STABILITY PROPERTIES OF RESET SYSTEMS

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Abstract: Stability properties for a class of reset systems, such as systems containing a Clegg integrator, are investigated. We present Lyapunov based results for verifying $\mathcal{L}_2$ and exponential stability of reset systems. Our results generalize the available results in the literature and can be easily modified to cover $\mathcal{L}_p$ stability for arbitrary $p \in [1, \infty]$. Several examples illustrate that introducing resets in a linear system may reduce the $\mathcal{L}_2$ gain if the reset controller parameters are carefully tuned.

Keywords: Hybrid systems, Lyapunov, nonlinear, stability.

1. INTRODUCTION

It is a well known fact that linear control suffers from certain fundamental performance limitations. These limitations may sometimes be alleviated by nonlinear or hybrid feedback (Feuer *et al.*, 1997). Reset controllers are an example of nonlinear controllers that may overcome some of the fundamental performance limitations of linear controllers (Beker *et al.*, 2001).

Reset controllers are motivated by the so-called Clegg integrator introduced in (Clegg, 1958). This device is a particular type of a nonlinear integrator that operates in the same manner as the linear integrator whenever its input and output have the same sign and it resets its output to zero otherwise (modeling of the Clegg integrator was analyzed in detail in (Zaccarian *et al.*, 2004)). Its describing function has the same magnitude plot as the linear integrator but it has a phase lag of only 38.1° compared to the lag of 90° for a linear integrator (see (Clegg, 1958) for details). This feature can be used to provide more flexibility in controller design. A more general reset element is the so called *First Order Reset Element (FORE)*. A FORE operates in the same way as the Clegg integrator except that it contains a more general first order linear filter instead of an integrator.

Early designs of reset controllers that use respectively Clegg integrators and FOREs can be found in (Krishnan and Horowitz, 1974) and (Horowitz and Rosenbaum, 1975). First attempts to rigorously analyze stability of reset systems with Clegg integrators can be found in (Hu *et al.*, 1997; Hollot *et al.*, 1997). In particular an integral quadratic constraint was proposed in (Hollot *et al.*, 1997)

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to analyze stability of reset systems. However, the proposed condition was conservative as it was independent of reset times. Stability analysis of reset system consisting of a second order plant and a FORE was conducted in (Chen *et al.*, 2001) (see also (Chen *et al.*, 2000*b*)). Stability analysis of general reset systems can be found in (Beker *et al.*, 2004) (see also (Hollot *et al.*, 2001; Chen *et al.*, 2000*a*)), where Lyapunov based conditions for asymptotic stability of general reset systems were presented. Moreover, the authors proposed computable conditions for quadratic stability based on linear matrix inequalities (LMIs). Bounded-input bounded-state stability of general reset systems was obtained as a consequence of quadratic stability. Finally, an internal model principle was proved for tracking of and disturbance rejection.

In this paper we present Lyapunov based conditions for $\mathcal{L}_2$ stability of general reset systems. We emphasize that the same proof technique can be used to prove $\mathcal{L}_p$ stability for arbitrary $p \in [1, \infty]$. Moreover, a similar Lyapunov condition is presented for exponential stability that generalizes the stability condition in (Beker *et al.*, 2004, Theorem 1) in several directions. First, our results use locally Lipschitz Lyapunov functions, including piecewise quadratic Lyapunov functions, as opposed to continuously differentiable Lyapunov functions that were used in (Beker *et al.*, 2004). Second, we use a model of reset systems, proposed in (Zaccarian *et al.*, 2004), that allows us to considerably relax the Lyapunov conditions. For instance, in (Beker *et al.*, 2004, Theorem 1) the authors require existence of a Lyapunov function that decreases along solutions of the system in absence of resets everywhere in the state space. Our condition, on the other hand, requires such a decrease only in a strict subset of the state space. This allows us to obtain sharper stability bounds and input/output gains and, as a result, we obtain interesting new insights into design of reset systems with Clegg integrators and FOREs (see also (Zaccarian *et al.*, 2004)).

The results of this paper provide a framework for the analysis of exponential and input/output stability of reset systems and will be useful in the development of systematic reset controller design procedures. For instance, the results of this paper are used in (Zaccarian *et al.*, 2004) to derive LMI based tools for the construction of piecewise quadratic Lyapunov functions that establish $\mathcal{L}_2$ and exponential stability of reset systems with Clegg integrators and FOREs. We believe that further such developments will be made possible using the results of this paper.

The paper is organized as follows. In Sections 2 and 3 we present respectively preliminaries and the class of reset systems that we consider. Section 4 contains the main results. Examples are presented in Section 5. Summary and conclusions are given in the last section.

Notation. The sets of positive integers (including zero) and real numbers are respectively denoted as $\mathbb{N}_0$ and $\mathbb{R}$. Given vectors x_1, x_2 we use the notation $(x_1, x_2) := [x_1^T \;\; x_2^T]^T$. Given an integer $p \in [1, \infty)$ and a Lebesgue measurable function $d : [t_1, t_2] \to \mathbb{R}^d$, we use the notation $\|d[t_1, t_2]\|_{\mathcal{L}_2} := \left(\int_{t_1}^{t_2} |d(\tau)|^2 d\tau \right)^{\frac{1}{2}}$. If $\|d[0, +\infty)\|_{\mathcal{L}_2}$ is bounded, then we write $d \in \mathcal{L}_2$.

2. PRELIMINARIES

We use here the approach from (Goebel *et al.*, 2004) to define the solutions of hybrid systems. The hybrid time domain is defined as a subset of $[0, \infty) \times \mathbb{N}_0$, given as a union of finitely or infinitely many intervals $[t_i, t_{i+1}] \times \{i\}$ where the numbers $0 = t_0, t_1, \ldots,$ form a finite or infinite nondecreasing sequence. The last interval is allowed to be of the form $[t_i, T)$ with T finite or $T = +\infty$. Let two closed sets $\mathcal{C}$ and $\mathcal{D}$ be given such that $\mathcal{C} \cup \mathcal{D} = \mathbb{R}^n$ and functions $f : \mathcal{C} \to \mathbb{R}^n$ and $g : \mathcal{D} \to \mathbb{R}^n$. A solution of the hybrid system $x(\cdot, \cdot)$ is a function defined on the hybrid time domain, such that

$$\begin{aligned} \dot{x}(t,i) = f(x(t,i)) & \left.\right\} \begin{array}{l} \text{only if } x(t,i) \in \mathcal{C} \\ \text{and } t \in (t_i, t_{i+1}) \end{array} \\ \begin{array}{r} x(t_{i+1}, i+1) \\ = g(x(t_{i+1}, i)) \end{array} & \left.\right\} \begin{array}{l} \text{only if } x(t_{i+1}, i) \in \mathcal{D} \\ \text{and } i \in \mathbb{N}_0 \; . \end{array} \end{aligned} \tag{1}$$

To shorten notation, we omit the time arguments and write (1) as:

$$\begin{aligned} \dot{x} &= f(x) && \text{only if } x \in \mathcal{C} \\ x^+ &= g(x) && \text{only if } x \in \mathcal{D} \; . \end{aligned} \tag{2}$$

Given (t, N) such that $t \in [t_N, t_{N+1}]$ we define:

$$\int_0^t x(\tau) d\tau := \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} x(\tau, i) d\tau + \int_{t_N}^t x(\tau, N) d\tau \; .$$

In the next section we will use sets $\mathcal{C}$ and $\mathcal{D}$ of a special form that are defined next. Let $\epsilon \geq 0$ and $M = M^T$ and denote

$$\mathcal{C}_\epsilon := \{x \in \mathbb{R}^n : x^T M x + \epsilon x^T x \geq 0\} \tag{3}$$

$$\mathcal{D}_\epsilon := \{x \in \mathbb{R}^n : x^T M x + \epsilon x^T x \leq 0\} \tag{4}$$

and $\mathcal{C} := \mathcal{C}_0$ and $\mathcal{D} := \mathcal{D}_0$.

3. RESET SYSTEMS

In the sequel we concentrate on the following class of hybrid models:

$$\left.\begin{aligned} \dot{x} &= Ax + Bd \\ \dot{\tau} &= 1 \end{aligned}\right\} \begin{array}{r} \text{only if } \;\; x \in \mathcal{C} \\ \text{or } \tau \leq \rho \end{array} \tag{5}$$

$$\left.\begin{aligned} x^+ &= \widetilde{A}x \\ \tau^+ &= 0 \end{aligned}\right\} \begin{array}{r} \text{only if } \;\; x \in \mathcal{D} \\ \text{and } \;\; \tau \geq \rho \end{array} \tag{6}$$

$$y = Cx \; , \tag{7}$$

where $x \in \mathbb{R}^n$, $d \in \mathbb{R}^{n_d}$, $\tau \geq 0$ and $\rho > 0$.

The role of the variable τ to achieve "time regularization" in the sense of (Johansson *et al.*, 1999) in order to avoid Zeno solutions. Indeed, it is obvious that the reset times satisfy $t_{i+1} - t_i \geq \rho$ for all $i \in \mathbb{N}_0$ and, hence, Zeno solutions can not occur.

It was shown in (Zaccarian *et al.*, 2004) that the class of models (5), (6), (7) can be used to describe general (linear) reset systems, as the following example illustrates.

Example 1. (Zaccarian *et al.*, 2004) The block diagram of the Clegg integrator controlling an integrator via a unity feedback is given in Figure 1.

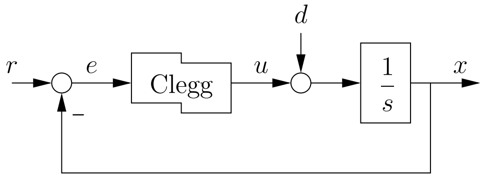

Fig. 1. Clegg integrator controlling an integrator.

The model of the closed loop system can be written as follows:

$$\left.\begin{aligned} \dot{x}_r &= r - x \\ \dot{x} &= k \cdot x_r + d \\ \dot{\tau} &= 1 \end{aligned}\right\} \text{ only if } (x, x_r) \in \mathcal{C} \text{ or } \tau \leq \rho$$
$$\left.\begin{aligned} x_r^+ &= 0 \\ \tau^+ &= 0 \end{aligned}\right\} \text{ only if } (x, x_r) \in \mathcal{D} \text{ and } \tau \geq \rho\ .$$

where $\rho > 0$; $\mathcal{C}$ and $\mathcal{D}$ are defined in (3), (4) with $M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and $\epsilon = 0$; x_r and x are respectively the controller (reset) and plant states; d and r are the disturbance and reference inputs.

Remark 1. It is important to note the difference between our model and the model in (Beker *et al.*, 2004), which has the following form:

$$\dot{x} = A_{cl}x + B_{cl}d \qquad \text{if } x \notin \mathcal{M}(t) \tag{8}$$
$$x^+ = A_R x \qquad \text{if } x \in \mathcal{M}(t)\ , \tag{9}$$

where $\mathcal{M} := \{x : C_{cl}x = 0,\ (I - A_R)x \neq 0\}$ for some matrix $C_{cl} \in \mathbb{R}^{p \times n}$. There are three main differences between our model (5), (6) and the model (8), (9):

1. In the model (8), (9) resets are only possible on the hyperplane $C_{cl}x = 0$ (as long as some flow has occurred since the last reset), whereas in our model (5), (6) resets are enforced on a sector $\mathcal{D}$.

2. Our model (5), (6) uses time regularization to avoid Zeno solutions whereas there is no time regularization in the model (8), (9). Instead, (Beker *et al.*, 2004, Theorem 1) states existence of solutions for (8), (9). Despite this result, it is not clear what they mean by solution for some states. Indeed, for the reset system (8), (9) without disturbances it is not clear how to define solutions for the initial conditions satisfying $C_{cl}x_0 = 0$, $(I - A_R)x_0 = 0$ and where, following the differential equation for arbitrarily small time yields $Cx(t) = 0$ and $(I - A_R)x(t) \neq 0$. As an example, consider the initial condition $x_0 = (0, a, 0)$, $a > 0$ for the system with $A_{cl} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$, $A_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $C_{cl} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. Note that we have $C_{cl}x_0 = 0$, $(I - A_R)x_0 = 0$ for the given initial condition (thus $x_0 \notin \mathcal{M}$ and the reset is not possible at $t = 0$, which means that the dynamics can only be governed by the flow equation (8) for small $t \geq 0$). Moreover, integrating the in differential equation (8) from the same initial condition yields $C_{cl}x(t) = 0$ for all t and $(I - A_R)x(t) = \begin{bmatrix} 0 & 0 & x_3(t) \end{bmatrix}'$, which is initially zero but is nonzero for all small t (thus $x(t) \in \mathcal{M}$ for $t > 0$ and thus flowing from the initial condition is not possible). Note that the conditions of (Beker *et al.*, 2004, Theorem 1) hold for this example (use $V(x) = |x|^2$, which yields $\dot{V} = -2V$ and $\Delta V \leq 0$).

3. The set $\mathcal{M}$ and its complement are not closed whereas the sets $\mathcal{C}$ and $\mathcal{D}$ are always closed. Moreover, the sets $\mathcal{M}$ and its complement are disjoint, whereas the sets $\mathcal{C}$ and $\mathcal{D}$ have a common boundary and, hence, they overlap.

4. MAIN RESULTS

In this section we state our main results. Sufficient $\mathcal{L}_2$ and exponential stability conditions for the system (5), (6) are presented respectively in Theorems 1 and 2. In all our results we will rely on the following assumption:

Assumption 1. For the system (5), (6), the reset map $\tilde{A}$ is such that

$$x \in \mathcal{D} \implies \tilde{A}x \in \mathcal{C}\ . \tag{10}$$

Condition (10) is quite natural to assume for reset systems. This condition guarantees that after each reset time the solutions will be mapped to the set $\mathcal{C}$ where the dynamics are governed by the differential equation (5) so that flowing is possible from there. Without this condition, due to the time regularization, defective trajectories may correspond to solutions that keep flowing and jumping within the set $\mathcal{D}$, so that it would be impossible to establish that all solutions flow only in the set $\mathcal{C}_\epsilon$. This last property is a key tool for exploiting the advantages of resets within the Lyapunov framework, thereby establishing our main results.

Theorem 1. Suppose that Assumption 1 holds and that there exists a locally Lipschitz Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, strictly positive numbers $a_1, a_2, a_3, a_4, \gamma, \epsilon$ and a matrix $M = M^T$ such that the following holds for all $d \in \mathbb{R}^{n_d}$:

$$a_1|x|^2 \leq V(x) \leq a_2|x|^2,\ \forall x \in \mathbb{R}^n; \tag{11}$$

$$\frac{\partial V}{\partial x}(Ax+Bd) \leq -a_3|y|^2+\gamma|d|^2, \text{for a.a. } x \in \mathcal{C}_\epsilon \tag{12}$$

$$V(\widetilde{A}x) - V(x) \leq 0 \qquad \forall x \in \mathcal{D}; \tag{13}$$

$$\left|\frac{\partial V}{\partial x}\right| \leq a_4\,|x|\,, \text{ for a.a. } x \in \mathbb{R}^n\ . \tag{14}$$

Then, for any $L > 1$ there exists $\rho^* > 0$ such that [4] for all $\rho \in (0, \rho^*)$ the solutions of the system (5), (6), (7) satisfy:

$$\int_0^t |y(\tau)|^2 d\tau \leq \frac{La_2}{a_3}|x_0|^2 + \frac{\gamma}{a_3}\int_0^t |d(\tau)|^2 d\tau\ ,$$

for all $t \geq 0$, $\tau(0,0) = \tau_0 \geq 0$, $x(0,0) = x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_2$. ■

Remark 2. A results similar to Theorem 1 can be stated for the case of $\mathcal{L}_p$ stability for arbitrary $p \in [1, \infty]$. The conditions of Theorem 1 need to be changed slightly and the proofs modified in a straightforward manner. We did not state this result due to space constraints and for simplicity.

Remark 3. Sufficient conditions for $\mathcal{L}_\infty$ (bounded input bounded state) stability of reset systems were presented in (Beker *et al.*, 2004) for general reset systems. Theorem 1 presents for the first time results on $\mathcal{L}_2$ stability of reset systems.

It is instructive to note that $\mathcal{L}_p$ stability from w to y for some $p \in [1, \infty)$ implies exponential stability of the system in the absence of disturbances. Therefore, if we have an appropriate $\mathcal{L}_p$ detectability from y to x, we can conclude $\mathcal{L}_p$ stability from w to x from Theorem 1. Then, under mild technical conditions this implies exponential stability in the absence of disturbances. This result can be proved using results of (Teel *et al.*, 2002) and it is very similar to (Nešić and A.R.Teel, 2004). A special case of the required detectability property is when there exists $\mu > 0$ such that $\mu^2|x|^2 \leq |y|^2$. We formally state this case in the next theorem, while additional results relying on more general detectability conditions will be not covered here.

Theorem 2. Consider the system (5), (6) without disturbances. Suppose that Assumption 1 holds and that there exists a locally Lipschitz Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, strictly positive numbers $a_1, a_2, a_3, a_4, \epsilon$ and a matrix $M = M^T$ such that the following holds:

$$a_1|x|^2 \leq V(x) \leq a_2|x|^2,\ \forall x \in \mathbb{R}^n; \tag{15}$$

$$\frac{\partial V}{\partial x}Ax \leq -a_3|x|^2,\ \text{for a.a. } x \in \mathcal{C}_\epsilon; \tag{16}$$

$$V(\widetilde{A}x) - V(x) \leq 0,\ \forall x \in \mathcal{D}; \tag{17}$$

$$\left|\frac{\partial V}{\partial x}\right| \leq a_4\,|x|\,, \text{ for a.a. } x \in \mathbb{R}^n; \tag{18}$$

Then, there exist $\rho^*, K > 0$ such that for all $\rho \in (0, \rho^*)$ the solutions of the system (5), (6) satisfy:

$$|x(t,i)| \leq K \exp\left(-\frac{a_3}{2a_2}t\right)|x_0|\ ,$$

for all $t \in [t_i, t_{i+1}]$, $i \geq 0$, $\tau(0,0) = \tau_0 \geq 0$ and $x(0,0) = x_0 \in \mathbb{R}^n$. ■

Remark 4. Note that conditions (12) and (16) need to hold only on the set $\mathcal{C}_\epsilon$, which is a subset of $\mathbb{R}^n$. Moreover, the closure of $\mathcal{C}_\epsilon$ is typically a proper subset of $\mathbb{R}^n$; hence conditions (12) and (16) are much weaker than requiring stability of $\dot{x} = Ax + Bd$ that was required in (Beker *et al.*, 2004, Theorem 1) to guarantee stability of the reset system. Hence, Theorems 1 and 2 relax the stability conditions used in (Beker *et al.*, 2004). Finally, we note that in general we can not replace $\mathcal{C}_\epsilon$ by $\mathcal{C}$ in (12). However, when $V(\cdot)$ has extra properties (as in the following Proposition 1), this can be possible.

Remark 5. Our conditions (14) and (18) allow for non-differentiable Lyapunov functions $V(\cdot)$, which is another relaxation of the conditions in (Beker *et al.*, 2004, Theorem 1), where continuous differentiability of $V(\cdot)$ was required. This generalization allows us, among other things, to consider piecewise quadratic Lyapunov functions which were not possible to handle using the results of (Beker *et al.*, 2004, Theorem 1). It turns out that piecewise quadratic Lyapunov functions are a key tool for exploiting convex optimization tools such as LMIs when trying to obtain tight estimates of $\mathcal{L}_2$ gains for this class of systems, as illustrated in (Zaccarian *et al.*, 2004).

Theorems 1 and 2 provide a theoretical framework for analysis and design of reset systems. A typical analysis problem consists in finding an appropriate Lyapunov function satisfying the conditions of the theorems for a given system (5), (6). Computational approaches via LMIs that use piecewise quadratic Lyapunov functions are given in (Zaccarian *et al.*, 2004). For instance, Theorem 1 can be used to prove the following result on $\mathcal{L}_2$ stability via quadratic Lyapunov functions $V(x) = x^T P x$.

Proposition 1. (Zaccarian *et al.*, 2004) Consider the reset control system (5), (6), (7), where the sets $\mathcal{C}$ and $\mathcal{D}$ are defined by the matrix M via (3), (4). If the following linear matrix inequalities in the variables $P = P^T > 0$, τ_F, $\tau_R \geq 0$, $\gamma > 0$ are feasible:

$$\begin{pmatrix} A^TP + PA + \tau_F M & PB & C^T \\ \star & -\gamma I & 0 \\ \star & \star & -\gamma I \end{pmatrix} < 0, \tag{19}$$
$$\widetilde{A}^T P \widetilde{A} - P - \tau_R M \leq 0,$$

[4] The explicit value of ρ^* is omitted due to space constraints.

then, there exists ρ^* such that, for all $\rho \in (0, \rho^*)$, the reset system (5), (6), (7) has a finite $\mathcal{L}_2$ gain from d to y that is smaller than γ. ■

We note that using quadratic Lyapunov functions is often too restrictive for reset systems and more general theorems based on piecewise quadratic Lyapunov functions from (Zaccarian *et al.*, 2004) are often needed.

Sketch of Proof of Theorem 1: The proof is based on Lemmas 1-3 (see below) that are stated without a proof. Denote the reset times as t_i where we use the convention that $t_0 = 0$ and $t_N = t$ even though the times 0 and t may not be reset times. Lemma 2 gives us an appropriate bound on the time interval $[t_0, t_1]$. Because of Assumption 1 we have that $x(t_i, i) \in \mathcal{C}$ for all $i \geq 1$ and Lemma 1 gives us appropriate bounds on the intervals $[t_i, t_{i+1}]$ for $i = 1, \ldots, N-1$. Lemma 3 guarantees that the value function does not increase at reset times. Hence, by concatenating the intervals $[t_i, t_{i+1}]$ we can add the bounds in Lemmas 1 and 2 to prove the result. More details can be found in the journal version of this paper (Nešić *et al.*, 2004). ■

Lemma 1. Suppose that the conditions of Theorem 1 hold. Then, there exists $\rho^* > 0$ such that for all $\rho \in (0, \rho^*)$ we have that if $x(t_i, i) \in \mathcal{C}$ and $d \in \mathcal{L}_2$ then $a_3 \int_{t_i}^{t} |y(\tau, i)|^2 d\tau \leq V(x(t_i, i)) - V(x(t, i)) + \gamma \int_{t_i}^{t} |d(\tau)|^2 d\tau$ for all $t \in [t_i, t_{i+1}]$. ■

Lemma 2. Suppose that the conditions of Theorem 1 hold. Then, for any $L > 1$ there exists $\rho^* > 0$ such that for any $\rho \in (0, \rho^*)$, $x(0,0) = x_0$, $\tau(0,0) \geq 0$ and $d \in \mathcal{L}_2$ we have that $a_3 \int_{t_0}^{t} |y(\tau, t_0)|^2 d\tau \leq LV(x(t_0, 0)) - V(x(t, 0)) + \gamma \int_{t_0}^{t} |d(\tau)|^2 d\tau$ for all $t \in [t_0, t_1]$. ■

Lemma 3. Under the conditions of Theorem 1, for any $i \geq 0$ we have that $V(x(t_{i+1}, i+1)) \leq V(x(t_{i+1}, i))$. ■

5. EXAMPLES

Constructing Lyapunov functions for general reset systems that satisfy the conditions of Theorems 1 and 2 is typically hard. It is easier to do so for systems containing FOREs. In (Zaccarian *et al.*, 2004) we presented a method based on Linear Matrix Inequalities to construct piecewise quadratic Lyapunov functions to check $\mathcal{L}_2$ stability for a class of reset systems containing FOREs. In this section, we use results from (Zaccarian *et al.*, 2004) to analyze the $\mathcal{L}_2$ stability of systems with reset controllers. In particular, we show how changing parameters in the FORE affects the gain of the reset closed-loop system.

Example 2. Consider an integrator (plant) controlled by a FORE:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + d \\ \dot{x}_2 &= -x_1 + \beta x_2 \\ \dot{\tau} &= 1 \end{aligned}\right\} \text{only if } x_1 x_2 \leq 0 \text{ or } \tau \leq \rho \tag{20}$$

$$\left.\begin{aligned} x_2^+ &= 0 \\ \tau^+ &= 0 \end{aligned}\right\} \text{only if } x_1 x_2 \geq 0 \text{ and } \tau \geq \rho \tag{21}$$

and assume that the output is $y = x_1$. Here, x_1 and x_2 respectively denote the state of the scalar plant and of the FORE. We computed the $\mathcal{L}_2$ gain from d to y for the system (20), (21) using the LMI method from (Zaccarian *et al.*, 2004). The gain has been computed for the limit case as $\rho \to 0$. (Larger values of ρ correspond, in general, to larger gains due to the fact that $\mathcal{C}_\epsilon$ would be larger.) The gain is plotted as a function of the parameter β that determines the pole of the FORE. This plot is represented by the dashed line in Figure 2. Moreover, we considered the linear system without resets:

$$\begin{aligned} \dot{x}_1 &= x_2 + d \\ \dot{x}_2 &= -x_1 + \beta x_2 \\ y &= x_1. \end{aligned} \tag{22}$$

The full line in Figure 2 shows the $\mathcal{L}_2$ gain of the linear system (22) as a function of the parameter β. Note that adjusting the parameter β in the linear controller can not produce a gain smaller than ≈ 1.5. Moreover, as β tends to zero the $\mathcal{L}_2$ gain of the linear system tends to infinity. For positive values of β the linear system (22) is unstable and does not have a well defined $\mathcal{L}_2$ gain. On the other hand, the $\mathcal{L}_2$ gain of (20), (21) is well defined for all values of β. Moreover, as $\beta \to \infty$ the $\mathcal{L}_2$ gain of the reset system tends to zero. This example illustrates that reset controllers may have advantages over linear controllers.

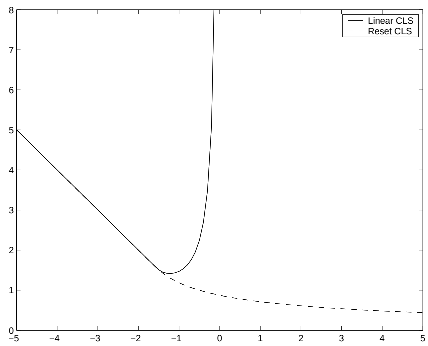

Fig. 2. $\mathcal{L}_2$ gains of linear and reset closed loops for Example 2, as a function of the pole of the FORE.

Example 3. We also address a second example borrowed from (Hollot *et al.*, 2001). In this example, a FORE element whose linear part is characterized by the transfer function $\frac{1}{s+1}$ controls via a negative unitary feedback a SISO plant whose transfer function is $\frac{X_r(s)}{E(s)} = \frac{s+1}{s(s+0.2)}$. For this example, the control system involving the FORE is shown in (Hollot *et al.*, 2001) to behave more desirably than the linear control system. It was shown in (Hollot *et al.*, 2001) that the reset system had only about 40% overshoot of the linear closed loop system while retaining the rise time of the linear design. This example can be further interpreted using our results. Indeed, when computing the $\mathcal{L}_2$ gain from the plant input to the plant output, the linear closed-loop system has an H_∞ norm around 5, while using the construction in (Zaccarian *et al.*, 2004, Theorem 3) and the main results of our paper we obtain that the $\mathcal{L}_2$ gain of the reset system is 3.82.

Figure 3 reports the $\mathcal{L}_2$ gains for the linear closed-loop and the reset closed-loop as a function of the pole of the FORE. Once again, for positive values of β (unstable fores) the linear closed-loop is unstable, while the reset closed-loop guarantees smaller gains. The case studied in (Hollot *et al.*, 2001) corresponds to the horizontal coordinate $\beta = -1$ in Figure 3.

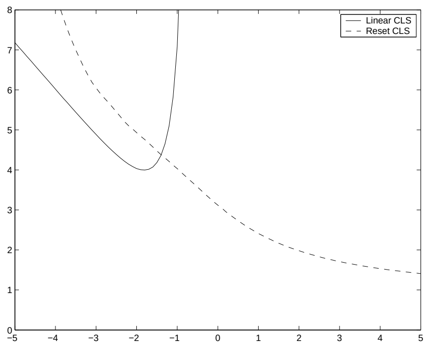

Fig. 3. $\mathcal{L}_2$ gains of linear and reset closed loops for Example 3, as a function of the pole of the FORE.

6. CONCLUSIONS

We provided Lyapunov like conditions that guarantee $\mathcal{L}_2$ stability and exponential stability of a class of reset systems, such as systems containing Clegg integrators. Our results provide a theoretical framework for systematic analysis and controller design of reset systems and they generalize the corresponding results in (Beker *et al.*, 2004). Examples illustrate that it is possible to improve the $\mathcal{L}_2$ gain of a linear controller by a simple introduction of resets.

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