LIMIT CYCLE BIFURCATION IN SISO CONTROL SYSTEMS WITH SATURATION [1]

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Abstract: The bifurcation of limit cycles in single-input single-output control systems with saturation is considered. Under some non-degeneracy conditions, a theorem characterizing such bifurcation is stated for the cases of dimension two and three. In terms of the deviation from the critical value of the bifurcation parameter, expressions in form of power series for the period, amplitude and the characteristic multipliers of the bifurcating limit cycle are obtained. These results are similar to the Hopf bifurcation theorem for differentiable systems, but they show some differences coming from the non-smooth character of the saturation characteristic. *Copyright© 2005 IFAC.*

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1. INTRODUCTION

Equilibrium points correspond with the simplest solutions of a dynamical system. The study of their location and their stability character is one of the first tasks in the analysis of a given nonlinear system, since the different equilibria and their attraction basins organize partially or completely the phase space. Next, periodic orbits constitute another class of relevant solutions to be considered. When a periodic orbit is isolated, that is, there exists some neighborhood of it where there are no other periodic orbits, it is said that it is a *limit cycle*. Obviously, the knowledge of limit cycles and their stability can be regarded as the second main step in nonlinear analysis.

Apart from approximate methods like the describing function method, few techniques are available in order to show the existence of limit cycles and their stability for non-smooth systems. Rigorous mathematical proofs are hard to obtain and cumbersome; see for instance (Moreno and Suárez, 2004). In the case of piecewise linear systems, the more popular they are the more necessary is to achieve sound theoretical results about their possible limit cycles. Trying to avoid this lack of results, here a new methodology for studying limit cycles is proposed, and explicit results are given for low dimensional piecewise linear systems. Some related results and more details of the followed approach can be seen in (Freire *et al.*, 1999), (Freire *et al.*, 2005) and (Carmona *et al.*, 2005).

It will be considered the following system in $\mathbb{R}^n$,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \tag{1}$$

subject to a nonlinear feedback $u = -\varphi(y)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd piecewise linear function, $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))^T$, $\mathbf{A}$ is an $n \times n$ constant real matrix, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, and $y(t) \in \mathbb{R}$ is the output variable as follows

$$y(t) = \mathbf{c}^T\mathbf{x}(t). \tag{2}$$

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In the common case of a characteristic with only three linear pieces, the nonlinearities are of the form

$$\varphi(y) = \begin{cases} k_2 y - (k_1 - k_2)v, & \text{if } y \le -v, \\ k_1 y, & \text{if } -v < y < v, \\ k_2 y + (k_1 - k_2)v, & \text{if } v \le y, \end{cases} \tag{3}$$

where $k_1 \neq k_2$ and $v > 0$. In fact, these piecewise linear characteristics can be normalized as follows. Take $\mathbf{x} = v\overline{\mathbf{x}}$, $\overline{\mathbf{A}} = \mathbf{A} - k_2\mathbf{b}\mathbf{c}^T$ and $\overline{\mathbf{b}} = (k_1 - k_2)\mathbf{b}$. Then, the control system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \varphi(\mathbf{c}^T\mathbf{x})\mathbf{b}$ with $\varphi(y)$ given by (3) can be transformed into $\dot{\overline{\mathbf{x}}} = \overline{\mathbf{A}}\overline{\mathbf{x}} - \overline{\varphi}(\mathbf{c}^T\overline{\mathbf{x}})\overline{\mathbf{b}}$ with

$$\overline{\varphi}(\overline{y}) = \begin{cases} -1, \text{ if } & \overline{y} \le -1, \\ \overline{y}, \text{ if } & -1 < \overline{y} < 1, \\ 1, \text{ if } & 1 \le \overline{y}. \end{cases} \tag{4}$$

From now on, only normalized saturation nonlinearities given by (4) will be considered. Thus, the control system (1)-(2) is a dynamical system defined by a symmetric piecewise continuous vector field with three linear zones and two parallel frontiers. Clearly, it is always possible to suppose that the frontiers are the planes $\Sigma_1 = \{\mathbf{x} \in \mathbb{R}^n : x_1 = 1\}$ and $\Sigma_{-1} = \{\mathbf{x} \in \mathbb{R}^n : x_1 = -1\}$. The regions of $\mathbb{R}^n$ where $x_1 < -1$, $|x_1| \le 1$ and $x_1 > 1$, respectively hold, will be denoted by L (left), C (central) and R (right) zones.

Therefore, the system object of analysis can be expressed as follows

$$\begin{aligned} \dot{\mathbf{x}} &= A_L\mathbf{x} - \mathbf{b}, \text{ if } x_1 < -1, \\ \dot{\mathbf{x}} &= A_C\mathbf{x}, \qquad \text{if } |x_1| \le 1, \\ \dot{\mathbf{x}} &= A_L\mathbf{x} + \mathbf{b}, \text{ if } x_1 > 1, \end{aligned} \tag{5}$$

where the continuity and symmetry of the vector field involved have been taken into account; in particular, the matrices A_L and A_C can only differ in their first columns. Note that the origin is always an equilibrium point.

From Proposition 16 of (Carmona *et al.*, 2002), under the generic condition of observability, that is, when the observability matrix

$$\mathcal{O} = \begin{bmatrix} \mathbf{e}_1^T \\ \hline \mathbf{e}_1^T A_C \\ \hline \vdots \\ \hline \mathbf{e}_1^T A_C^{n-1} \end{bmatrix},$$

has full rank, system (5) can be written in the generalized Liénard's form, namely

$$\dot{\mathbf{x}} = \begin{bmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & 0 & 0 & \cdots & -1 \\ a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{x} + \mathbf{b}\bar{\varphi}(x_1), \tag{6}$$

where, with a slight misuse of notation, the same symbols will be used for coordinates. It should be remarked that, in order to check the above observability condition, one can use equivalently the matrix A_L.

For these systems, it will be considered the case where for some critical values of parameters the matrix corresponding to the central zone has a pair of imaginary eigenvalues $\pm\beta i$, and there are no other eigenvalues on the imaginary axis of the complex plane, so that system (6) has a linear center contained in the central zone C. Furthermore, the outermost periodic orbit of the center will be tangent to both Σ_1 and Σ_{-1}. By varying some parameter, it will be analyzed the possible bifurcation of a symmetrical limit cycle from this center (obviously, it should be born from the outermost periodic orbit of the center).

The passing of the bifurcation parameter by the critical value implies that a complex pair of eigenvalues of the linear system in the zone C crosses the imaginary axis of the complex plane. Note the similarities with the classical Hopf bifurcation scenario. However, the piecewise linear character of the model makes that the change in the dynamics is not merely local.

The proposed bifurcation approach to show the existence of limit cycles has a local character with respect to the parameters and is based in the study of the so called *closing equations*. The method works very well for low-dimensional systems and can be generalized, but the complexity of the computations involved logically grows with the dimension of the system.

The paper is organized as follows. In Section 2, the method of closing equations as the main tool in the analysis is sketched. Next, in Section 3 the bifurcation result for planar systems is given and the corresponding three-dimensional case is outlined in Section 4. Some conclusions are offered at the end of the paper.

2. THE CLOSING EQUATIONS

Assume that system (6) has a symmetrical periodic orbit, see Figure 1. Let us denote by $\mathbf{x}^0$, $\mathbf{x}^1$, $\mathbf{x}^2$ and $\mathbf{x}^3$ the points where the periodic orbit intersect Σ_1 and Σ_{-1}, and let be τ_C the time spent by this orbit from $\mathbf{x}^0$ to $\mathbf{x}^1$ in zone C, and τ_L the time spent by this orbit from $\mathbf{x}^1$ to $\mathbf{x}^2$ in zone L.

As system (6) is linear in every zone, it is possible to obtain explicitly its solutions, and to identify symmetrical periodic orbits of the system living in the three zones with the solutions of the equations

$$\begin{aligned} e^{A_C\tau_C}\mathbf{x}^0 - \mathbf{x}^1 &= \mathbf{0}, \\ e^{A_L\tau_L}\mathbf{x}^1 - \int_0^{\tau_L} e^{A_L(\tau_L - s)}\mathbf{b}\, ds + \mathbf{x}^0 &= \mathbf{0}, \end{aligned} \tag{7}$$

where τ_C and τ_L are the times spent by the semi-orbit in each zone, and

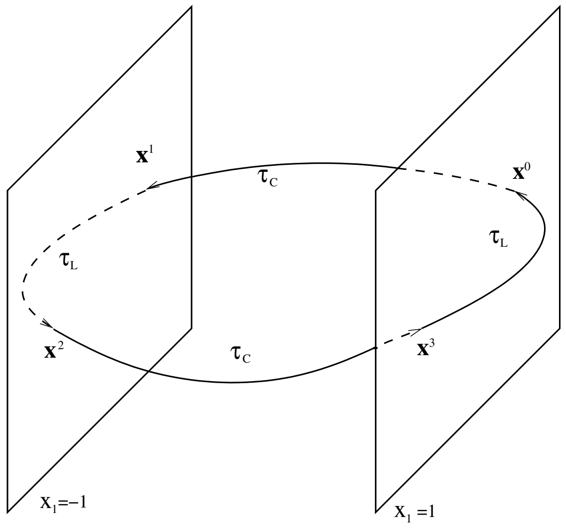

Fig. 1. Sketch of a 3D symmetric periodic orbit using the three linear zones of system (6).

$$\mathbf{x}^0 = \begin{bmatrix} 1 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}, \quad \mathbf{x}^1 = \begin{bmatrix} -1 \\ x_2^1 \\ \vdots \\ x_n^1 \end{bmatrix},$$

are two intersection points of the orbit with the planes Σ_1 and Σ_{-1}, respectively (from the symmetry, there will exist their symmetrical $\mathbf{x}^2 = -\mathbf{x}^0$ and $\mathbf{x}^3 = -\mathbf{x}^1$). The system formed by Eqs. (7) will be referred as closing equations. The use of these equations goes back to Andronov and coworkers (Andronov *et al.*, 1966) and, in the context of limit cycle bifurcations, it was firstly exploited in (Kriegsmann, 1987). The last quoted author studied the *rapid* bifurcation in the Wien bridge oscillator, later revisited in (Freire *et al.*, 1999).

Natural parameters for analyzing the closing equations are the coefficients of the two involved characteristic polynomials. It will be assumed that all these coefficients are constant, excepting the trace T of the matrix A_C, to be considered as the unique bifurcation parameter of the problem. Under straightforward additional assumptions, there will appear some critical value, say $T = T_c$, for which a linear center exists in the central zone, bounded by the boundary hyperplanes Σ_1 and Σ_{-1}.

Starting from this critical value and considering the outermost periodic orbit of the corresponding center configuration, the closing equations will be used in order to analyze what happens to such periodic orbit as T varies, keeping constant the other parameters. To achieve this goal, first it will be determined the non-transversal solution of closing equations (7) that corresponds to the outermost periodic orbit of the center. This orbit is tipically associated to the values $\tau_L = 0$ and $\tau_C = \pi/\beta$, where β stands for the frequency of the linear oscillations of the center. The key idea is, by varying T, to follow the branch of solutions of the closing equations that emanates from the above point. The points of this branch will satisfy $\tau_L > 0$ and $\tau_C < \pi/\beta$, and are already associated to actual nonlinear periodic solutions, generically isolated.

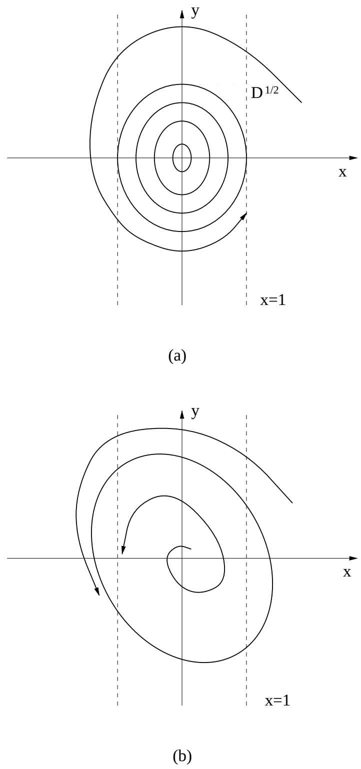

Fig. 2. (a) Phase plane of the piecewise linear system for $T = 0$. The center configuration is constrained to the middle region. (b) Phase plane for $T > 0$ showing a stable limit cycle lying in the three regions, which comes from the outermost orbit of the center.

Such kind of *path following* problem can be solved numerically, of course, but it is also possible to parameterize the closing equations solutions analytically, at least in a local neighborhood of the starting solution. This can be done by means of an adequate application of the implicit function theorem, that previously requires to remove the singularities imposed by the mentioned non-transversality.

Thus, for the desired branch that springs from the aforementioned solution, it is possible to obtain power series in τ_L for the remaining variables of the closing equations. Other technical issue is to pass from the series in τ_L to series in the bifurcation parameter. In order to know about the stability of the bifurcating limit cycle, it is useful to estimate the characteristic multipliers of the limit cycle, that is, the eigenvalues of the derivative of a Poincaré return map defined in an adequate section of the phase space. Such computation is also included in the analysis.

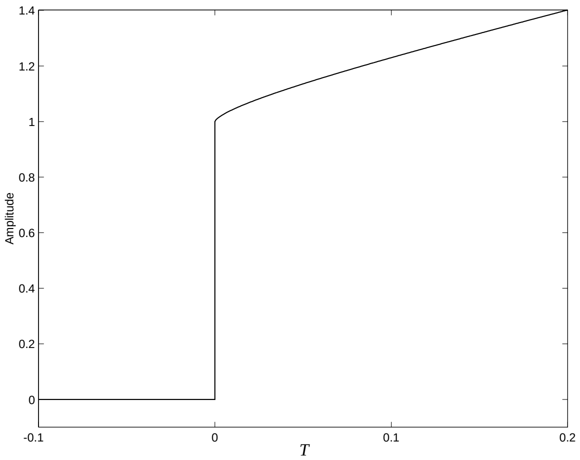

Fig. 3. Limit cycle amplitude (measured as the maximum value of $|x_1|$) versus the bifurcation parameter T, as predicted by Theorem 1, considering the first three non-null terms in the series. The depicted case is for $d = D = 1$, $t = -1$. The vertical line stands for the existence of the center configuration of the central zone.

The theoretical results obtained by the application of this method in dimension two and three appear in next sections. More details of this method can be found in (Freire *et al.*, 1999), (Ros, 2003), (Freire *et al.*, 2005) and (Carmona *et al.*, 2005).

3. THE TWO-DIMENSIONAL CASE

Under the generic condition of observability for the planar case of (5), every system of the above form can be written in the Liénard's form

$$\frac{d}{d\tau}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t & -1 \\ d & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} T-t \\ D-d \end{bmatrix}\overline{\varphi}(x_1). \tag{8}$$

As a necessary hiphothesis for the existence of a linear center at the origin, is that the origin be a topological focus (that is, $4D - T^2 > 0$. The most interesting situation for the scope of this paper is the one where the origin is the only equilibrium point and $t > 0$, which corresponds to an unstable open loop system and, if the system is desired to be closed loop stable, the conditions $D > 0$ and $T < 0$ must hold. On the contrary, from the point of view of design of oscillators, it is required $t < 0$, $D > 0$ and $T > 0$, see Figure 2.

The following result gives a quantitative description of the generation of a limit cycle when the value of T passes from negative to positive values. The phenomenon has been described as a rapid bifurcation in (Kriegsmann, 1987), and the following result has been proved in (Freire *et al.*, 1999) and generalized in (Ros, 2003) and (Freire *et al.*, 2005).

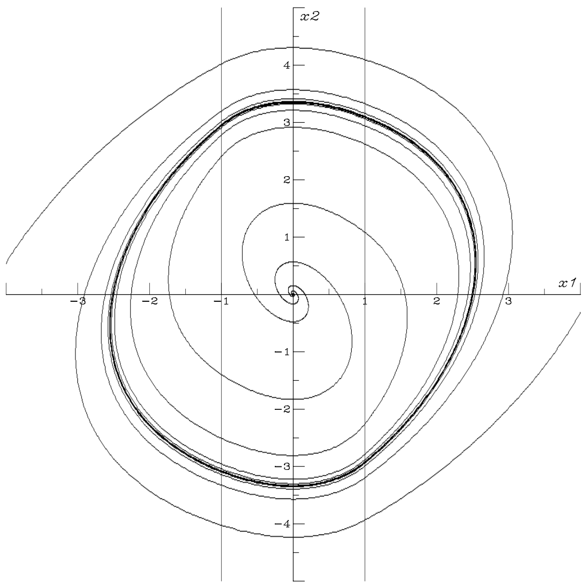

Fig. 4. Phase portrait for the system (8) for $T = -1$, $t = 1$, $D = 2$ and $d = 1$. For theses values the limit cycle predicted by Theorem 1 that bifurcated for $T = 0$ still remains. Note that the limit cycle is unstable and defines the basin of attraction of the stable equilibrium at the origin.

Theorem 1. Assume for system (8) that $D > 0$, $T^2 < 4D$ and $t \neq 0$. For $T = 0$ the system undergoes a focus-center-limit cycle bifurcation, that is, from the focus configuration at the origin, that exists for $Tt > 0$, the system has a linear center restricted to the central zone for $T = 0$, that gives places to an hyperbolic limit cycle for $tT < 0$ and T sufficiently small, symmetrical respect the origin and that intersects transversally Σ_1 and Σ_{-1}.

For $t < 0$, the bifurcating limit cycle appears for $T > 0$ and is orbitally asymptotically stable, while for $t > 0$, the bifurcating limit cycle appears for $T < 0$ and is unstable. The amplitude a (measured as the maximum in $|x_1|$), the period P of the periodic oscillation and the logarithm of the characteristic multiplier ρ of the limit cycle are analytic functions at 0, in the variable $T^{\frac{1}{3}}$, for $T > 0$ and sufficiently small. Furthermore, the series in $T^{\frac{1}{3}}$ are

$$\begin{aligned} a &= 1 + \frac{(6\pi)^{2/3}}{8t^{2/3}}T^{2/3} + \\ &+ \frac{(6\pi^4)^{1/3}(120D - 2t^2 - 21d)}{960t^{4/3}}T^{4/3} + \\ &+ \frac{(6\pi)^{2/3}}{12t^{5/3}}T^{5/3} + O(T^2), \\ P &= \frac{2\pi}{\sqrt{D}} + \frac{\pi(d-D)}{D^{\frac{3}{2}}t}T + \\ &- \frac{(6\pi^5)^{1/3}((d-D)^2 + t^2 D)}{10D^{5/2}t^{5/3}}T^{5/3} + O(T^2), \end{aligned}$$

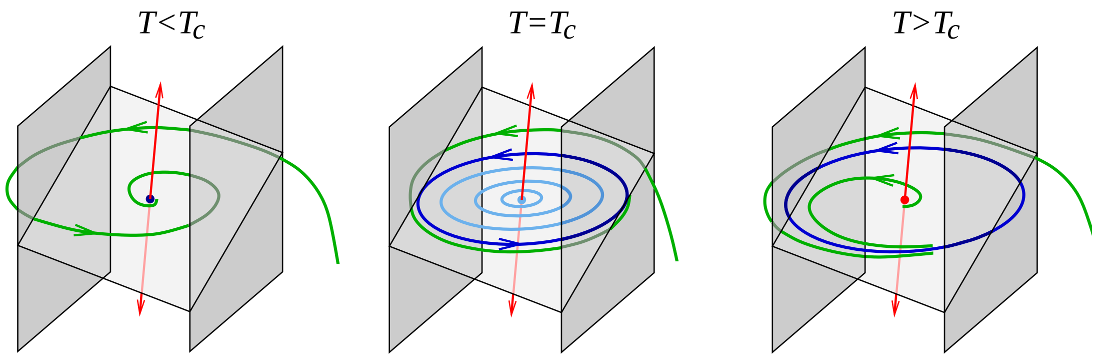

Fig. 5. The focus-center-limit cycle bifurcation in the case $D > 0$, $\gamma > 0$. The focal plane and the complementary one-dimensional invariant manifold at the origin are shown, along with the two parallel planes which separate the three linear regions. In the situation sketched, as deduced from Theorem 2, the bifurcating limit cycle is of saddle type.

$$\rho = -2\,(6\pi)^{1/3}\, t^{2/3} T^{1/3} + \frac{\pi}{15}\left(12d + 15 - t^2\right) T + \\ + \frac{4\,(6\pi)^{1/3}}{3t^{1/3}} T^{4/3} + O(T^{5/3}).$$

Note that the amplitude behavior versus the bifurcation parameter is rather different from the one in the case of the Hopf bifurcation for smooth systems, where the amplitude should evolve as $O(T^{\frac{1}{2}})$, without having the jump that it is observed here for piecewise linear systems, see Figure 3. Nevertheless, the bifurcation involved resembles the Hopf bifurcation, as the imaginary axis crossing of an eigenvalue pair for the equilibrium is accompanied by the appearance of a limit cycle, whose period also evolves as $2\pi/\sqrt{D} + O(T)$.

Note that a possible application of this two-dimensional case when $t > 0$ is the prediction of unstable limit cycles for $T > 0$ whose size can be estimated, and then this limit cycle is the boundary of the attraction basin of the origin, see Figure 4.

4. THE TRIDIMENSIONAL CASE

Under the generic condition of observability for system (5), every system of the above form can be written in the generalized Liénard's form

$$\frac{d}{d\tau}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t & -1 & 0 \\ m & 0 & -1 \\ d & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \\ + \begin{bmatrix} T-t \\ M-m \\ D-d \end{bmatrix}\overline{\varphi}(x_1), \qquad (9)$$

so that, regarding system (5), it turns out

$$A_L = \begin{bmatrix} t & -1 & 0 \\ m & 0 & -1 \\ d & 0 & 0 \end{bmatrix},$$

$$A_C = \begin{bmatrix} T & -1 & 0 \\ M & 0 & -1 \\ D & 0 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} T-t \\ M-m \\ D-d \end{bmatrix}.$$

Choosing again T as the bifurcation parameter, for the critical value $T_c = D/M$ with $M > 0$, system (9) has a linear center in the zone C, see Fig. 5, that is, the matrix A_C have a pair of pure imaginary eigenvalues. Then, it is a natural issue to analyze whether a limit cycle bifurcates from this configuration when the bifurcation parameter T moves. Due to geometrical reasons, the logarithms of the characteristic multipliers will be denoted by μ_r and μ_a, from radial and axial respectively. By means of thorough analysis of the closing equations, the following result can be obtained.

Theorem 2. Let us consider system (9) with $M > 0$, $T_c = D/M$ and

$$\gamma = DM - Dm + dM - tM^2 \neq 0.$$

For $T = T_c$ the system undergoes a focus-center-limit cycle bifurcation, that is, from the lineal center configuration in the central zone, that exists for $T = T_c$, one limit cycle appears for $\gamma(T - T_c) > 0$ and $T - T_c$ sufficiently small.

The amplitude a (measured as the maximum in $|x_1|$), the period P and logarithms of the characteristic multipliers μ_r and μ_a of the periodic orbit are analytic functions at 0, in the variable $(T - T_c)^{1/3}$, namely

$$
\begin{aligned}
a &= 1 + \frac{(6\pi)^{2/3} M^{4/3}}{8\gamma^{2/3}} (T - T_c)^{2/3} + \\
&+ \frac{(6\pi^4)^{1/3} a_4}{960 M^{1/3} \gamma^{7/3}} (T - T_c)^{4/3} + \\
&+ O(T - T_c)^{5/3}, \\
P &= \frac{2\pi}{\sqrt{M}} + \frac{\pi(M - m)\sqrt{M}}{\gamma} (T - T_c) + \\
&- \frac{6^{2/3} \pi^{5/3} M^{5/6} P_5}{20\gamma^{8/3}} (T - T_c)^{5/3} + \\
&+ O(T - T_c)^2, \\
\mu_r &= -\frac{(48\pi)^{1/3} M^{7/6} \gamma^{2/3}}{D^2 + M^3} (T - T_c)^{1/3} + \\
&+ O(T - T_c)^{2/3}, \\
\mu_a &= \frac{2\pi D}{M^{3/2}} + \mu_{a,1} (T - T_c)^{1/3} + O(T - T_c)^{2/3},
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{a,1} &= \frac{(48\pi)^{1/3}}{M^{5/6}} \left(\frac{Mt - D}{\gamma^{1/3}} + \frac{M^2 \gamma^{2/3}}{D^2 + M^3} \right), \\
a_4 &= -120tM^5 + \left(120D + 2t^3 + 21mt + 72d\right) M^4 \\
&+ \left[-\left(93m + 27t^2\right) D + \left(27m - 2t^2\right) d \right] M^3 \\
&+ \left(2t^2 m + 25dt - 27m^2\right) DM^2 \\
&+ \left[25D^3 + 23(mt - d) D^2\right] M - 25mD^3, \\
P_5 &= \left[M(M - m)^2 + (Mt - d)^2 \right] (Mt - D).
\end{aligned}
$$

In particular, if $\gamma > 0$ and $D < 0$, then the limit cycle bifurcates for $T > T_c$ and is orbitally asymptotically stable.

The coefficient γ allows a complete characterization of the bifurcation when it does not vanish. Its role is analogous to the coefficient of cubic term in the Poincaré-Andronov-Hopf normal form. When $\gamma = 0$ the bifurcation is of higher codimension, requiring a specific treatment that will appear elsewhere. In such cases, it can be rigorously shown the existence of two limit cycles in certain parameter regions.

5. CONCLUDING REMARKS

The given theorems describe codimension one bifurcations, similar to the Hopf bifurcation of differentiable dynamics, see (Chow and Hale, 1982), but it should be noted some differences. In particular, the expressions characterizing the bifurcation are in terms of the parameter to the power one third instead of the power one half, and, what is more important, the limit cycle amplitude's leading order is $O(1)$, which indicates that the appearance of the limit cycles is rather rapid as they are born with a significant size.

Note that this bifurcation, although it could be thought of a border-collision bifurcation in a broad sense, it is not so strictly speaking due to several reasons. First, there is no limit cycle before the bifurcation but only a non-hyperbolic periodic orbit belonging to a linear center and this periodic orbit only exists for the critical value of the bifurcation parameter. Second, there is no discontinuity surface in the vector field which is continuous everywhere.

It should be also remarked that it is possible, with the same techniques, to obtain similar bifurcation results for the asymmetric case of single-sided saturation. Thus, the proposed methodology is able to cope with a wider class of piecewise linear systems.

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