A SIMPLE SEPARATION RESULT FOR CONTROL AFFINE SYSTEMS

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Abstract: The purpose of this paper is to establish a simple global separation result for nonlinear control systems which are affine in the input. In particular, it is shown that a globally asymptotically stabilizing inverse optimal state feedback with an asymptotic polynomial growth rate in conjunction with a globally asymptotically stable observer leads to a globally asymptotically stable closed-loop. For example, this asymptotic growth rate condition may be helpful in case of polynomial control systems with a polynomial stage cost.

Keywords: Separation principle, nonlinear systems, cascades, polynomial control systems.

1. INTRODUCTION

A separated design of a globally stabilizing state feedback and of a globally stabilizing observer does not automatically lead to a stable closed-loop in nonlinear feedback design. Additional effort is necessary to guarantee global asymptotic stability. For example, either to redesign the observer or to redesign the state feedback. This is usually done by making the observer sufficiently fast, i.e, to use high-gain observers (Atassi and Khalil, 2000), or by making the state feedback sufficiently robust, e.g., to use (i)ISS-like concepts (Angeli *et al.*, 2004). Both strategies are quite successful in control theory. But often, one would like to design the state feedback and the observer completely independent from each other. For example, in control practice one would like to replace an "old" state feedback by a "new" one without modifying the observer. However, a true modular design of the state feedback and the observer, i.e., a certainty-equivalence implementation, is in the general nonlinear case not possible. Due to this lack of a general nonlinear separation principle, one has to assume at least an inherent property in one of the two components (feedback or observer), in order to guarantee stability when the loop is going to be closed. Since in control practice, often optimal feedbacks with respect to a certain performance measure are applied, it makes sense to assume that the state feedback which is part of the control loop satisfies a certain performance measure. Therefore, the present work exploits the inherent robustness of optimal state feedback to establish a simple global separation result for nonlinear control systems. More precisely, a global separation result for nonlinear control systems which are affine in the input is established. It is assumed that the globally stabilizing state feedback is (inverse) optimal with respect to the classical integral performance measure: "u-squared plus a positive definite function of the states". Such performance measures are well known for example from LQR theory, and often used in control practice. Typical examples are feedback design methologies which are based on inverse optimal design or on model predictive control (Sepulchre *et al.*, 1997; Magni *et al.*, 2001). Furthermore, it is assumed there, that the feedback has to satisfy an asymptotic polynomial growth condition. For example, this asymptotic growth rate condition may be helpful in case of polynomial control systems with a polynomial stage cost. Finally, it is assumed that the observer in the control loop is globally asymptotically stable. As a result, global asymptotic stability of the closed-loop is established.

The remainder of the paper is organized as follows: In Section 2, the problem formulation is presented. In Section 3 the basic idea, an overview about existing approaches in the literature, and the main results are established. The main results of this paper are a global separation result for control affine systems that is based on inverse optimality and on an asymptotic polynomial growth rate condition. Concluding remarks are given in Section 4.

NOTATIONS. A function $V : \mathbb{R}^n \to \mathbb{R}$ is called positive definite, if $V(0) = 0, V(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ and A matrix $P \in \mathbb{R}^{n\times n}$ with entries $(P)_{ij}$ is positive definite if $x^T Px > 0, \forall x \in \mathbb{R}^n \setminus \{0\}$. The row vector $V_x(x) = \nabla V(x) = (\partial V/\partial x)(x)$ denotes the derivative of V with respect to x. Let $\mathbb{R}_+$ denote the set of positive real numbers, then $\mathcal{K}$ is the class of functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ which are zero at zero, strictly increasing, and continuous. $\mathcal{K}_\infty$ is the subset of class-$\mathcal{K}$ functions that are unbounded. The Euclidian norm of $x \in \mathbb{R}^n$ is denoted by $\|x\|$. 0 denotes a scalar zero, a zero vector, or a zero matrix respectively.

2. PROBLEM FORMULATION

The question studied in this paper is the following:

(a) Given a nonlinear control system of the form

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^q$ is the input and $y \in \mathbb{R}^p$ the output. $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times q}$, and $h : \mathbb{R}^n \to \mathbb{R}^p$, are assumed to be sufficiently smooth with $f(0) = 0$, $h(0) = 0$.

(b) Given a globally asymptotically stabilizing state feedback

$$u = k(x) = -\frac{1}{2}R(x)^{-1}G^T(x)V_x^T(x), \tag{2}$$

for the control system (1) which is assumed to be (inverse) optimal with respect to the following performance measure:

$$J = \int_0^\infty q(x(t)) + u^T(t)R(x(t))u(t)\, dt, \tag{3}$$

i.e., the following Hamilton-Jacobi-Bellman (HJB) equation is satisfied:

$$\begin{aligned} V_x(x)f(x) + V_x(x)G(x)k(x) + \\ q(x) + k(x)^T R(x)k(x) = 0, \end{aligned} \tag{4}$$

where q is a positive definite function and R is a positive definite matrix function with $R(x) = r(x)\underline{R}$, $r(x) \geq 1$ and $\underline{R}$ is a positive definite matrix. V is assumed to be a positive definite, radially unbounded C^2 function.

(c) Given a state observer

$$\dot{\hat{x}} = \hat{f}(\hat{x}, y, u) \tag{5}$$

for the control system (1) such that the observer error $e = x - \hat{x}$ is globally asymptotically stable. More precisely, it is assumed that there exists a Lyapunov function W such that

$$W_e(e)a(e, x) < -\alpha(W(e)), \tag{6}$$

where $\dot{e} = a(e, x)$ is the observer error dynamics and α is a differentiable, radially unbounded and positive definite function. For example observers which have a linear error dynamics in appropriate coordinates satisfy this assumption. Alternatively and less restrictive is the assumption that for an appropriately chosen initial value, lets say $\hat{x}_0 = 0$, the observer error $e(t) = x(t) - \hat{x}(t)$ converges to zero for $t \to \infty$.

Question: Under which additional assumptions is the closed-loop (1),(5),(2) ($u = k(\hat{x})$) (see Fig. 1), globally asymptotically stable?

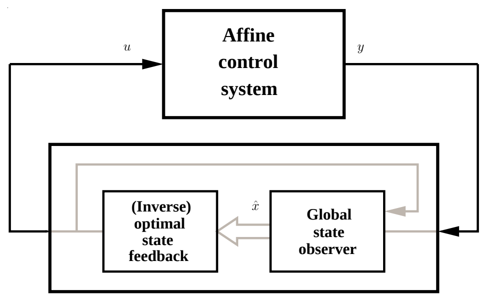

Fig. 1. Problem setup.

3. MAIN RESULTS

In this section, the main results are established, starting with some motivating considerations as well as some overview of existing approaches in the literature.

In the literature, there exists at least three concepts to establish a separation result: The high-gain concept (Atassi and Khalil, 2000), the (i)ISS-like concept (Angeli *et al.*, 2004), and the concept based on cascades (Loria, 2004; Arcak *et al.*, 2002; Sepulchre *et al.*, 1997). In the present work, the latter point of view is taken into account. In particular, the basic idea of the proposed separation principle is based on the point of view to consider the observer error as an exogenous error system. For this, the closed-loop is considered as cascade in (x, e)-coordinates:

$$\begin{aligned} \dot{x} &= f(x) + G(x)k(x + e) \\ \dot{e} &= a(e, x). \end{aligned} \tag{7}$$

As stated in Section 2, the e-subsystem is assumed to be asymptotically stable with a Lyapunov function W for the e-subsystem such that $W(e) > 0$ and

$$W_e(e)a(e,x) < -\alpha(W(e)), \tag{8}$$

for all nonzero e, x. A rather simple approach to establish asymptotic stability of the closed-loop (7) is to use a Lyapunov function candidate for the closed-loop which is separated in x and e,

$$V(x) + W(e) > 0, \tag{9}$$

and to try to establish that the derivative of the Lyapunov function candidate with respect to the closed-loop trajectories is negative definite, i.e.,

$$V_x(x)(f(x) + G(x)k(x+e)) + W_e(e)a(e,x) < 0.$$

Observe now, if W is a Lyapunov function of the e-subsystem (8), so is $c_1W + c_2W^2$, $c_i > 0$ a Lyapunov function. Or more generally, if s is a differentiable strictly monotonically increasing function, then $s(W(e))$ is a Lyapunov function of the e-subsystem. This rescaling argument is not new and is frequently used in Lyapunov-based designs, cf. e.g. (Sepulchre *et al.*, 1997; Praly and Arcak, 2004). Hence by using the Lyapunov function candidate $V(x) + s(W(e))$, one gets for the derivative

$$\begin{aligned} V_x(x)(f(x) + G(x)k(x+e)) + \\ s'(W(e))\underbrace{W_e(e)a(e,x)}_{-\alpha(e)} < 0, \end{aligned} \tag{10}$$

where s' denotes the derivative of $s = s(W)$ with respect to W. The fact that s' can be made large suggests that the expression $-s'(W(e))\alpha(W(e))$ can be made arbitrarily negative [1]. Hence, one could pose the problem of closed-loop stability as follows: If there exists a differentiable, positive definite function ρ, such that

$$V_x(x)f(x) + V_x(x)G(x)k(x+e) < \rho(W(e)) \tag{11}$$

holds for all x with $\rho(W(e)) \leq s'(W(e))\alpha(W(e))$, then the closed-loop is globally asymptotically stable. Alternatively, this holds also if that the function

$$\tilde{\rho}(e) := \max_{x\in\mathbb{R}^n} V_x(x)(f(x) + G(x)k(x+e)) \tag{12}$$

and the rescaling function s, defined via $\tilde{\rho}(e) \leq s'(W(e))\alpha(W(e))$, is well-defined. A closer look on (11), which is $\dot{V}(x) < \rho(W(e))$, reveals some connections to results in the literature, namely:

(i) In case $\rho(W(e)) \leq M$, boundedness of the solutions $x = x(t)$, which is a crucial building block in global separation results, is guaranteed. For example, this was exploited in (Praly and Arcak, 2004), but in a more general setup.

(ii) In case of replacing ρ in (11), with a class-$\mathcal{K}_\infty$ function, and by adding a positive definite function $-\gamma(\|x\|)$ on the right side, the inequality turns into an iISS (integral-input-to-state stability) conditions. ISS and iISS are often used concepts to guarantee boundedness of solutions (Arcak *et al.*, 2002).

(iii) If one integrates (11), that is,

$$V(x(t)) - V(x(0)) < \int_0^t \rho(W(e(\tau)))d\tau, \tag{13}$$

and if the value of the integral has a finite value for $t \to \infty$, then boundedness of the solutions $x = x(t)$ is guaranteed, in case of $e = e(t)$ exists. High-gain concepts in combination with local Lipschitz assumptions, which avoid finite escape phenomena, allow to make the value of the integral arbitrarily small under certain condition, like no peaking etc.

(vi) That boundedness of the solutions is necessary and sufficient, is also justified by following appealing statement from cascaded systems results (Seibert and Suarez, 1990) (Theorem 1.1): "boundedness + global asymptotic stability of each subsystem of the cascade implies global asymptotic stability of the cascaded system".

This shows that boundedness plays a central role and appears in one or the other form as assumption in most if not all separation theorems. As already mentioned, many results in the literature can be seen from this particular view point. Related conditions on forward completeness, i.e., existence of the solutions for all $t \geq 0$, can be found in (Angeli and Sontag, 1999). Hence, a central question is: How can one establish an easy-to-use setup to guarantee boundedness of the solutions. The purpose of the next theorem, which is the main result, is to show that an (inverse) optimal state feedback with an asymptotic polynomial growth rate in conjunction with an asymptotic stable observer is sufficient to guarantee existence of the solution and global asymptotic stability of the closed loop. This is done by utilizing (2), (4), (6), (7) and (11).

Theorem 1. Suppose all assumptions made in Section 2 hold. Moreover, suppose there exists a polynomial function $p = p(\lambda, x)$ of minimal degree in λ for a given (fixed) x such that the feedback $u = k(x)$ defined by (2) grows asymptotically less than polynomial, i.e.,

$$\lim_{\lambda\to\infty} \frac{\|k(\lambda x)\|}{p(\lambda, x)} = 0, \tag{14}$$

for any given (fixed) x and suppose the derivative of (2) satisfies

$$\lim_{\lambda\to\infty} \frac{\|\lambda k'(\lambda x)\|}{p(\lambda, x)} = 0, \tag{15}$$

for any given (fixed) x. Then the closed-loop (7) is globally asymptotically stable.

[1] For generalizations, one can additionally rescale also V.

Proof. First, it is shown that there exists a positive definite function ρ, such that

$$V_x(x)f(x) + V_x(x)G(x)k(x+e) < \rho(W(e)) \quad (16)$$

holds for all x. Adding and subtracting $k(x)$ in (16) leads to

$$\begin{aligned} &V_x(x)f(x) + V_x(x)G(x)k(x) + \\ &V_x(x)G(x)(k(x+e) - k(x)) < \rho(W(e)). \end{aligned} \quad (17)$$

From the HJB equation (4) and from (2), one arrives at

$$\begin{aligned} &-q(x) - k(x)^T R(x)k(x) \\ &-2k(x)^T R(x)(k(x+e) - k(x)) < \rho(W(e)). \end{aligned} \quad (18)$$

Invoking Hadamard's Lemma (see Appendix), i.e., a mean value theorem for vector-valued functions, leads to

$$\begin{aligned} &-q(x) - k(x)^T R(x)k(x) \\ &-2k(x)^T R(x)\left[\int_0^1 k_x(x+\theta e)d\theta\right] e < \rho(W(e)). \end{aligned} \quad (19)$$

It has to be shown now, that the left hand side of the inequality (19) is bounded from above for any given (fixed) e. The idea is to show that $-k(x)^T R(x)k(x)$ dominates $-2k(x)^T R(x)\left[\int_0^1 \dots\right] e$ for $\|x\|$ sufficiently large. For this, the functions p and k are considered as functions of λ which are parametrized in x. Notice that it is enough to parameterize all x-directions, i.e., to parametrize the compact set $\{x \mid \|x\| = 1\}$. From the assumptions made in Theorem 1, i.e., Equation (14) and (15), the derivative of the state feedback $k'(\lambda x) = k_x(\lambda x)x$ grows with a growth rate of one degree less than that of the state feedback $k(\lambda x)$. This follows from

$$\lim_{\lambda\to\infty} \frac{\|\lambda k(\lambda x)\|}{p(\lambda, x)} \neq 0. \quad (20)$$

since p is assumed to be of minimal degree in λ for a given (fixed) x and from

$$\lim_{\lambda\to\infty} \frac{\|\lambda k'(\lambda x)\|}{p(\lambda, x)} = 0. \quad (21)$$

Therefore, because of this asymptotic polynomial behavior of $u = k(x) = -\frac{1}{2}R(x)^{-1}G^T(x)V_x^T(x)$,

$$\begin{aligned} &-k(\lambda x)^T R(\lambda x)k(\lambda x) \\ &-2k(\lambda x)^T R(\lambda x)\left[\int_0^1 k_x(\lambda x+\theta e)d\theta\right] e < 0, \end{aligned} \quad (22)$$

is satisfied for any given (fixed) x, e and for λ sufficiently large. Notice that R is diagonal and bounded from below by $\underline{R}$. The integration in the second term does not affect the polynomial growth rate of k_x, since the integration is w.r.t. θ which varies between zero and one and e is given (fixed). Therefore, the left hand side of (19) is bounded from above for a given fixed e and thus, ρ is well-defined. Notice also if k grows less than a linear function, then it follows from the Assumptions that the derivative k' has to approach asymptotically to zero. Second, it is shown that the closed-loop (7) is stable. This is done, via inequality (10). One can define, for example, a desired rescaling function s by

$$s'(\omega) = \begin{cases} \dfrac{\rho(\omega)}{\alpha(\omega)} + \epsilon & : \quad \omega > 1 \\ 1 & : \quad \omega \leq 1, \end{cases} \quad (23)$$

with $\epsilon > 0$, which is well-defined for all ω. Note that s' can be also chosen in such a way, that s is smooth. However, from such a rescaling function follows, that inequality (10) holds for all e with $W(e) > 1$. To establish boundedness of the closed-loop solutions $x = x(t), e = e(t)$, it is shown now that (10) even holds for all e with $W(e) \leq 1$ in case of $\|x\|$ being sufficiently large. In particular, from (22) follows immediately that (10) holds for all e with $W(e) \leq 1$ in case of $\|x\|$ being sufficiently large. More precisely, there exists a $\bar{\lambda}$ such that for all $(\lambda x, e)$ with (x, e) in the compact set $\{(x,e) \mid \|x\| = 1, W(e) \leq 1\}$ and with $\lambda > \bar{\lambda}$, the inequality (22) is satisfied. Therefore (10), i.e., the derivative of $V(x) + s(W(e))$ holds for all e and all x with $\|x\| \geq \bar{\lambda}$. Thus the closed-loop solutions of (7) are bounded. Finally, by the result of (Seibert and Suarez, 1990) (Theorem 1.1: "boundedness + global asymptotic stability of the subsystems $\Rightarrow$ global asymptotic stability of the cascade"), asymptotic stability of the closed-loop (7) follows. □

From this result follows immediately:

Corollary 1. Suppose all assumptions made in Section 2 hold. Moreover, suppose that the state feedback $u = k(x)$ defined by (2) is a polynomial function in x. Then the closed-loop (7) is globally asymptotically stable.

Remark 1. An alternative growth condition to (14) and (15) is the following condition:

$$\lim_{\|x\|\to\infty} \frac{k_x(x+c)}{1 + \|k(x)\|} = 0, \quad (24)$$

for any given (fixed) $c \in \mathbb{R}^n$. This follows by multiplying the inequality (22) with $\frac{1}{1+\|k(x)\|}$. In particular

$$\underbrace{\frac{k(\lambda x)}{1 + \|k(\lambda x)\|}}_{\not\to 0(\lambda\to\infty)}, \underbrace{2\int_0^1 \frac{k_x(\lambda x+\theta e)e}{1 + \|k(\lambda x)\|} d\theta}_{\to 0(\lambda\to\infty)}. \quad (25)$$

Remark 2. Although the asymptotic polynomial growth condition is not the least restrictive assumption (see Remark 1), it is worthwhile to note that the asymptotic nature of polynomials is easy to verify. In particular,

(14) ensures that the state feedback does not grow asymptotically faster than a polynomial and (15) ensure that the derivative of the state feedback grows slower than the state feedback. Roughly speaking, equation (15) avoids asymptotic oscillating behavior of the state feedback. This polynomial behavior at infinity may be of special interest for polynomial control systems, which have gained a lot of attention in recent years due to the fact that certain numerical tools, like the sum of squares decomposition, allow a computer-aided design. Already in (Sepulchre *et al.*, 1997) and in (Seibert and Suarez, 1990) special attention on polynomial growth conditions in cascaded systems was paid. Furthermore, also in (Panteley *et al.*, 1998; Loria, 2004) one can already find such results for cascades, but not in a setup as presented here.

As an immediate consequence of Theorem 1 is the following question: When or for which class of control systems is the growth condition (14) and (15) satisfied? A quite interesting question is given in the next statement. Namely, if of all functions f, G, q, R can be asymptotically bounded by polynomial functions, are then the asymptotic polynomial growth rate condition (14) and (15) of the inverse optimal state feedback satisfied?

Statement 1. Suppose all assumptions made in Section 2 hold. Moreover, suppose $u \in \mathbb{R}$ and the functions f, q, R, G in (1) and (3) with $\|G(x)\| \geq g_0 > 0$ can be asymptotically bounded by polynomial functions in the sense of (14) and (15). Then the closed-loop (7) is globally asymptotically stable.

In the following, ideas and discussions to Statement 1 are given which may be helpful for a future proof. In the case of $u \in \mathbb{R}$, denote $R(x) = r(x), G(x) = g(x)$, $\|g(x)\| \geq g_0 > 0$. Then, the HJB equation (4) has the following form:

$$V_x(x)f(x) - \frac{1}{4r(x)}(V_x(x)g(x))^2 + q(x) = 0. \quad (26)$$

Using the identity $a^T b = \|a\|\|b\|cos(a, b)$ and skipping the arguments, one can write (26) as

$$\begin{aligned} &\|V_x\|\|f\|cos(V_x, f) \quad (27) \\ &-\frac{1}{4r}\|V_x\|^2\|g\|^2cos^2(V_x, g) + q = 0. \end{aligned}$$

Now, one has to make sure that $\|V_x\|$ has polynomial growth at infinity, in case $\frac{1}{r(x)}V_x(x)g(x) \not\to 0$ for $\|x\| \to \infty$. In the other case, the optimal feedback (2) $u = k(x) = -\frac{1}{2r(x)}V_x(x)g(x)$ would converge to 0, which implies, of course, polynomial growth at infinity. Let's assume that $\|V_x\|$ grows faster than polynomial at infinity, then by dividing (27) through $\|V_x\|$, one obtains

$$\begin{aligned} &\|f\|cos(V_x, f) \quad (28) \\ &-\frac{1}{4r}\|V_x\|\|g\|^2cos^2(V_x, g) + \frac{q}{\|V_x\|} = 0. \end{aligned}$$

Since $\frac{q}{\|V_x\|}$ goes to 0 for $\|x\| \to \infty$, in case the limit exists, one gets in the limit

$$\|f\|cos(V_x, f) - \frac{1}{4r}\|V_x\|\|g\|^2cos^2(V_x, g) = 0. \quad (29)$$

Moreover, since $\|g\| \geq g_0 > 0$ and since it was assumed that $\frac{1}{r}cos(V_x, g)$ does not converge to 0, i.e., $\frac{1}{r}V_xg \not\to 0$, $\|V_x\|$ cannot grow faster than polynomial at infinity, because of $\|f\|cos(V_x, f)$ can be bounded by a polynomial function. Finally, one could argue that $\|V_x\|$ becomes (exponentially) large whenever $cos(V_x, g)$ goes to zero. Although V is assumed to be continuously differentiable, this can indeed happen. In this case the step from (28) to (29) is not valid in general, since the limit for $\|x\| \to \infty$ may not exist, due to these "oscillating" behavior. However, V is a control Lyapunov function and hence $cos(V_x, f) < 0$ for $cos(V_x, g) = 0$. Therefore, if $cos(V_x, g) = 0$, then $\|V_x\|\|f\|cos(V_x, f)$ must has polynomial growth because of $cos(V_x, f) < 0$, $\|f\| > 0$ and all function are at least C^2. Hence the optimal feedback grows asymptotically like a polynomial. For the second part, one has to show that (15) holds. For this, one may consider (26) along the ray λx for any given fixed x with $\|x\| = 1$. In other words, replace in (26) the argument x by λx:

$$V_x(\lambda x)f(\lambda x) - \frac{(V_x(\lambda x)g(\lambda x))^2}{4r(\lambda x)} + q(\lambda x) = 0. \quad (30)$$

Next differentiate (30) w.r.t. λ:

$$\begin{aligned} &x^TV_{xx}[f - \frac{1}{2r}(V_xg)g] \quad (31) \\ &+x^TR(V_x, f_x, g, g_x, r, r_x, q_x) = 0, \end{aligned}$$

where R is a (vector-valued) function which contains all the remaining functions with asymptotic polynomial growth. Notice that $f_{cl} = f - \frac{1}{2r}(V_xg)g$ is the closed loop. By using again the identity $a^Tb = \|a\|\|b\|cos(a, b)$, one obtains

$$\begin{aligned} &\|V_{xx}f_{cl}\|cos(x, V_{xx}f_{cl}) \quad (32) \\ &+\|R\|cos(x, R) = 0. \end{aligned}$$

Since $\|f_{cl}\|$ is positive definite and of asymptotic polynomial growth and since $\|R\|$ is also of asymptotic polynomial growth, one would expect that $\|V_{xx}f_{cl}\|$ is also of asymptotic polynomial growth in case one assumes $\|f_{cl}\| \not\to 0$ for $\|x\| \to \infty$. Basically, the problem in equation (31), (32) is similar as in equation (28), (29). One has to show (or to make sure by certain assumptions that pathological situations like oscillation may not appear) that there exists no (smooth) function with asymptotic exponential growth such that multiplied with a (smooth) function with asymptotic polynomial growth results in a (smooth) function with asymptotic polynomial growth. However, further investigations are necessary in this direction. Finally, it should be also noticed that there is a relation between equation (31) and the Hamiltonian system: $\dot{x} = H_y^T$, $\dot{y} = -H_x^T$, $H = yf - \frac{1}{4r}(yg)^2 + q$. Hence, (31) is related with the equation $V_{xx}H_y + H_x = 0$ with $H_x = R$. From this observe that in case H_x and $V_{xx}H_y$ vanishes, one obtains $y = V_x = const$ since $\dot{x} = H_y^T$, $\dot{y} = -H_x^T$. $\Box$

4. OUTLOOK AND SUMMARY

Future research will focus on following questions. (a) A proof to Statement 1 under certain additional conditions, e.g., V is convex, or a characterization for which class of control systems the growth rate condition is satisfied. (b) A more thorough study of the connections to existing results, i.e., how is the present result related with separation principles based on (i)ISS and cascades. (c) Generalization of Theorem 1, e.g., by imposing a certain growth rate condition on q.

Summarizing, in the present paper a new separation result for nonlinear control systems is established. It was shown that an inverse optimal state feedback with certain asymptotic growth rates in conjunction with an observer leads to globally asymptotically stability, in case of the state feedback control loop and the observer are globally asymptotically stable. The established separation result is based on following assumptions:

(i) control system is affine in the input
(ii) robustness of the inverse optimal state feedback
(iii) polynomial growth rate condition

Hence, the separation result do not use standard assumptions, like (i)ISS stability and high-gain arguments, which are often used in the literature. To the best knowledge of the authors, there are no results available which are based on the same assumptions. Nevertheless, there may exist generically results, e.g. results based on (i)ISS and cascades, which contains this setup. However, these results are often hard to verify and not easy to apply in control practice. Furthermore, although the separation result is quite simple to establish, the assumption used in the main theorems, i.e., "inverse optimal state feedback + stable observer", are often satisfied in control practice. Therefore, a simple separation result was established in this paper which also justifies why in control practice such a combination of (inverse) optimal feedback and observer often lead to satisfactory results.

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6. APPENDIX

Hadamard's Lemma (Petrovski, 1966). Let $k : \mathbb{R}^n \to \mathbb{R}^p$ be a (twice) continuously differentiable function, then for any x, e holds:

$$k(x + e) - k(x) = \left[\int_0^1 k_x(x + \theta e) d\theta \right] e \qquad (33)$$