DYNAMICAL CONTROL IN OLIGOPOLIES

Sandor Molnar[1],Ferenc Szidarovszky[2], Mark Molnar[3]

[1]*Department of Informatics, Szent Istvan University,* [2]*Department of System Engineering, University of Arizona, USA, Hungary,* [3]*Systemexpert Consulting Ltd., Hungary*

Abstract. This paper examines the controllability of dynamic oligopoly models. Both discrete and continuous time scales are considered, and sufficient and necessary conditions are derived for the complete controllability of the outputs of the fims.

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I. INTRODUCTION

Oligopoly models are the most intensively investigated economic games. The existence and uniqueness of the Nash equilibria is the central problem in static oligoplies, and the asymptotical behavior of the equilibium is investigated by many authors in the dynamic case. A comprehensive summary of earlier results on single-product models is given in Okuguchi (1976), and their generalizations to multi-product firms are discussed in Okuguchi and Szidarovszky (1999), where the different variants of oligopoly models are also intoduced and examined. The classical Cournot model can be formuled as follows. Assumes that n firms produce the same product and/or offer identical services in the same market. Let x_k be the production output of firm k, $c_k(x_k)$ its cost function, and let

$$f(\sum_{l=1}^{n} x_l)$$

denote the unit price of the product (or service). Then the profit of firm k is given as

$$j_k(x_1,...,x_n) = x_k f(\sum_{l=1}^{n} x_l) - c_k(x_k). \quad (1)$$

The decision space of firm k is an interval $[0, L_k]$, where L_k is its capacity limit. The classical Cournot oligoply is a noncooperative n-person game, where the firms are the players, $x_k = [0, L_k]$ and f_k are the strategy set and payoff function of player x_k, respectively.

In the case of oligopolies with product differentiation we assume that the firms produce different items (or offer different services), and the price of each prodct (or service) depends on the outputs of all firms: $f_k(x_1, x_2, \dots, x_n)$, so the profit of firm k is now given as

$$f_k(x_1, \dots, x_n) = x_k f_k(x_1, \dots, x_n) - c_k(x_k). \quad (2)$$

In the case of labor-manages oligopies assume that w is the competitive wage rate and $c_k(x_k)$ is the laborindependent production cost. If $h_k(x_k)$ is the amount of labor needed for output x_k, then the profit of firm k per labor is

$$j_k(x_1,...,x_n) = \left\{ x_k f(\sum_{l=1}^{n} x_l) - wh_k(x_k) - c_k(x_k) \right\} \Big/ h_k(x_k). \quad (3)$$

In the case of rent-seeking games we assume that n agents compete for a rent. Let x_k denote the effort of agent k, and $c_k(x_k)$ be his cost function, and assume that the probability of winning the rent is

$$x_k \Big/ \sum_{l=1}^{n} x_l ,$$

which provides a profit of P$ to the agent who actually wins the rent, Then the expected profit of agent k is

$$j_k(x_1,\ldots,x_n)=\frac{x_k}{\sum_{l=1}^{n}x_l}\cdot P-c_k(x_k). \qquad (4)$$

The multi-product extensions of there models are straightforward, the output of each firm is a production vector, and the unit price function is replaced by a price vector.

The actual forms of the payoff functions are similar to the single-product case. In this paper we asumed that tha market is controlled by a central (e.g. government) agency via subsidies, tax cuts, etc. We will examine the complete controllabilty of the resulting dynamic systems.

II. THE CONTROL MODELS

For the sake of simplicity consider the classical Cournot model with price function

$$f\left(\sum_{l=1}^{n}x_l\right)=b-A\sum_{l=1}^{n}x_l$$

and cost functions $c_k(x_k) = c_k(x_k) + d_k$. Here b, A, c_k, and d_k are all positive constants. Assume that the control effects the costs of the firms, than the profit of firm k can be formulated as follows:

$$j_k(x_1,\ldots,x_n)=x_k\left(b-A\sum_{l=1}^{n}x_e\right)-\left(c_k x_k u+d_k\right), \qquad (5)$$

where u is the control variable affecting unit production costs. Other types of controls can be examined in a similar way.

In developing the relevant dynamic models assume first that the time scale is discrete. At each time period each firm maximizes its predicted profit functions

$$x_k(b-Ax_k-A\sum_{l\neq k}^{n}x_l(t-1))-\left(c_k x_k u+d_k\right) \qquad (6)$$

where we assume that each firm expects all rivals to produce the same output as they produced in the previous time period. This type of expectations is called static in the economic literature. Other types of expectations such as adaptive extrapolative, and delayed can be examined in a similar way. It is also assumed that each firm selects the profit maximizing output of each time period. Assuming that it is positive, simple differentiation shows that

$$x_k=-\frac{1}{2}\sum_{l\neq k}x_l(t-1)+\frac{b-c_k u(t-1)}{2A}. \text{ (k=1, 2, …, n)} \qquad (7)$$

We can rewrite these equations into the usual systems form by introducing the new state variables

$$z_k(t)=x_k(t)-\frac{b}{(N+1)A} \qquad (8)$$

to have

$$z_k(t)=-\frac{1}{2}\sum_{l\neq k}z_k(t-1)-\frac{c_k}{2A}u(t-1). \qquad (9)$$

This is a linear system with

$$\begin{pmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \cdots & -\frac{1}{2} \\ \vdots & \vdots & & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & \cdots & 0 \end{pmatrix} \text{ and } \underline{b}\begin{pmatrix} -\frac{c_1}{2A} \\ -\frac{c_2}{2A} \\ \vdots \\ -\frac{c_n}{2A} \end{pmatrix}$$

systems coefficients.

If the different firm are controlled differently, then the cost term in equation (6) is ($c_k x_k u_k + d_k$), so the equation (9) modify as

$$z_k(t)=-\frac{1}{2}\sum_{l\neq k}z_l(t-1)-\frac{c_k}{2A}u_k(t-1) \qquad (10)$$

with system coefficients

$$\underline{A}=\begin{pmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \cdots & -\frac{1}{2} \\ \vdots & \vdots & & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & \cdots & 0 \end{pmatrix}$$

and

$$\underline{B}=diag\left(-\frac{c_1}{2A},-\frac{c_2}{2A},\ldots,-\frac{c_n}{2A}\right).$$

Assume next that the time-scale is continuous, and that at each time period each firm adjusts its output proportionally to the marginal profit. If the firms are controlled in the same way, then the dynamic model has the form

$$\dot{x}_k(t)=S_k\left(-2Ax_k(t)-A\sum_{l\neq k}x_l(t)+b-c_k u(t)\right) \quad \text{(k=1,2,…,n)} \qquad (11)$$

where $S_k > 0$ is given speed of adjustment of firm k. Notice that

$$\underline{A} = \underline{S}\begin{pmatrix} -2A & -A & \cdots & -A \\ -A & -2A & \cdots & -A \\ \vdots & \vdots & & \vdots \\ -A & -A & \cdots & -2A \end{pmatrix} \quad \text{and} \quad \underline{b} = \underline{S}\begin{pmatrix} -c \\ -c_2 \\ \vdots \\ -c_n \end{pmatrix}$$

with S=diag(S_1 , S_2 , … , S_n).

If the firms are controlled diferently, then the cost term in equation (11) is q_k u_k instead of q_k u , and therefore the system coefficient are

$$\underline{A} = \underline{S}\begin{pmatrix} -2A & -A & \cdots & -A \\ -A & -2A & \cdots & -A \\ \vdots & \vdots & & \vdots \\ -A & -A & \cdots & -2A \end{pmatrix}$$

and

$$\underline{B} = diag(-S_1c_1, -S_2c_2, \dots, -S_nc_n).$$

III. COMPLETE CONTROLLABILITY

Consider first the dynamic system (9). It is well known from linear systems theory that the system is completely controllable if and only if the Kalman-controllability matrix K=(b, A b, A^2 b , … , A^{n-1} b) has full rank (see for example Szidarovszky and Bahill, 1998).

<u>Theorem 1.</u>

System (9) is completely controllable if and only if n=2 and c_1 ? c_2

<u>Proof</u>

Assume first that n=2. Then

$$\underline{K} = (\underline{b}, \underline{Ab}) = \begin{pmatrix} -\frac{c_1}{2A} & \frac{c_2}{4A} \\ -\frac{c_2}{2A} & \frac{c_1}{4A} \end{pmatrix} \tag{12}$$

with

$$\det(\underline{K}) = -\frac{c_1^2}{8A^2} + \frac{c_2^2}{8A^2}. \tag{13}$$

with positive A, c_1 **,** c_2 His determinant is nonzero if and only if c_1 ? c_2 **.**

Assume next that n = 3. We will show that the rank of matrix K is always less than n. Observe first that

$$A = \frac{1}{2}(\underline{I} - \underline{E}), \tag{14}$$

where I is the n x n identity matrix and E is the n x n matrix with all elements equal unity. Since $\underline{E}^2 = n\underline{E}$,

$$A^2 = \frac{n-1}{4}\underline{I} + \frac{2-n}{2}\underline{A} \tag{15}$$

and

$$A^2b = \frac{n-1}{4}\underline{b} + \frac{2-n}{2}\underline{AB}. \tag{16}$$

Hence the columns of K are linearly dependent, therefore rank (K) < n so the system is not completely controllable,

We will next show that system (10), when different firms may be controlled differently, is always completely controllable.

<u>Theorem 2.</u>

System (10) is always completely controllable for all n = 2 and arbitrary positive constants A and c_k (k = 1, 2, …, n).

<u>Proof.</u>

In this case the Kalman contollability matrix is the following :

K=(B, A B, A^2 B , A^{n-1} B).

Observe that B is a diagonal matrix with nonzero diagonal elements, so the columns of B are linearly independent. Therefore tha rank of K is at least n, and since it has only n rows, rank(K) = n.

Lets turn our attention next to the continous time scales models. System (11) is completely controllable if and only if the following Kalman controllability matrix has full sank:

K = (S c, S H S c, S H S H S c , … , S H S . . . S H S c)

with

$$\underline{c} = \begin{pmatrix} -c_1 \\ -c_2 \\ \vdots \\ -c_n \end{pmatrix} and\ \underline{H} = A\cdot\begin{pmatrix} -2 & -1 & \cdots & -1 \\ -1 & -2 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & -2 \end{pmatrix} = A\cdot(-I-E).$$

It is very difficult to see a simple condition in the general case, therefore two important special cases will be discussed.

<u>Theorem 3.</u>

(i) In the case of n = 2, system (11) is completely controllable if and only if

$$S_1c_1^2 + 2(S_2 - S_1)c_1c_2 - S_2c_2^2 \neq 0.$$

(ii) Assume that $S_1 = S_2 = \dots = S_n$ and $n = 3$. Then system (11) is not completely controllable.

Proof

(i) Assume first that n = 2 then,

$$\underline{K} = \begin{pmatrix} -S_1c_1 & A(2S_1^2c_1 + S_1S_2c_2) \\ -S_2c_2 & A(S_1S_2c_1 + 2S_2^2c_2) \end{pmatrix} \quad (17)$$

and therefore det($\underline{K}$) ? 0 if and only if

$$S_1c_1^2 + 2(S_2 - S_1)c_1c_2 - S_2c_2^2 \neq 0$$

(ii) If $S_1 = S_2 = \dots = S_n = S$, then

$$\underline{A} = -As(\underline{I} + \underline{E})$$

and simple calculation shows that

$$\underline{A} = -(n+1)A^2S^2\underline{I} - (n+2)AS\underline{A}, \quad (18)$$

so $\underline{A}^2$ $\underline{b}$ is a linear combination of $\underline{b}$ and $\underline{A}$ $\underline{b}$. Therefore rank ($\underline{K}$) < n.

Remark.

Assume n=2 and $S_1 = S_2$. Then system (11) is completely controllable if and only if c_1 ? c_2. It is clear, that if $c_1=c_2$, then the system must not be completely controllable. Select a symmetric initial state, then for all t = 0, $z_1(t) = z_2(t)$, so the state cannot be controlled to nonsymmetric final states.
Consider finally the case when different forms maybe controlled differently. Then

$$K = (\underline{B}, \underline{A}\,\underline{B}, \underline{A}^2\underline{B}, \dots, \underline{A}^{n-1}\underline{B})$$

where $\underline{B}$ = diag ($-S_1 c_1$, $-S_2 c_2$, … , $-S_n c_n$) with linearly independent columns. Therefore rank ($\underline{K}$) = n, and hence we have the following result:

Theorem 4.

Assume that in the case of continuous time scales different firms maybe controlled differently. Then the resulting dynamic system is always completely controllable.
In the above results only the classical single-product Cournot model without profit differentation was considered, and only static expectations were assumed. Other kinds of expectations and other model varriants can be investigated in an analogous manner. Their controllablity properties will be discussed in a future paper.

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