{(x^\top R^+ x - \lambda_{max})^2}{4} & \text{if } x^\top R^+ x > \lambda_{max} \\ \frac{(x^\top R^- x - \lambda_{min})^2}{4} & \text{if } x^\top R^- x < \lambda_{min} \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

where λ_{max} (λ_{min}) is the maximum (minimum) eigenvalue of R^+ (R^-). Figure 2 shows a three-dimensional plot of V_0^r . The region in the (x_1, x_2) plane comprised between the curves labelled A and B in that plot, will be denoted \mathcal{S}_1 . According to (11), $V_0^r = 0$ in \mathcal{S}_1 .

The ellipse $x^\top R^+ x = \lambda_{max}$ (labelled A in Fig. 2) has been chosen because it is the smallest level curve of V^+ that entirely encloses $\Gamma = 0$. The converse applies for the ellipse $x^\top R^- x = \lambda_{min}$ (B in Fig. 2). As a consequence, it is clear that $\mathcal{S}_1 = \{x : V_0^r(x) = 0\}$.

The function V_0^r has continuous partial derivatives and its shape is similar to that of Fig. 1, but as can be seen in Fig. 2, the level sets are ellipses congruent with $x^\top R^+ x$ in the outer part of the desired orbit, and with $x^\top R^- x$ in the inner part.

Remark 2. Observe that

$$R^+ = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} R^- = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. they have the same eigenvalues and the major axes of the ellipses represented by both matrices are rotated $\pi/2$ with respect to each other.

Now observe that the derivative of the compound Lyapunov functions $V = V_0 + V_0^r$ along the trajectories of system (8) yields

$$\dot{V} = - \underbrace{k_0 |\Gamma| x_2^2}_{nominal} - \underbrace{\sqrt{V_0^r} x^\top x}_{robust} \quad (12)$$

In this expression, the first term guarantees asymptotic convergence to $\Gamma = 0$ in the absence of disturbances, but when x_2 is close to zero, it

can be dominated by any unmodeled dynamics. Conversely, the term labelled *robust* is strictly negative outside \mathcal{S}_1 and hence it provides some robustness, but it is annihilated on the whole area \mathcal{S}_1 and hence it is not sufficient to ensure nominal convergence to $\Gamma = 0$.

Minimal band of convergence. We are unable to provide a strict Lyapunov function for all x such that $\Gamma(x) \neq 0$. Instead, we have $V_0^r = 0$ in the whole set \mathcal{S}_1 . Then, the robustness approach will only guarantee, in practical cases, convergence to a set $\mathcal{S} \supset \mathcal{S}_1$ where the effect of disturbances are no longer dominated by the derivative of V_0^r .

Hence, the narrower \mathcal{S}_1 is, the smaller the bounds on the deviation from the target orbit will be. But the boundaries of \mathcal{S}_1 are concentric ellipses rotated $\pi/2$ with respect to each other. This geometric misadjustment hampers the reduction of \mathcal{S}_1 (see Fig. 2).

Robust and exponential convergence to periodic orbits. The previous arguments give a theoretical basis to analyze exponential convergence of (3)-(4) towards a band $\mathcal{S} \supset \mathcal{S}_1$ containing $\Gamma = 0$ for a class of disturbances. We are also interested in the existence of periodic orbits in \mathcal{S} . The following proposition analyzes the exponential convergence to the limit oscillations.

Proposition 2. Consider a perturbed closed-loop oscillating system of the form

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1(x) \\ \dot{x}_2 &= -x_1 - k_0 \text{sgn}(\Gamma) x_2 + w_2(x) \end{aligned} \quad (13)$$

where the disturbances (w_1, w_2) are smooth functions such that outside a set $\mathcal{S} \supset \mathcal{S}_1$

$$\frac{\partial V_0^r}{\partial x_1} w_1 + \frac{\partial V_0^r}{\partial x_2} w_2 < \sqrt{V_0^r} \|x\|^2.$$

Further assume that there are no equilibrium points within \mathcal{S} . Then, the system reaches a periodic orbit in \mathcal{S} . Moreover, the trajectories of the nominal system ($w_{1,2} = 0$) exponentially converge to \mathcal{S}_1 .

Proof. The convergence to \mathcal{S} is implied by the fact that outside that set the trajectories of (13) are such that

$$\dot{V}_0^r = -\sqrt{V_0^r} \|x\|^2 + \frac{\partial V_0^r}{\partial x_1} w_1 + \frac{\partial V_0^r}{\partial x_2} w_2 < 0$$

The existence of the perturbed oscillation stems from the Poincaré-Bendixson theorem, see (Khalil, 2002) and the absence of equilibrium points in \mathcal{S} .

Finally, the exponential convergence of the nominal system will be studied. Using that for $\Gamma < 0$ the norm $\|x(t)\|^2$ is non-decreasing, there exists some constant $\alpha > \min(\|x(0)\|^2, 1)$ such that

$\|x(t)\|^2 > \alpha \forall t$, except for the trajectories starting at the origin. Moreover, it is easy to see that for all $x(t)$ such that $\|x(t)\|^2$ is bounded away from zero, there is another positive constant β such that

$$\|x(t)\|^2 > \frac{\beta}{2} \max\{|x^\top R^+ x - \lambda_{max}|, |x^\top R^- x - \lambda_{min}|\}$$

and this implies, from (11), that $\|x(t)\|^2 > \beta \sqrt{V_0^r}$ and hence

$$\dot{V}_0^r = -\sqrt{V_0^r} \|x\|^2 < -\beta V_0^r$$

◁

Clearly, the convergence rate β is a conservative estimate that should be computed for every initial condition.

4. HIGHER-ORDER SINGLE INPUT SYSTEMS AND BACKSTEPPING

So far we have demonstrated the stability of oscillations in the second-order nominal case, as well as robustness to unmatched dynamics that would result in (boundedly) perturbed limit cycles. The method can be extended to systems with a dimension greater than 2. As will be seen, the nominal stability result is easily extended to arbitrary order systems in strictly cascaded form, while the robustness analysis is still valid.

First, we will focus on applying the method to higher order systems in the “nominal” case (global asymptotic stability of the “perfect orbit”). The main idea stems from an extension of the *backstepping* method (Krstic *et al.*, 1995; Sepulchre *et al.*, 1997), introduced in (Aracil *et al.*, 2004). Consider the following class of n^{th} -order systems

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1(x_1, x_2) + g_1(x_1, x_2)h_1(x_3) \\ \dot{x}_3 &= f_2(x_1 \dots x_3) + g_2(x_1 \dots x_3)h_2(x_4) \\ &\dots \\ \dot{x}_n &= u \end{aligned} \quad (14)$$

with $h'_i(\cdot), g_i(\cdot) \neq 0$, $i = 1 \dots n-2$ on the whole domain of interest. Then, defining the Lyapunov functions

$$V_0 = \frac{\Gamma^2}{4}, \quad V_i \triangleq V_0 + \frac{1}{2} \sum_{j=1}^i z_j^2 \quad i = 1 \dots n-2 \quad (15)$$

where

$$\begin{aligned} z_1 &\triangleq h_1(x_3) - h_1(u_0) \\ z_2 &\triangleq h_2(x_4) - h_2(u_1) \\ &\dots \\ z_{n-2} &\triangleq h_{n-2}(x_n) - h_{n-2}(u_{n-1}) \end{aligned}$$

and based on the iterative application of the backstepping method (Aracil *et al.*, 2004), there

is a set of recursively defined control laws of the form²

$$\begin{aligned} u_0 &= -x_1 - \frac{2k_0}{\pi} \arctan(\sigma\Gamma)x_2 \\ u_i &= h_{i+1}^{-1} \left(\frac{\tilde{u}_i - f_{i+1}(x_1 \dots x_{i+2})}{g_{i+1}(x_1 \dots x_{i+2})} \right) \end{aligned} \quad (16)$$

where $i = 1 \dots n-2$, $k_i > 0$, and

$$\begin{aligned} \tilde{u}_i &= \frac{1}{h'_i(x_{i+2})} \left[h'_i(u_{i-1})\dot{u}_{i-1} - \frac{\partial V_{i-1}}{\partial x_{i+1}} g_i - k_i z_i \right] \\ \dot{u}_i &= \left(\frac{\partial u_i}{\partial (x_1 \dots x_{i+2})} \right)^\top [\dot{x}_1 \dots \dot{x}_{i+2}], \end{aligned}$$

such that $u = u_{n-2}$ stabilizes system (14) at $x_2\Gamma = 0$, $z_1 = 0 \dots z_n = 0$ as can be viewed from the fact that the Lyapunov function

$$V = V_0 + \frac{1}{2} z^T z \quad (17)$$

with $z = [z_1 \dots z_{n-2}]^\top$ is monotonically decreasing,

$$\dot{V} = -k_0|\Gamma|x_2^2 - k_1z_1^2 - k_2z_2^2 \dots - k_{n-2}z_{n-2}^2, \quad (18)$$

thus reaching the invariant set $x_2\Gamma = 0$ studied in (Gordillo *et al.*, 2002), and hence converging to the orbit $\Gamma(x_1, x_2) = 0$.

4.1 Strict Lyapunov functions in high order systems.

From the first term in the right hand side of (18), we see that the Lyapunov function obtained in the last section is non-strict. To tackle this, we will introduce in the first step of the backstepping procedure the compound Lyapunov function $V_0 + V_0^r$ used for second order systems. Indeed, replacing V_0 by $\Gamma^2/4 + V_0^r$ in (15), we obtain a strict Lyapunov for the closed-loop system with dimension greater than 2. For instance, in system (14) with $n = 3$, in closed loop with u_1 computed as in (16) but redefining \tilde{u}_1 as

$$\tilde{u}_1 = \frac{h'_1(u_0)\dot{u}_0 - \left(\frac{\partial V_0}{\partial x_2} + \frac{\partial V_0^r}{\partial x_2} \right) g_1 - k_1 z_1}{h'_1(x_3)}, \quad (19)$$

the compound Lyapunov function

$$V_1 = \frac{\Gamma^2}{4} + V_0^r + \frac{z_1^2}{2}$$

has time derivative³

$$\dot{V}_1 = -k_0|\Gamma|x_2^2 - \sqrt{V_0^r} \|x\|^2 - k_1 z_1^2 \quad (20)$$

² As u_0 must be differentiable for backstepping, the $\text{sgn}(\cdot)$ function of (7) is approximated by $\arctan(\sigma\Gamma)$ with $\sigma > 0$ an adjustable parameter. The reader should observe that (i) nominal stability is preserved and (ii) the robust target region \mathcal{S}_1 might be slightly enlarged depending on the value of σ , because the arguments of Section 3.2 are only applicable where the \arctan function becomes constant (at a certain distance from $\Gamma = 0$).

³ Observe that for $x^\top R^+ x > \lambda_{max}$, we have $\sqrt{V_0^r} = (x^\top R^+ x - \lambda_{max})/2$, while for $x^\top R^- x < \lambda_{min}$, $\sqrt{V_0^r} = (x^\top R^- x - \lambda_{min})/2$

which has all the terms required for nominal and robust convergence to an extension of \mathcal{S} containing

$$\mathcal{S}^+ \supset \{(x, z) | x \in \mathcal{S}, \|z_1\| < \delta\}$$

for some $\delta > 0$ small (depending on the disturbance). For higher order systems, the rest of the terms in the recursion (16) are left unchanged. Hence for $n = 3$ and greater we will always obtain

$$\dot{V}_{n-2} = -k_0|\Gamma|x_2^2 - \sqrt{V_0^r}\|x\|^2 - k_1z_1 \cdots - k_{n-2}z_{n-2} \quad (21)$$

which guarantees strict decay of the Lyapunov function outside a target set closely containing \mathcal{S}^+ , excluding trajectories starting at the origin of states.

Remark 3. (Differentiability of the control law). In order to apply the proposed method it is necessary that u_0 is differentiable. The control law

4.2 Existence of oscillations in high dimension

In dimensions higher than 2, the Poincaré-Bendixon theorem does not apply and even if the trajectories lack of equilibrium points on the limit set \mathcal{S}^+ and are bounded, any behavior at all should be accounted for, including chaos.

Nevertheless, as long as we can find a sufficiently small bound $\|z\| < \delta$ in the limit set there is a criterion to check if the trajectories viewed in the plane (x_1, x_2) turn around the origin with a bounded pseudo-period, in an oscillating way. In fact, deriving with respect to time the polar angle of the point $(x_1(t), x_2(t))$ in \mathbb{R}^2 , we have

$$\begin{aligned} \dot{\theta} &= \frac{d}{dt} \arctan\left(\frac{x_2}{x_1}\right) = \frac{x_2\dot{x}_1 - x_1\dot{x}_2}{x_1^2 + x_2^2} \\ &= 1 - \underbrace{\frac{k_0\Gamma x_1 x_2 - g_1(x_1, x_2)z_1}{x_1^2 + x_2^2}}_{\Delta} \end{aligned}$$

Now if the underbraced term $\Delta(x_1, x_2, z_1)$ is strictly less than 1, i.e. $\Delta < \epsilon < 1$, it is guaranteed that the trajectories of the system projected on the plane (x_1, x_2) will make turns around the origin in finite time. For this, we observe that

$$\left| \frac{k_0\Gamma x_1 x_2 - g_1(x_1, x_2)z_1}{x_1^2 + x_2^2} \right| < \frac{1}{2}|k_0\Gamma_{max}| + \left| \frac{g_{max}}{r_{min}^2} \delta \right|$$

where Γ_{max} , g_{max} are the maximum values of $\Gamma(x_1, x_2)$ and $g_1(x_1, x_2)$; r_{min}^2 the minimum of $x_1^2 + x_2^2$ all in \mathcal{S}^+ . But if the upper bound $\|z\| < \delta$ that defines the geometry of the target band is such that $\delta < r_{max}^2/g_{min}$, and as long as there is a value of k_0 that guarantees⁴ that the sum less than 1, $\theta(t)$ is monotonic and the sustained oscillation exists. The pseudo-period T of the

⁴ Note that the region \mathcal{S}_1 shrinks with smaller values of k_0 , hence it can be lowered without an increment of Γ_{max} .

oscillation is estimated with the integral of $\theta(t)$ along one complete turn,

$$\begin{aligned} \int_0^T \dot{\theta} dt &= 2\pi = \int_0^T (1 + \Delta(x_1, x_2, z_1)) dt \\ \Rightarrow 2\pi &= T + \int_0^T \Delta(x_1, x_2, z_1) dt \end{aligned}$$

now from the Mean Value Theorem we have that

$$\int_0^T \Delta(x_1, x_2, z_1) dt = T\Delta(x_1^*, x_2^*, z_1^*)$$

where (x_1^*, x_2^*, z_1^*) are the values of the coordinates (x_1, x_2, z_1) at some instant $t^* \in [0, T]$ unknown. Clearly, these coordinates are in \mathcal{S}^+ , hence if we compute the (existing) upper bound

$$\Delta_{max} = \max_{(x_1, x_2, z_1) \in \mathcal{S}^+} |\Delta(x_1, x_2, z_1)|$$

then, the estimated period is such that

$$\frac{2\pi}{1 + \Delta_{max}} < T < \frac{2\pi}{1 - \Delta_{max}}$$

5. CONCLUSIONS

We have addressed the problem of building strict and smooth Lyapunov functions for robust stabilization of oscillations in arbitrary order nonlinear systems. The problem has been tackled by introducing a partition of the state space in a second order oscillating subsystem, and then solving the Lyapunov equation on each part. The price paid is that the new Lyapunov function is only positive *outside* a thick ring-shaped manifold that closely wraps around the target limit cycle.

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REFERENCES

- Aracil, J., E. Ponce and F. Gordillo (2004). Stabilization of oscillations through backstepping in high-dimensional systems. *IEEE Tr. on Automat. Control* (To appear).
- Barreiro, A., J. Aracil, F. Gordillo and F. Gómez-Estern (2004). Domain of attraction of autonomous oscillations in underactuated systems. In: *6th IFAC Symposium on Nonlinear Control Systems NOLCOS*. Stuttgart, Germany.
- Gómez-Estern, F., J. Aracil and F. Gordillo (2002). The hopf bifurcation and controlled oscillations in electromechanical systems. In: *10th Mediterranean Conference on Control and Automation MED'02*. Lisbon, Portugal.
- Gordillo, F., J. Aracil and F. Gómez-Estern (2002). Generation of stable oscillations in nonlinear systems. In: *Proceedings of the 41th conference on Decision and Control CDC'02*. Las Vegas, USA.
- Khalil, H.K. (2002). *Nonlinear Systems*. Prentice Hall.
- Kristic, M., I. Kanellakopoulos and P. Kokotovic (1995). *Nonlinear and Adaptive Control Design*. Wiley.
- Mees, A.I. and L.O. Chua (1979). The Hopf bifurcation theorem and its applications to nonlinear oscillations in circuits and systems. *IEEE Trans. on CAS* **26**, No. 4, 235–254.
- Sepulchre, R., M. Jankovic and P. Kokotovic (1997). *Constructive Nonlinear Control*. Springer-Verlag.