QUADRATIC PERFORMANCE ANALYSIS FOR FINITE-HORIZON SYSTEMS

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Abstract: A finite dimensional condition is derived to test whether an integral quadratic constraint holds or not for a finite-horizon system with boundary conditions. A related parameter search problem is also considered and a cutting hyperplane generated by an infeasible parameter is derived.

Keywords: finite-horizon systems, integral quadratic constraints, boundary conditions

1. PROBLEM FORMULATION

Consider a state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

with a boundary condition

$$\Omega x(0) + \Upsilon x(1) = 0 \tag{2}$$

satisfying that

$$\Xi := \Omega + \Upsilon \mathrm{e}^{A} \tag{3}$$

is nonsingular, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\Omega \in \mathbb{R}^{n\times n}$, and $\Upsilon \in \mathbb{R}^{n\times n}$. The regularity of Ξ is required for the well-posedness of (1) and (2). In fact (1) and (2) has a unique solution $x = 0$ for $u = 0$ if and only if Ξ is nonsingular (Mirkin and Palmor, 1999).

The following is the first problem we study in this paper:

Problem 1. Let a real symmetric matrix $\Pi = \Pi^* \in \mathbb{R}^{(n+m)\times(n+m)}$ be given. Determine whether

$$\int_0^1 \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \Pi \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t < 0 \tag{4}$$

holds for all $u \in \mathbf{L}_2[0, 1]$, $u \neq 0$ or not.

We remark that (4) is a finite-horizon IQC (integral quadratic constraint), and Problem 1 is motivated by the important role of (infinite-horizon) IQCs in recent robust control theory (Megretski and Rantzer, 1997; Rantzer and Megretski, 1997). The norm computation of finite-horizon systems, which is a special case of Problem 1, is required in $\mathbf{H}_\infty$ analysis and synthesis of delay systems (e.g. (Zhou and Khargonekar, 1987)) and sampled-data systems (e.g. (Chen and Francis, 1995)), and in the computation of the spatio-temporal frequency response of a class of spatially invariant systems (e.g. (Jovanović and Bamieh, 2003)). Hence it is expected that Problem 1 is required to be solved in order to develop a robust control theory based on IQCs for the systems mentioned above.

There are several analysis tools for infinite-horizon IQCs including the Kalman-Yakubovich-Popov lemma (Rantzer, 1996). This paper indends to provide a counterpart for finite-horizon IQCs based on the approach in (Dullerud, 1999; Fujioka, 2004), where norm computation of finite horizon systems is considered.

We also remark that a special case ($\Omega = -\Upsilon$) of Problem 1 arises in robustness analysis of periodic systems (Kao *et al.*, 2001; Jönsson *et al.*, 2003).

As in the infinite-horizon case, we also consider the following parameter search problem, which will be important for reduction of conservativeness of robustness analysis:

Problem 2. Let real symmetric matrices $\hat{\Pi}_k = \hat{\Pi}_k^* \in \mathbb{R}^{(n+m)\times(n+m)}$ $(k = 0, 1, \ldots, q)$ and $\Lambda \subseteq \mathbb{R}_+^q$ be given, where $\mathbb{R}_+^q$ denotes the non-negative orthant of $\mathbb{R}^q$, Find a $\lambda \in \Lambda$ such that

$$\int_0^1 \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \hat{\Pi}(\lambda) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t < 0 \tag{5}$$

holds for all $u \in \mathbf{L}_2[0, 1]$, $u \neq 0$ if exists, where

$$\hat{\Pi}(\lambda) := \hat{\Pi}_0 + \sum_{k=1}^{q} \lambda_k \hat{\Pi}_k. \tag{6}$$

2. QUADRATIC PERFORMANCE TEST

In this section, we provide a solution to Problem 1 as a condition on a matrix.

We introduce a partition of Π:

$$\Pi = \Pi^* = \begin{bmatrix} \Pi_1 & \Pi_3 \\ \Pi_3^* & \Pi_2 \end{bmatrix} \tag{7}$$

where $\Pi_1 \in \mathbb{R}^{n\times n}$, $\Pi_2 \in \mathbb{R}^{m\times m}$, and $\Pi_3 \in \mathbb{R}^{n\times m}$. Then we have a condition so that the answer to Problem 1 is negative:

Proposition 3. There exists a $u \in \mathbf{L}_2[0, 1]$, $u \neq 0$ which violates (4) if Π_2 is not strictly negative-definite.

The proof is found in Appendix A

Hence in the sequel we consider the case of $\Pi_2 < 0$ where the following Hamiltonian matrix H is well-defined:

$$H := \begin{bmatrix} -A^* & -\Pi_1 \\ 0 & A \end{bmatrix} - \begin{bmatrix} -\Pi_3 \\ B \end{bmatrix} \Pi_2^{-1} \begin{bmatrix} B \\ \Pi_3 \end{bmatrix}^*.$$

The following theorem provides a solution to Problem 1:

Theorem 4. Suppose that $\Pi_2 < 0$. The following two statements are equivalent:

(i) (4) holds for all $u \in \mathbf{L}_2[0, 1]$, $u \neq 0$.
(ii) $\Phi < 0$ where the matrix Φ is defined as follows:

Step 1: Fix $\theta \in (-\pi, \pi]$ such that

$$\mathrm{e}^{\mathrm{j}\theta} \notin \mathrm{eig}(\mathrm{e}^{A}), \quad \mathrm{e}^{\mathrm{j}\theta} \notin \mathrm{eig}(\mathrm{e}^{H}).$$

Step 2: Define M by

$$M := R^* \begin{bmatrix} Q & (\mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^{A})^* \\ \mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^{A} & 0 \end{bmatrix} R, \tag{8}$$

where

$$Q := \int_0^1 \mathrm{e}^{A^* t}\, \Pi_1\, \mathrm{e}^{At}\ \mathrm{d}t,$$

$$R := \begin{bmatrix} \Xi^{-1}(\Omega\, \mathrm{e}^{-\mathrm{j}\theta} + \Upsilon) & 0 \\ 0 & I \end{bmatrix}.$$

Define also W_∞ as in the bottom of this page where

$$J := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

Step 3: Case 1) $\mathrm{eig}(H) \cap \mathrm{j}\mathbb{R} \neq \emptyset$: In this case

$$\eta := \max\{\, |\omega| : \omega \in \mathrm{eig}(H) \cap \mathrm{j}\mathbb{R}\} \geq 0$$

is well-defined. Fix N as a nonnegative integer satisfying

$$|\omega_{N+1}| > \eta, \quad |\omega_{N+2}| > \eta$$

where $\{\omega_i\}_{i=0}^{\infty}$ is defined by

$$\omega_i := 2\pi v_i + \theta, \quad \{v_i\}_{i=0}^{\infty} := \{0, 1, -1, 2, -2, \ldots\}.$$

Then Φ is defined by

$$\Phi := \begin{bmatrix} K & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} L^* \\ V_{N+1}^* \end{bmatrix} M \begin{bmatrix} L & V_{N+1} \end{bmatrix}$$

where

$$K := \begin{bmatrix} P_0^* \Pi P_0 & & 0 \\ & \ddots & \\ 0 & & P_N^* \Pi P_N \end{bmatrix}, \quad L := \begin{bmatrix} S_0 & \cdots & S_N \end{bmatrix},$$

$$P_i := \begin{bmatrix} (\mathrm{j}\omega_i I - A)^{-1} B \\ I \end{bmatrix}, \tag{9}$$

$$S_i := \begin{bmatrix} -(\mathrm{j}\omega_i I - A)^{-1} B \\ (\mathrm{j}\omega_i I - A)^{-*} \begin{bmatrix} \Pi_1 & \Pi_3 \end{bmatrix} P_i \end{bmatrix}. \tag{10}$$

V_{N+1} is a column full rank matrix defined by a factorization:

$$V_{N+1} V_{N+1}^* = W_\infty - \sum_{i=0}^{N} \bar{W}_i$$

where $\bar{W}_i$ is given at the bottom of this page.

$$W_\infty := \frac{1}{2} \begin{bmatrix} I & 0 \\ 0 & \left(\mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^{A}\right)^{-1} \end{bmatrix}^* \begin{bmatrix} 0 & -(\mathrm{e}^{\mathrm{j}\theta} I + \mathrm{e}^{A}) \\ -(\mathrm{e}^{\mathrm{j}\theta} I + \mathrm{e}^{A})^* & 2Q \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \left(\mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^{A}\right)^{-1} \end{bmatrix} - \frac{1}{2} J \left(\mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^{H}\right)^{-1} \left(\mathrm{e}^{\mathrm{j}\theta} I + \mathrm{e}^{H}\right).$$

$$\bar{W}_i := \left(\begin{bmatrix} \Pi_1 & (\mathrm{j}\omega_i I - A)^* \\ \mathrm{j}\omega_i I - A & 0 \end{bmatrix} + \begin{bmatrix} -\Pi_3 \\ B \end{bmatrix} \Pi_2^{-1} \begin{bmatrix} -\Pi_3 \\ B \end{bmatrix}^* \right)^{-1} - \begin{bmatrix} I & 0 \\ 0 & (\mathrm{j}\omega_i I - A)^{-1} \end{bmatrix}^* \begin{bmatrix} 0 & I \\ I & -\Pi_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (\mathrm{j}\omega_i I - A)^{-1} \end{bmatrix}.$$

Case 2) $\mathrm{eig}(H) \cap \mathrm{j}\mathbb{R} = \emptyset$: In this case Φ is defined by

$$\Phi := V_0^* M V_0 - I, \quad V_0 V_0^* = W_\infty.$$

where V_0 has its column full rank.

The proof is found in Appendix B.

3. SPECIAL CASES

Notice formally that M in Theorem 4 is equal to 0 if

$$\Omega \mathrm{e}^{-\mathrm{j}\theta} + \Upsilon = 0. \tag{11}$$

Hence Theorem 4 is further simplified when (11) holds. Since both Ω and Υ are real matrices, (11) implies either (a) $\Omega = -\Upsilon$ and $\theta = 0$, or (b) $\Omega = \Upsilon$ and $\theta = \pi$.

In fact we can take $\theta = 0$ when $\Omega = -\Upsilon$, and $\theta = \pi$ when $\Omega = \Upsilon$: In the proof of Theorem 4, M is constructed for θ satisfying $\mathrm{e}^{\mathrm{j}\theta} \notin \mathrm{eig}(\mathrm{e}^A)$. In addition, once we find $M = 0$, we do not need additional conditions on θ like $\mathrm{e}^{\mathrm{j}\theta} \notin \mathrm{eig}(\mathrm{e}^H)$. On the other hand, the regularity of Ξ requires that $1 \notin \mathrm{eig}(\mathrm{e}^A)$ when $\Omega = -\Upsilon$, and $-1 \notin \mathrm{eig}(\mathrm{e}^A)$ when $\Omega = \Upsilon$, respectively.

In this section we will show reduced versions of Theorem 4 for the cases of $\Omega = -\Upsilon$ and $\Omega = \Upsilon$. We will also point out that both cases are related to periodic solutions of infinite horizon systems.

3.1 *Case of* $\Omega = -\Upsilon$

Noting the regularity of Ξ, the boundary condition in this case is

$$x(0) = x(1).$$

Then we can study periodic solutions (with period 1) of infinite horizon systems governed by (1). The reduced version of Theorem 4 for this case is given as follows:

Corollary 5. Suppose that $\Pi_2 < 0$ and $\Omega = -\Upsilon$. Then the following two statements are equivalent:

(i) (4) holds for all $u \in \mathbf{L}_2[0, 1]$, $u \neq 0$.
(ii) $\mathrm{eig}(H) \cap \mathrm{j}\mathbb{R} = \emptyset$, otherwise $P_i^* \Pi P_i < 0$ for all $i \in \{0, 1, \ldots, N\}$ where N is defined as in Theorem 4 for $\theta = 0$.

This case is closely related to (Kao *et al.*, 2001; Jönsson *et al.*, 2003). Moreover the approach in this paper is also closely related to the Fourier domain analysis in (Jönsson *et al.*, 2003), where they derive a finite dimensional condition for time-varying A, B, and Π under a certain assumption.

3.2 *Case of* $\Omega = \Upsilon$

In this case, the boundary condition is

$$x(0) = -x(1),$$

which is related to periodic signals f with period 2 satisfying

$$f(t) = -f(t+1), \quad f(t) = f(t+2).$$

The reduced version of Theorem 4 for this case is given as follows:

Corollary 6. Suppose that $\Pi_2 < 0$ and $\Omega = \Upsilon$. Then the following two statements are equivalent:

(i) (4) holds for all $u \in \mathbf{L}_2[0, 1]$, $u \neq 0$.
(ii) $\mathrm{eig}(H) \cap \mathrm{j}\mathbb{R} = \emptyset$, otherwise $P_i^* \Pi P_i < 0$ for all $i \in \{0, 1, \ldots, N\}$ where N is defined as in Theorem 4 for $\theta = \pi$.

4. RELATED FEASIBILITY PROBLEM

In this section we consider Problem 2. We here derive a cutting hyperplane generated from an infeasible parameter, with which one can easily construct a concrete cutting plane algorithm to solve Problem 2, as in (Kao *et al.*, 2001; Jönsson *et al.*, 2003).

Let us introduce a partition of $\hat{\Pi}_k$ ($k = 0, 1, \ldots, q$):

$$\hat{\Pi}_k = \hat{\Pi}_k^* = \begin{bmatrix} \hat{\Pi}_{k1} & \hat{\Pi}_{k3} \\ \hat{\Pi}_{k3}^* & \hat{\Pi}_{k2} \end{bmatrix}$$

where $\hat{\Pi}_{k1} \in \mathbb{R}^{n\times n}$, $\hat{\Pi}_{k2} \in \mathbb{R}^{m\times m}$, and $\hat{\Pi}_{k3} \in \mathbb{R}^{n\times m}$. The following theorem provides a cutting hyperplane:

Theorem 7. Given $\check{\lambda} \in \Lambda$ such that

- (5) is violated by $\lambda = \check{\lambda}$ and some $u \in \mathbf{L}_2[0, 1]$, $u \neq 0$, and
- $\Pi_2 < 0$ where $\Pi_2 \in \mathbb{R}^{m\times m}$ is given in (7) for Π defined by
$$\Pi = \hat{\Pi}(\check{\lambda}).$$

The following two statements are equivalent:

(i) There exists a $\lambda \in \Lambda$ such that (5) holds for all $u \in \mathbf{L}_2[0, 1]$, $u \neq 0$.
(ii) There exists a $\lambda \in \Lambda \cap \{\lambda : \alpha + \beta^* \lambda \leq 0\}$ such that (5) holds for all $u \in \mathbf{L}_2[0, 1]$, $u \neq 0$, where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^q$ are defined as follows:

Step 1: Fix $\theta \in (-\pi, \pi]$ as in Theorem 4.

Step 2: Define $\hat{M}_k$ by

$$\hat{M}_k := R^* \begin{bmatrix} \hat{Q}_k & (\mathrm{e}^{j\theta} I - \mathrm{e}^A)^* \\ \mathrm{e}^{j\theta} I - \mathrm{e}^A & 0 \end{bmatrix} R$$

where

$$\hat{Q}_k := \int_0^1 \mathrm{e}^{A^* t} \hat{\Pi}_{k1} \mathrm{e}^{At}\, \mathrm{d}t.$$

Define also $\hat{\mho}_{k\infty}$ and $\hat{\Gamma}_{k\infty}$ by

$$\hat{\mho}_{k\infty} := \acute{\mho}_{k\infty} + \grave{\mho}_{k\infty} + \grave{\mho}_{k\infty}^* + \breve{\mho}_{k\infty},$$
$$\hat{\Gamma}_{k\infty} := W_\infty + \acute{\mho}_{k\infty} + \grave{\mho}_{k\infty}$$

respectively, where W_∞ is defined in Theorem 4 and

$$\acute{\mho}_{k\infty} := \begin{bmatrix} 0_n & 0 \\ 0 & -(\mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^A)^{-*}(\hat{Q}_k - Q)(\mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^A)^{-1} \end{bmatrix},$$

$$\grave{\mho}_{k\infty} := \frac{1}{2}\begin{bmatrix} 0_{n,2n} \\ \grave{C}\left((\mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^{\grave{A}_k})^{-1}(\mathrm{e}^{\mathrm{j}\theta} I + \mathrm{e}^{\grave{A}_k})\right)\grave{B} \end{bmatrix},$$

$$\breve{\mho}_{k\infty} := \frac{1}{2}\breve{C}\left((\mathrm{e}^{\mathrm{j}\theta} I - \mathrm{e}^{\breve{A}_k})^{-1}(\mathrm{e}^{\mathrm{j}\theta} I + \mathrm{e}^{\breve{A}_k})\right)\breve{B},$$

$$\left[\begin{array}{cc} \grave{A}_k & \grave{B} \\ \grave{C} & * \end{array}\right] := \left[\begin{array}{cc|c} -A^* & F_k & 0 \\ 0 & H & I_{2n} \\ \hline I_n & 0 & \end{array}\right],$$

$$\left[\begin{array}{cc} \breve{A}_k & \breve{B} \\ \breve{C} & * \end{array}\right] := \left[\begin{array}{cc|c} -H^* & E^*(\hat{\Pi} - \Pi)E & 0 \\ 0 & H & I_{2n} \\ \hline -I_{2n} & 0 & \end{array}\right],$$

$$E := \begin{bmatrix} 0 & I_n \\ \Pi_2^{-1}B^* & \Pi_2^{-1}\Pi_3^* \end{bmatrix},$$

$$F_k := \begin{bmatrix} \hat{\Pi}_{k1} - \Pi_1 & \hat{\Pi}_{k3} - \Pi_3 \end{bmatrix} E.$$

Step 3: Case 1) $\mathrm{eig}(H) \cap \mathrm{j}\mathbb{R} \neq \emptyset$: Define $\hat{\Phi}_k$ by

$$\hat{\Phi}_k := \begin{bmatrix} \hat{K}_k & 0 \\ 0 & \mho_{k(N+1)} \end{bmatrix} + \begin{bmatrix} \hat{L}_k^* \\ \Gamma_{k(N+1)}^* \end{bmatrix} \hat{M}_k \begin{bmatrix} \hat{L}_k & \Gamma_{k(N+1)} \end{bmatrix}$$

where

$$\hat{K}_k := \begin{bmatrix} P_0^*\hat{\Pi}_k P_0 & & 0 \\ & \ddots & \\ 0 & & P_N^*\hat{\Pi}_k P_N \end{bmatrix},$$

$$\hat{L}_k := \begin{bmatrix} \hat{S}_{k0} & \cdots & \hat{S}_{kN} \end{bmatrix},$$

$$\hat{S}_{ki} := \begin{bmatrix} -(\mathrm{j}\omega_i I - A)^{-1}B \\ (\mathrm{j}\omega_i I - A)^{-*}\begin{bmatrix} \hat{\Pi}_{k1} & \hat{\Pi}_{k3} \end{bmatrix} P_i \end{bmatrix},$$

$$\mho_{k(N+1)} := V_{N+1}^\dagger \left(\hat{\mho}_{k\infty} - \sum_{i=0}^{N} \bar{\mho}_{ki}\right)(V_{N+1}^*)^\dagger - I,$$

$$\Gamma_{k(N+1)} := \left(\hat{\Gamma}_{k\infty} - \sum_{i=0}^{N} \bar{\Gamma}_{ki}\right)(V_{N+1}^*)^\dagger$$

P_i, ω_i, N, and V_{N+1} are defined in Theorem 4, and

$$\bar{\mho}_{ki} := \acute{\mho}_{ki} + \grave{\mho}_{ki} + \grave{\mho}_{ki}^* + \breve{\mho}_{ki},$$

$$\bar{\Gamma}_{ki} := \bar{W}_i + \acute{\mho}_{ki} + \grave{\mho}_{ki},$$

$$\acute{\mho}_{ki} := \begin{bmatrix} 0_n & 0 \\ 0 & (j\omega_i I - A)^{-*}(\hat{\Pi}_{k1} - \Pi_1)(j\omega_i I - A)^{-1} \end{bmatrix},$$

$$\grave{\mho}_{ki} := \begin{bmatrix} 0_{n,2n} \\ -(j\omega_i I - A)^{-*}F_k(j\omega_i I - H)^{-1} \end{bmatrix}.$$

$$\breve{\mho}_{ki} := (\mathrm{j}\omega_i I - H)^{-*}E^*(\hat{\Pi} - \Pi)E(\mathrm{j}\omega_i I - H)^{-1}.$$

Case 2) $\mathrm{eig}(H) \cap \mathrm{j}\mathbb{R} = \emptyset$: Define $\hat{\Phi}_k$ by

$$\hat{\Phi}_k := \mho_{k\infty} - I + \Gamma_{k\infty}^* \hat{M}_k \Gamma_{k\infty},$$

$$\mho_{k\infty} := V_0^\dagger \hat{\mho}_{k\infty}(V_0^*)^\dagger, \quad \Gamma_{k\infty} := \hat{\Gamma}_\infty (V_0^*)^\dagger.$$

where V_0 is defined in Theorem 4.

Step 4: α and β are given by

$$\alpha := p^*(\hat{\Phi}_0 - \Phi)p, \quad \beta_k := p^*\hat{\Phi}_k p$$

where Φ is defined in Theorem 4, and p is a vector satisfying

$$p^*\Phi p \geq 0.$$

The proof is omitted for the paper brevity.

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Appendix A. PROOF OF PROPOSITION 3

Define an operator G on $\mathbf{L}_2[0, 1]$ by

$$G : u \mapsto \begin{bmatrix} x \\ u \end{bmatrix}$$

where x is governed by (1) and (2). Consider the unitary operator Ψ: $\mathbf{L}_2[0, 1] \to \ell_2$ mapping $f \mapsto \{\varphi_i\}_{i=0}^\infty$ defined by

$$\varphi_i := \int_0^1 \mathrm{e}^{-\mathrm{j}\omega_i t} f(t)\,\mathrm{d}t$$

which is a key tool in (Dullerud, 1999). Identifying the matrix Π and the corresponding multiplication operator on $\mathbf{L}_2[0, 1]$, we have the following lemma (Fujioka, 2004):

Lemma 8. Assume that $\mathrm{e}^{\mathrm{j}\theta} \notin \mathrm{eig}(\mathrm{e}^A)$. The (k, ℓ)-th block of the matrix expression of $\Psi G^*\Pi G\Psi^*$ is given by

$$\delta_{k\ell} P_k^*\Pi P_\ell + S_k^* M S_\ell.$$

where P_k, S_i and M are defined in (9), (10), and (8), respectively.

The proof completes by noting that

$$\lim_{i\to\infty}(P_i^*\Pi P_i + S_i^* M S_i) = \Pi_2.$$

Appendix B. PROOF OF THEOREM 4

By using G and Ψ defined in Appendix A, the purpose of Problem 1 is to check whether

$$G^*\Pi G < 0$$

holds or not.

Suppose that we have a unitary operator $U\colon \mathbf{L}_2[0,1] \to \mathbb{R}^n \oplus X$ for a Hilbert space X such that $UG^*\Pi GU^*$ is expressed as the sum of a block-diagonal and a finite rank operators:

$$\begin{bmatrix} K_0 & 0 \\ 0 & \mathscr{K} \end{bmatrix} + \begin{bmatrix} L_0^* \\ \mathscr{L}^* \end{bmatrix} M_0 \begin{bmatrix} L_0 & \mathscr{L} \end{bmatrix}$$

where $K_0\colon \mathbb{R}^{\tilde n} \to \mathbb{R}^{\tilde n}$, $\mathscr{K}\colon X \to X$, $M_0\colon \mathbb{R}^{\tilde m} \to \mathbb{R}^{\tilde m}$, $L_0\colon \mathbb{R}^{\tilde n} \to \mathbb{R}^{\tilde m}$, $\mathscr{L}\colon X \to \mathbb{R}^{\tilde m}$, and furthermore $\mathscr{K} < 0$ holds. Then $G^*\Pi G < 0$ is equivalent to

$$\begin{bmatrix} K_0 & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} L_0^* \\ \mathscr{V}^* \end{bmatrix} M_0 \begin{bmatrix} L_0 \\ \mathscr{V} \end{bmatrix} < 0$$

where $\mathscr{V} := \mathscr{L}(-\mathscr{K})^{-\frac{1}{2}}$. This turns to

$$I - \begin{bmatrix} I & 0 \\ 0 & \mathscr{V}^* \end{bmatrix} \Theta \begin{bmatrix} I & 0 \\ 0 & \mathscr{V} \end{bmatrix} > 0,$$

$$\Theta := \begin{bmatrix} I+K_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_0^* \\ I \end{bmatrix} M_0 \begin{bmatrix} L_0 & I \end{bmatrix}.$$

We then have an equivalent condition:

$$\rho\left(\Theta \begin{bmatrix} I & 0 \\ 0 & \tilde W \end{bmatrix}\right) < 1 \qquad \text{(B.1)}$$

where $\tilde W := \mathscr{V}\mathscr{V}^* = \mathscr{L}(-\mathscr{K})^{-1}\mathscr{L}^*$.

(B.1) is a finite dimensional condition since $\tilde W\colon \mathbb{R}^{\tilde m} \to \mathbb{R}^{\tilde m}$. With a (matrix) factorization of $\tilde W = VV^*$, (B.1) turns to

$$I - \left(\begin{bmatrix} I+K_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_0^* \\ V^* \end{bmatrix} M_0 \begin{bmatrix} L_0 & V \end{bmatrix}\right) > 0,$$

and hence

$$\begin{bmatrix} K_0 & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} L_0^* \\ V^* \end{bmatrix} M_0 \begin{bmatrix} L_0 & V \end{bmatrix} < 0.$$

The rest of the proof is a derivation of concrete formulas for K_0, L_0, M_0 and V, which is similar to that in (Fujioka, 2004), so it is omitted.

Appendix C. PROOF OF THEOREM 7

Let (5) be violated by $u = u_0$ when $\lambda = \check\lambda$. Then α and β are given by

$$\alpha = \sigma_{\hat\Pi_0}(u_0) - \sigma_\Pi(u_0), \quad \beta_k = \sigma_{\hat\Pi_k}(u_0)$$

since (5) is affine in λ and

$$\sigma_{\hat\Pi_0}(u_0) + \beta^*\lambda = \sigma_\Pi(u_0) \geq 0$$

where $\sigma_\Pi\colon \mathbf{L}_2[0,1] \to \mathbb{R}$ is defined by

$$\sigma_\Pi(u) := \int_0^1 \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \Pi \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t$$

and x is determined by (1) and (2). Hence our task here is to characterize u_0 and to derive formulas for $\sigma_{\hat\Pi_k}(u_0)$ and $\sigma_\Pi(u_0)$.

With symbols used in Appendix B, we have

$$\sigma_\Pi\left(\Psi^{-1}\begin{bmatrix} I & 0 \\ 0 & \mathscr{C} \end{bmatrix} p\right) = p^*\Phi p$$

for any compatible vector p by using the following facts:

$$-I = \mathscr{C}^*\mathscr{K}\mathscr{C}, \quad V = \mathscr{L}\mathscr{C}$$

where

$$\mathscr{C} := -\mathscr{K}^{-1}\mathscr{L}^*(V^*)^\dagger.$$

Hence we can characterize u_0 by

$$u_0 = \Psi^{-1}\begin{bmatrix} I & 0 \\ 0 & \mathscr{C} \end{bmatrix} p$$

by taking p as a vector satisfying $p^*\Phi p \geq 0$. Note that such a vector p exits due to Theorem 4.

We have already seen that $\sigma_\Pi(u_0)$ is given by $p^*\Phi p$. Hence we derive a computational formula for $\sigma_{\hat\Pi_k}(u_0)$ in the sequel. For the purpose we compute

$$\tilde\mho_k := \mathscr{L}\mathscr{K}^{-1}\hat{\mathscr{K}}_k\mathscr{K}^{-1}\mathscr{L}^*$$

and

$$\tilde\Gamma_k := -\hat{\mathscr{L}}_k\mathscr{K}^{-1}\mathscr{L}^*$$

since

$$\sigma_{\hat\Pi_k}\left(\Psi^{-1}\begin{bmatrix} I & 0 \\ 0 & \mathscr{C} \end{bmatrix} p\right)$$
$$= p^*\left(\begin{bmatrix} \hat K_k & 0 \\ 0 & V^\dagger\tilde\mho_k(V^*)^\dagger \end{bmatrix} + \begin{bmatrix} \hat L_k^* \\ V^\dagger\tilde\Gamma_k^* \end{bmatrix} \hat M_k \begin{bmatrix} \hat L_k \\ \tilde\Gamma_k(V^*)^\dagger \end{bmatrix}\right) p$$

where $\hat{\mathscr{K}}_k$ and $\hat{\mathscr{L}}_k$ are respectively defined similarly to $\mathscr{K}$ and $\mathscr{L}$ but replacing Π by $\hat\Pi_k$.

We get

$$\tilde\mho_k = \sum_{i=i_0}^{\infty} S_i(P_i^*\Pi P_i)^{-1}(P_i^*\hat\Pi_k P_i)(P_i^*\Pi P_i)^{-1}S_i^*,$$

$$\tilde\Gamma_k = -\sum_{i=i_0}^{\infty} \hat S_{ki}(P_i^*\Pi P_i)^{-1}S_i^*.$$

where i_0 is determined as in Appendix B.

We compute $\tilde{\mho}_k$ first. It is readily to see that

$$\tilde{\mho}_k = \sum_{i=i_0}^{\infty} S_i(P_i^*\Pi P_i)^{-1}(P_i^*(\hat{\Pi}_k-\Pi)P_i)(P_i^*\Pi P_i)^{-1}S_i^* \\ -\tilde{W}.$$

Noting that

$$\begin{aligned}&(P_i^*\Pi P_i)^{-1}S_i^* \\ &= \left(-\Pi_2^{-1}\tilde{C}(\mathrm{j}\omega_i I-H)^{-1}\tilde{B}\Pi_2^{-1}+\Pi_2^{-1}\right)\tilde{C}(\mathrm{j}\omega_i I-\tilde{A})^{-1} \\ &= \Pi_2^{-1}\tilde{C}\left(-(\mathrm{j}\omega_i I-H)^{-1}\tilde{B}\Pi_2^{-1}\tilde{C}+I\right)(\mathrm{j}\omega_i I-A)^{-1} \\ &= \Pi_2^{-1}\tilde{C}(\mathrm{j}\omega_i I-H)^{-1}(\mathrm{j}\omega_i I-A)(\mathrm{j}\omega_i I-A)^{-1} \\ &= \Pi_2^{-1}\tilde{C}(\mathrm{j}\omega_i I-H)^{-1},\end{aligned}$$

we have

$$\begin{aligned}&P_i(P_i^*\Pi P_i)^{-1}S_i^* \\ &= \begin{bmatrix}(\mathrm{j}\omega_i I-A)^{-1}B \\ I\end{bmatrix}\Pi_2^{-1}\tilde{C}(\mathrm{j}\omega_i I-H)^{-1} \\ &= \begin{bmatrix}I_n & 0 \\ 0 & \Pi_2^{-1}\tilde{C}\end{bmatrix}\left(\mathrm{j}\omega_i I-\begin{bmatrix}A & B\Pi_2^{-1}\tilde{C} \\ 0 & H\end{bmatrix}\right)^{-1}\begin{bmatrix}0 \\ I_{2n}\end{bmatrix} \\ &= \begin{bmatrix}I_n & \begin{bmatrix}0 & I_n\end{bmatrix} \\ 0 & \Pi_2^{-1}\tilde{C}\end{bmatrix}\left(\mathrm{j}\omega_i I-\begin{bmatrix}A & 0 \\ 0 & H\end{bmatrix}\right)^{-1}\begin{bmatrix}\begin{bmatrix}0 & -I_n\end{bmatrix} \\ I_{2n}\end{bmatrix} \\ &= \begin{bmatrix}0 & -(\mathrm{j}\omega_i I-A)^{-1} \\ 0 & 0\end{bmatrix}+E(\mathrm{j}\omega_i I-H)^{-1}. \qquad \text{(C.1)}\end{aligned}$$

Substituting (C.1) we get

$$S_i(P_i^*\Pi P_i)^{-1}(P_i^*(\hat{\Pi}_k-\Pi)P_i)(P_i^*\Pi P_i)^{-1}S_i^* = \bar{\Omega}_{ki}$$

and hence

$$\tilde{\Omega}_k = \sum_{i=0}^{\infty}\bar{\Omega}_{ki}-\sum_{i=0}^{i_0-1}\bar{\Omega}_{ki}-\tilde{W}$$

We also get

$$\sum_{i=0}^{\infty}\acute{\Omega}_{ki}=\acute{\Omega}_{k\infty}, \quad \sum_{i=0}^{\infty}\grave{\Omega}_{ki}=\grave{\Omega}_{k\infty}, \quad \sum_{i=0}^{\infty}\breve{\Omega}_{ki}=\breve{\Omega}_{k\infty}.$$

Consequently we have

$$\sum_{i=0}^{\infty}\bar{\mho}_{ki}=\hat{\mho}_{k\infty}.$$

Next we move to computation of $\tilde{\Gamma}_k$: We have

$$\hat{S}_{ki}=S_i+\begin{bmatrix}0 \\ (\mathrm{j}\omega_i I-A)^{-*}\begin{bmatrix}\hat{\Pi}_{k1}-\Pi_1 & \hat{\Pi}_{k3}-\Pi_3\end{bmatrix}P_i\end{bmatrix}.$$

Noting (C.1), we have

$$\begin{aligned}&\begin{bmatrix}0 \\ (\mathrm{j}\omega_i I-A)^{-*}\begin{bmatrix}\hat{\Pi}_{k1}-\Pi_1 & \hat{\Pi}_{k3}-\Pi_3\end{bmatrix}P_i\end{bmatrix} \\ &\times(P_i^*\Pi P_i)^{-1}S_i^* \\ &= -\acute{\mho}_{ki}-\grave{\mho}_{ki}.\end{aligned}$$

Hence we get

$$\tilde{\Gamma}_k=\sum_{i=i_0}^{\infty}\bar{\Gamma}_{ki}$$

and

$$\hat{\Gamma}_{k\infty}=\sum_{i=0}^{\infty}\bar{\Gamma}_{ki}.$$

This completes the proof of Theorem 7.

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