A ROBUST REPETITIVE CONTROL SCHEME WITH RELAXED MINIMUM TIME CRITERION

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Abstract: A repetitive control scheme is proposed for constrained nonlinear optimal control problems. The lower level algorithm adjusts switching times for bang control arcs and parameters of interval polynomial approximations for interior control arcs. It is based on a linearization of optimal controller and performs reduced optimization with changes of control structure. The upper level finds the optimal control and recalculates the linearization each time the deviation from the optimal solution becomes too large. The linearized controller is analytically derived. The upper level uses the MSE method to determine the reference optimal control structure. Simulation and experimental tests show that the proposed approach yields an optimizing nonlinear controller, able both to ensure close to minimum-time point-to-point transition as well as to stabilize the state.

Keywords: minimum time control, nonlinear systems, robust control.

1. INTRODUCTION

The purpose of this paper is to improve the adaptive control algorithm, first presented in (Korytowski, *et al.*, 2001). It is an extension of the *neighboring optimum feedback* (Bryson Jr., 1999; Pesch, 1989a, b). The controller reacts to state disturbances on two levels. The lower level adjusts control parameters basing on a linearization of optimal controller and performs reduced optimization whereas the upper level detects changes in the optimal control structure and recalculates the linearization each time the deviation of trajectory from the optimal one becomes too large. Both the reduced and full optimization rely on forced changes of control structure, which take place if suitable tests are satisfied. The linearized controller is described by an explicit linear relationship between the measured or computed deviations of state and the corrections of control parameters. It is analytically derived with the use of discontinuous matrix solutions of the canonical variational system. The modifications of the results of (Korytowski, *et al.*, 2001) include:

- introduction of the horizon as a decision variable,
- limitation of amplitude of control variations in the vicinity of the target state,
- improved convergence due to stronger stabilization requirements.

The sequence of boundary/interior arcs determines the *structure* of optimal control. Pesch's approach (Pesch, 1989b), known as the *repeated correction method*, relies on the assumption of fixed control structure. This limitation is overcome using pre-computed neighboring extremals (Pesch, 1989a), at a high computational cost. The approach proposed below applies also to varying control structure. This is achieved by combining the linearized feedback scheme with the *monotonous structure evolution* (MSE) method (Szymkat, *et al.*, 2003), which makes it possible to generate or reduce arcs without considerable computational effort. Comparisons with the repeated correction scheme (Pesch, 1989b) and full repetitive optimization show that the new method gives good disturbance rejection at a low computational cost.

2. OPTIMAL CONTROL PROBLEM

Consider the minimum time problem of steering the state of the system

$$\dot{x}(t) = f(x(t),u(t)) \equiv f^0(x(t)) + f^1(x(t))u(t)$$

$$t \ge t_0\,,\quad x(t_0) = x_0\,,\quad x(t) \in \boldsymbol{R}^n\,,\quad u(t) \in \boldsymbol{R} \qquad (1)$$

to an ε-neighborhood of a given state x^f **(** $\varepsilon > 0$ **)**

$$(x(T) - x^f)^\top (x(T) - x^f) \le \varepsilon\,. \qquad (2)$$

$T \ge 0$ is the control horizon. The functions f^0 and f^1 are twice continuously differentiable. The set of admissible controls U consists of all right-continuous functions $u : [t_0, \infty[\ \to [-1, 1]$. Define the Hamiltonian $H(x,\psi,u) = \psi^\top f(x,u)$ where the adjoint variable $\psi(t) \in \boldsymbol{R}^n$ satisfies

$$\dot{\psi} = -H_x(x,\psi,u), \quad t \in [0,T] \qquad (3)$$

$$\psi(T) = x^f - x(T)\,. \qquad (4)$$

Let

$$g(x,\psi) = H_u(x,\psi,u) = \psi^\top f^1(x)\,. \qquad (5)$$

The respective switching function is defined as

$$\phi(t) = g(x(t),\psi(t))\,. \qquad (6)$$

Its projection onto U at an admissible point u is given by

$$\phi^U(t) = \begin{cases} 0, & u(t)\,\mathrm{sgn}\,\phi(t) = 1 \\ \phi(t), & \text{otherwise.} \end{cases} \qquad (7)$$

By the Maximum Principle, ϕ^U is identically zero on an optimal control. The optimal control satisfies

$$u = v(x,\psi) = \mathrm{sgn}\, g(x,\psi),\ \ g(x,\psi) \ne 0\,. \qquad (8)$$

The switching function ϕ is continuously differentiable (Korytowski, *et al.*, 2001). We assume that ϕ takes zero value at most at a finite number of points (switching times) $\tau_1, \ldots, \tau_m$, $t_0 < \tau_1 < \ldots < \tau_m < T$, and its derivative is different from zero at every point τ_i, $i = 1,\ldots,m$.

The canonical system of equations is obtained by substituting the control (8) in (1) and (3)

$$\dot{X} = F(X, v(X))\,,\quad t \in [0,T]\,,\ X = \mathrm{col}(x,\psi)\,,$$

$$F(X,u) = \mathrm{col}(f(x,u), -H_x(x,\psi,u))$$

with the boundary conditions as in (1), (4).

The variational equation for the canonical system, with jump conditions at switching moments was given in (Lastman, 1978)

$$\delta\dot{X}(t) = J(t)\,\delta X(t) \qquad (9)$$

$$J(t) = \nabla_X F(X(t),u(t))^\top = \begin{bmatrix} A(t) & 0 \\ B(t) & -A(t)^\top \end{bmatrix} \qquad (10)$$

where $t \in]t_0, T[\setminus \{\tau_1, \tau_2, \ldots, \tau_m\}$,

$$A(t) = \nabla_x f(x(t),u(t))^\top$$

$$B(t) = -\nabla^2_{xx} H(x(t),\psi(t),u(t))\,. \qquad (11)$$

The terminal condition results from (4)

$$\begin{bmatrix} I & I \end{bmatrix} \delta X(T) = 0\,. \qquad (12)$$

δX is in general discontinuous at $\tau_1, \tau_2, \ldots, \tau_m$

$$\delta X(\tau_i \pm) = Z_{i\mp}\delta X(\tau_i \mp)\,,\quad i = 1,2,\ldots,m \qquad (13)$$

$$Z_{i\mp} = I \pm \frac{\Delta F_i\, \nabla g(X(\tau_i))^\top}{\psi(\tau_i)^\top [f^0, f^1](x(\tau_i))}$$
$$\Delta F_i = F(X(\tau_i), u(\tau_i -)) - F(X(\tau_i), u(\tau_i +)) \qquad (14)$$

where $[f^0, f^1]$ is the Lie bracket. Obviously $Z_{i-} = (Z_{i+})^{-1}$. The dependence of the switching time variation $\delta\tau_i$ on δX is determined from the identity $g(X(\tau_i)) \equiv 0$, see (Korytowski, *et al.*, 2001)

$$\delta\tau_i = \frac{\nabla g(X(\tau_i))^\top \delta X(\tau_i \pm)}{\psi(\tau_i)^\top [f^0, f^1](x(\tau_i))}\,. \qquad (15)$$

Note that the function $t \mapsto \nabla g(X(t))^\top \delta X(t)$ is continuous at every switching time.

Define a $2n \times n$ matrix solution V of the variational equation, satisfying

$$\dot{V}(t) = J(t)\,V(t)\,,\ t \in]0,T[\setminus \{\tau_1, \tau_2, \ldots, \tau_m\} \qquad (16)$$

$$V(T) = \mathrm{col}(I, -I)\,. \qquad (17)$$

The jumps at the switching moments are given by

$$V(\tau_i -) = Z_{i+} V(\tau_i +)\,,\quad i = 1,2,\ldots,m \qquad (18)$$

Let $V = \mathrm{col}(V_1, V_2)$ with square matrices V_1 and V_2. Thus for any solution of (9), (12), (13) and every t

$$\delta x(t) = V_1(t)\delta x(T)\,,\ \delta\psi(t) = V_2(t)\delta x(T)\,. \qquad (19)$$

3. LINEARIZED CONTROLLER

The construction of the linearized controller is based on (15) and the relationship between the variations of state and adjoint trajectories

$$\delta\psi(t) = V_2(t) V_1(t)^{-1} \delta x(t)\,. \qquad (20)$$

From now on it is assumed that $V_1(t)$ is nonsingular for every t. This crucial assumption is generically satisfied in practical control problems. The coefficient matrix $K(t) = V_2(t) V_1(t)^{-1}$ is symmetric for every t and differentiable everywhere, except for the switching times. It satisfies a linear Riccati equation (Korytowski, *et al.*, 2001). According to formula (19), the matrix

$$W(t) = V_1(t)^{-1} \qquad (21)$$

represents the *sensitivity* of the terminal state of the optimal solution with respect to the state at t.

Formulas (15) and (20) yield a relationship between the variation of the state trajectory and the variations of the switching times

$$\delta\tau_i = \Lambda_{i\pm}\delta x(\tau_i \pm)\,,\quad i = 1,2,\ldots,m \qquad (22)$$

$$\Lambda_{i\pm} = \frac{\psi(\tau_i)^{\mathsf{T}} \nabla f^1(x(\tau_i))^{\mathsf{T}} + f^1(x(\tau_i))^{\mathsf{T}} K(\tau_i \pm)}{\psi(\tau_i)^{\mathsf{T}} [f^0, f^1](x(\tau_i))} .$$

Suppose that the state increment δx satisfies the variational equation in the interval $[t,T]$ for some $t \in [t_0,T[$ and is the result of a perturbation of the state at t by a known value $\delta x(t)$. The values of $\delta x(\tau_i \pm)$ can be computed in advance by solving equations (9), (12), (13), and the respective corrections $\delta\tau_i$ of the switching times $\tau_i > t$ can be applied during the control process, provided

$$\tau_i + \delta\tau_i < \tau_{i+1} + \delta\tau_{i+1} \quad \text{if} \quad \tau_i + \delta\tau_i < t \tag{23}$$

for $i \in \{0,1,...,m\}$. By definition $\tau_{m+1} + \delta\tau_{m+1} = T$, $\tau_0 + \delta\tau_0 = t_0$. If $\tau_i > t$ and $\tau_i + \delta\tau_i < t$, the control at t should change sign. To avoid too frequent switchings, the intervals between the time moments t at which the control in $[t, T]$ is corrected, have a fixed length Δt.

From (19), $\delta x(\tau_i \pm) = V_1(\tau_i \pm) V_1(t)^{-1} \delta x(t)$. Putting this into (22) and using (21) we obtain the general form of the linearized switching controller

$$\delta\tau_i = \Pi_i W(t) \delta x(t) , \quad i = 1,2,...,m \tag{24}$$

$$\Pi_i = \frac{\nabla g(X(\tau_i))^{\mathsf{T}} V(\tau_i \pm)}{\psi(\tau_i)^{\mathsf{T}} [f^0, f^1](x(\tau_i))}$$

$$\nabla g(X(\tau_i))^{\mathsf{T}} V(\tau_i \pm) = \psi(\tau_i)^{\mathsf{T}} \nabla f^1(x(\tau_i))^{\mathsf{T}} V_1(\tau_i \pm) + f^1(x(\tau_i))^{\mathsf{T}} V_2(\tau_i \pm) .$$

The value of Π_i does not depend on which limit is taken in the right-hand side of (24). The variation of the horizon due to a variation of state at the time t is obtained in a similar way

$$\delta T = - \frac{(x(T) - x^f)^{\mathsf{T}} V_1(t)^{-1}}{(x(T) - x^f)^{\mathsf{T}} f(x(T), u(T))} \delta x(t) .$$

If $\delta x(t)$ is known at $t \in]\tau_{i-1} + \delta\tau_{i-1}, \tau_i + \delta\tau_i[$, for some i, $1 \le i \le m$, the correction $\delta\tau_i$ can be computed in another way. Define the continuous $n \times n$ matrix solution of the equation $\partial\Phi(t,s)/\partial t = A(t)\Phi(t,s)$, $\Phi(s,s) = I$, for every t,s in $[0,T]$. The equality $\delta\tau_i = \Lambda_{i\pm}\Phi(\tau_i, t)\delta x(t)$ is then equivalent to (24), where Λ_{i-} is taken for $t < \tau_i$ and Λ_{i+} for $t > \tau_i$. The decision about the value of the correction $\delta\tau_i$ should be taken as late as possible. This critical, last moment t fulfils $t - \tau_i = \Lambda_{i\pm}\Phi(\tau_i, t)\delta x(t) = \Pi_i W(t) \delta x(t)$.

4. BASIC REPETITIVE SCHEME

The overall repetitive computational scheme has two levels. On the lower level the linearized controller is applied, combined with reduced optimization. In each time step it additionally calculates two quantities for the remaining part of the control time interval. These are the norm of the projection $\| \phi^U \|$ in the control space, see (7), and the expected value of the auxiliary cost functional $\frac{1}{2}(x(T) - x^f)^{\mathsf{T}}(x(T) - x^f)$. When both of them exceed some predetermined thresholds, the upper level algorithm is activated. On the upper level, the MSE method (Szymkat, *et al.*, 2003) is adopted. This dynamic optimization algorithm in the variant applicable here uses switching times as decision variables. It automatically adjusts the control structure by generating and reducing switchings. The upper level algorithm is continued until the projection norm decreases below another threshold. If the auxiliary cost is not below its threshold value at that time, the MSE algorithm has to be reinitialized. After a successful completion of the upper level computations, the linearized controller is recalculated and the lower level algorithm restarted. The distinctive feature of this adaptive scheme is that the control structure is adapted in the course of the control process. This distinguishes the approach described here from the repeated correction method of (Pesch, 1989a, b).

Recall that the derivative of the performance index with respect to a switching time τ_i is equal to (Szymkat, *et al.*, 2003)

$$\nabla_{\tau_i} S = 2\phi(\tau_i) u(\tau_i) . \tag{25}$$

The repetitive control scheme consists of the following steps.

1^0 Set $t_0 := 0$.

2^0 Find optimal solution using MSE started with current control approximation, i.e. determine horizon T, reference control u and switching times τ_i, $i = 1,...,m$; calculate matrix V, and vectors Π_i for all switching times.

3^0 Choose time step $\Delta t \le T - t_0$ (*update interval*), execute control u in $[t_0, t_0 + \Delta t]$, and substitute $t_0 := t_0 + \Delta t$.

4^0 Determine state deviation $\Delta x(t_0)$. Calculate ΔT and $\Delta\tau_i = \Pi_i V_1(t_0)^{-1} \Delta x(t_0)$ for all $\tau_i > t_0$. Set corrected values $\tau_i := \tau_i + \Delta\tau_i$, $T := T + \Delta T$ and thus determine new control u.

5^0 Update initial state. Compute the norm of ϕ^U in control space (7), and the expected value of the auxiliary cost functional $\frac{1}{2}(x(T) - x^f)^{\mathsf{T}}(x(T) - x^f)$. If thresholds for both are exceeded, return to 2^0. Otherwise go to 6^0.

6^0 If $\phi(t_0) u(t_0) < 0$, i.e., the derivative (25) would be negative, add a control switching at t_0 and perform reduced (fixed structure) optimization. Stop when a switching time hits the boundary of the admissible set or the gradient norm termination conditions are met. Return to 3^0. ■

The introduction of an additional switching time at t_0 (step 6^0) does not initially change the control, but creates the possibility of improving the value of performance index by moving this switching time to the right. For a more detailed treatment of MSE including control switching generations and reductions, see (Szymkat, *et al.*, 2003). The restraining of optimization in step 6^0 to fixed structure (with the additional initial switching) yields relatively good results at low computational cost.

The area of application of the algorithm can be extended to cases where conditions (23) are not fulfilled in $[t_0, T]$ (step 4^0). Denote $i_0 = \min\{i : \tau_i > t_0\}$. For s increasing from 0 to 1, successively remove every switching for which the respective value of $\tau_i'(s) = \tau_i + s\Delta\tau_i$, $i_0 \le i \le m$, $T'(s) = T + s\Delta T$ hits the boundary of the admissible set. Every time the constraint $\tau_i'(s) > t_0$ is hit, the control initial value $u(t_0)$ changes its sign.

5. EXAMPLE

We show the application of the repetitive optimizing scheme to the well-known benchmark problem of steering a pendulum hinged on a cart, which is a strongly nonlinear fourth order system. Denote the cart position by x_1, its velocity by x_3, the angle between the upward direction and the pendulum by x_2, and the angular velocity of the pendulum by x_4. Put $x = \mathrm{col}(x_1,...,x_4)$, $f = \mathrm{col}(f_1,..., f_4)$, $s = \sin x_2$, $c = \cos x_2$, $S = \sin 2x_2$, $C = \cos 2x_2$, $\theta = \tanh 10x_3$, $w_1 = k_1x_3 + k_2u - k_3\theta - l x_4^2 s$, $w_2 = k_4 s + k_5 x_4$, $D = (1-ec)^{-1}$. Then

$$f_1(x,u) = x_3, \qquad f_2(x,u) = x_4$$

$$f_3(x,u) = D(w_1 + l w_2 c), \; f_4(x,u) = D(a w_1 c + w_2)$$

with $k_1 = -1.0785$, $k_2 = 6.6046$, $k_3 = 0.98794$, $k_4 = 22.432$, $k_5 = -0.057389$, $l = 0.043715$, $e = 0.099961$, $a = 2.2866$.

Consider the optimization problem of section 2 with $x_0 = \mathrm{col}(0, \pi, 0, 0)$ and open-loop unstable target state $x^f = 0$. Assume $\varepsilon = 5\cdot 10^{-6}$. By (8), the optimal control satisfies $u(t) = \mathrm{sgn}\,\phi(t)$ where $\phi = k_2 D(\psi_3 + ac\psi_4)$. The adjoint equation has the form $\dot\psi = -A^{\mathrm{T}}\psi$ where A defined in (11) has the following nonzero elements

$$A_{13} = A_{24} = 1,$$

$$A_{32} = lD(-x_4^2 c + k_4 C - k_5 x_4 s - a f_3 S)$$

$$A_{33} = D(k_1 - 10k_3(1-\theta^2))$$

$$A_{34} = lD(-2x_4 s + k_5 c)$$

$$A_{42} = D(-e x_4^2 c^2 - a w_1 s + k_4 c - e f_4 S)$$

$$A_{43} = ac A_{33}, \quad A_{44} = D(k_5 - alx_4 S).$$

The performance of the repetitive control scheme is evaluated in a series of simulation experiments, with modeling of time discretization of control (with constant hold intervals), and stochastic state disturbances. The length of the update interval in step 3^0 of the algorithm is constant, equal to 0.1. Gaussian state disturbances with zero mean generated by the MATLAB expression `p*[4,2,4,2].*randn(1,4)` are added to the current state at every time moment of update. Typical trajectories and controls are shown in Figures 1 and 2. For $p \le 0.009$ the periods of effective stabilization are long (Fig. 1 shows the results for $p = 0.009$), and for greater p, the probability of failure rapidly increases (see Fig. 2 where $p = 0.01$).

Observe that the process of Fig. 1 can be divided into two stages: swinging the pendulum up, and stabilizing it in the upper position. Although the use of bang-bang controls is fully justified in the first stage and guarantees a close to minimum time transition to the neighborhood of the unstable upper position, it may seem purposeful to seek a more relaxed control behavior in the stabilization stage, resulting in smaller oscillations. This will be realized by adding an integral of squared control to the auxiliary cost functional, with a weight factor growing as the distance to the target state decreases.

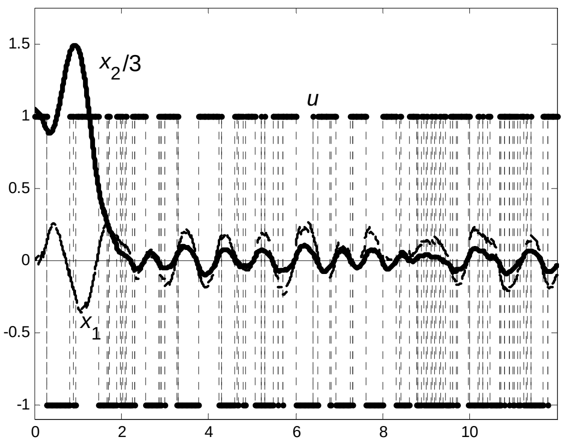

Fig. 1. Example of effective stabilization.

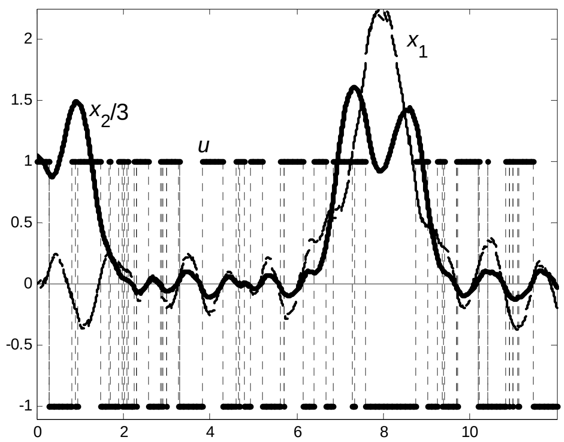

Fig. 2. Example of failure.

In most analyzed simulation runs and real-time experiments the above proposed control scheme ensures stabilization of the system in a vicinity of the open-loop unstable equilibrium. However, in some cases we observe large oscillations that cannot be solely explained by the disturbances. A more detailed analysis leads to the disclosure of a "trap phenomenon". It consists in the failure of the MSE algorithm employed in step 2^0 to properly identify the true minimizer of the auxiliary cost functional, due to the presence of a competing solution with a low value of the criterion for the given horizon T. Independently of control, this solution departs from the vicinity of target state shortly after T. In order to avoid such candidate solutions in the course of monotonous MSE search, a "tail term" in the form of an integral over an additional interval $[T, T+T_1]$ will be introduced.

To give some hints how to handle the trap phenomenon, consider the situation when a new starting point for the MSE algorithm has to be generated. This occurs, e.g., after a step with the optimal horizon smaller than the update interval. The proposed new value of the horizon has to be sufficiently large, to avoid local minima at which the target is missed. An appropriate increase of the penalty coefficient also proves helpful. However, such measures are problem specific, and not always efficient. A more general solution is suggested below.

6. ROBUSTIFIED REPETITIVE SCHEME

The auxiliary optimal control problem consists now in the minimization of the following criterion functional on the trajectories of (1)

$$S_\rho(u,T) = T + \tfrac{1}{2}\alpha(x_0)\int_0^T u(t)^2\,\mathrm{d}t + + \tfrac{1}{2}\rho\left(\|x(T)-x^f\|^2 + \int_T^{T+T_1} \|x(t)-x^f\|^2\,\mathrm{d}t \right). \quad (26)$$

The decision variables, that is, the control u and horizon T are subject to constraints: $T \ge t_0$,

$$|u(t)| \le 1 \text{ for } t \le T, \quad u(t) = 0 \text{ for } t > T. \quad (27)$$

The constant T_1 is nonnegative and ρ is positive. The weight factor $\alpha(x_0)$ monotonously decreases as x_0 departs from the target state, starting from a positive value. For $\|x_0 - x^f\|$ greater than a certain threshold value, it is identically zero. Such a construction guarantees appropriate regularity of the structure evolution for the transition from the point-to-point to the stabilizing feedback type control. The hamiltonian for the basic optimal control problem is as follows

$$H = \psi(t)^\mathsf{T} f(x(t),u(t)) - -\tfrac{1}{2}\alpha u(t)^2 - \tfrac{1}{2}\rho\,\sigma_T(t)\,\|x(t)-x^f\|^2 \quad (28)$$

where $\sigma_T(t) = 0$ for $t \le T$, and $\sigma_T(t) = 1$ for $t > T$.

The adjoint variable ψ satisfies the adjoint equation

$$\dot{\psi} = -f_x(x,u)\psi + \rho\,\sigma_T(x - x^f) \quad (29)$$

with a terminal condition $\psi(T+T_1) = 0$ and a jump

$$\psi(T_-) = \psi(T_+) + \rho(x(T) - x^f).$$

If $\alpha(x_0) = 0$, the extremal control, i.e., the control that maximizes the hamiltonian (28) subject to (27) is given by

$$u(t) = \mathrm{sgn}(\psi(t)^\mathsf{T} f^1(x(t)), \quad t \le T. \quad (30)$$

If $\alpha(x_0) > 0$, the extremal control satisfies

$$u(t) = \mathrm{sat}(\alpha^{-1}\psi(t)^\mathsf{T} f^1(x(t)), \quad t \le T. \quad (31)$$

The sat function is defined by

$$\mathrm{sat}(\xi) = \begin{cases} \xi, & |\xi| \le 1 \\ \mathrm{sgn}\,\xi, & \text{otherwise.} \end{cases} \quad (32)$$

The idea of the robustified repetitive scheme of section 4 remains generally unchanged with the following modifications. In step 2^0, each time the MSE procedure is recalled the value of $\alpha(x_0)$ is updated. If the current value is zero the rest of the algorithm is executed without any essential change for bang-bang type controls. If $\alpha(x_0)$ is nonzero the control is continuous for $t < T$, and may have both boundary and interior (non-saturated) arcs. Its first derivative may be discontinuous only at the ends of the boundary arcs. We assume that approximations of optimal control also have these properties. Let $t_0 = \sigma_0 < \sigma_1 < \ldots < \sigma_N = T$ be end points of subsequent control approximation arcs. Some σ_i coincide with τ_k dividing boundary and interior arcs. Let in every interior arc

$$u(t) = p_i^\mathsf{T} w(t, \sigma_{i-1}, \sigma_i) \quad (33)$$

where p_i is a vector of parameters and w, a vector of Hermite cubic polynomials

$$w_1(t,\sigma_{i-1},\sigma_i) = w_3(t,\sigma_i,\sigma_{i-1}) = (t-\sigma_i)^2(2t+\sigma_i-3\sigma_{i-1})/(\sigma_i-\sigma_{i-1})^3$$

$$w_2(t,\sigma_{i-1},\sigma_i) = w_4(t,\sigma_i,\sigma_{i-1}) = (t-\sigma_i)^2(t-\sigma_{i-1})/(\sigma_i-\sigma_{i-1})^2.$$

Thus $u(\sigma_{i-1}) = p_{i1}$, $\dot{u}(\sigma_{i-1}+) = p_{i2}$, $u(\sigma_i) = p_{i3}$, $\dot{u}(\sigma_i -) = p_{i4}$. To ensure continuity at division points and smoothness between neighboring interior arcs, some parameters p_{ik} are fixed or made identical.

Let Σ denote the performance index as a function of the parameters, division points and horizon. Its derivative w.r.t. p_{ik} reads

$$\nabla_{p_{ik}}\Sigma = -\int_{\Omega_{ik}} \nabla_u H(\psi,x,u)\nabla_{p_{ik}} u\,\mathrm{d}t \quad (34)$$

where the derivatives of u are determined by (33), and Ω_{ik} is the union of $[\sigma_{i-1},\sigma_i]$ and, possibly, one of its neighboring interior intervals. The derivative w.r.t. $\sigma_i \neq \sigma_N$, being the right-hand end of an interior interval is given by

$$\nabla_{\sigma_i}\Sigma = -\dot{u}(\sigma_i-)\nabla_{p_{i3}}\Sigma - \ddot{u}(\sigma_i-)\nabla^-_{p_{i4}}\Sigma - \ddot{u}(\sigma_i+)\nabla^+_{p_{i4}}\Sigma$$

where $\nabla^-_{p_{i4}}\Sigma$ and $\nabla^+_{p_{i4}}\Sigma$ are computed according to (34), but with Ω_{ik} equal to $[\sigma_{i-1},\sigma_i]$ and $[\sigma_i,\sigma_{i+1}]$, respectively. For the left-hand end points we have

$$\nabla_{\sigma_i}\Sigma =$$
$$-\dot{u}(\sigma_i+)\nabla_{p_{i+1,1}}\Sigma - \ddot{u}(\sigma_i-)\nabla^-_{p_{i+1,2}}\Sigma - \ddot{u}(\sigma_i+)\nabla^+_{p_{i+1,2}}\Sigma\,.$$

If σ_i is an end point of a boundary arc, the terms with vanishing control derivatives are dropped. The derivative w.r.t. horizon

$$\nabla_T\Sigma = 1 + \tfrac{1}{2}\alpha(x_0)u(T^-)^2 + \alpha(x_0)\int_{\sigma_{N-1}}^{\sigma_N} u(t)\nabla_T u(t)\mathrm{d}t$$
$$+\rho\|x(T)-x^f\|f(x(T),u(T^-))$$
$$+\tfrac{1}{2}\rho\left(\|x(T+T_1)-x^f\|^2 - \|x(T)-x^f\|^2\right).$$

For step 4^0 of the scheme in the case of $\alpha(x_0)\neq 0$ we consider controls parameterized with division points and vectors p_i (33) for segments within the interior arcs. A variational approach analogous to that described in section 3 can be employed to get linearized parametric controllers.

An example of solution optimal according to the performance index (26) is given in Fig. 3. For comparison, the minimum time solution for the same initial condition is shown in Fig. 4.

7. CONCLUSIONS

The construction of optimal closed-loop controller for systems with non-linear state equations is a complex computational task. The adaptive optimizing controller is a practical solution, which can be applied in real time, in a vicinity of a reference trajectory computed beforehand. A combination with repetitive optimization using the MSE method enlarges the area of application, at the cost of more on-line computations. An important observation is that the implementation of reduced optimization largely decreases the computational cost of the algorithm, with insignificant deterioration of its performance. The inclusion of the horizon into the optimization process improves the overall efficiency of the repetitive control scheme. The use of relaxed minimum time criterion (26) forces the computational procedure to reject certain "unsafe" controls and assures required robustness by restraining the control amplitude in the neighborhood of the target state.

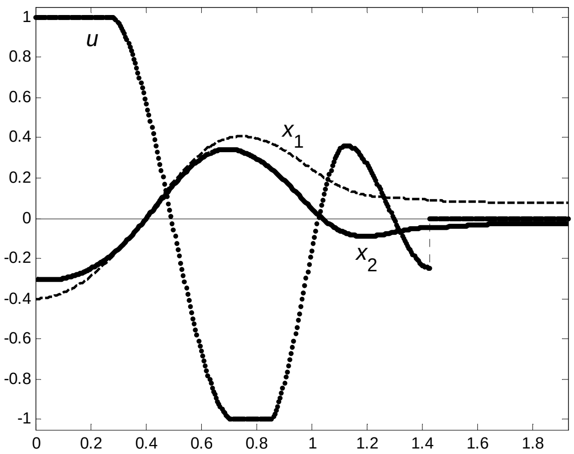

Fig. 3. Robustified optimal solution.

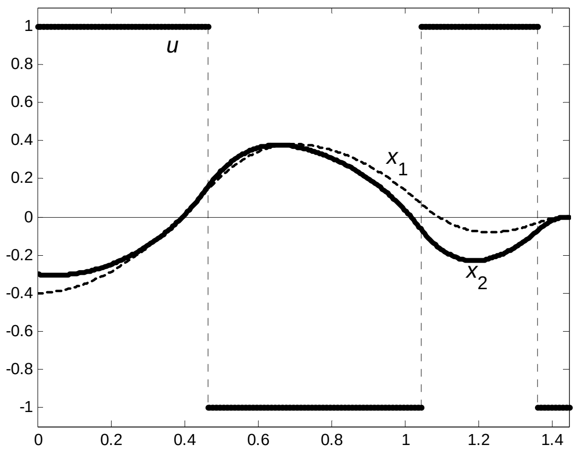

Fig. 4. Minimum time solution.

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