EQUILIBRIUM PRICE BIFURCATION IN WALRAS PRICE FORMATION MODEL WITH DELAYS

Natalia K. Obrosova

Dorodnicyn Computing Centre of the Russian Academy of Science
Vavilov st.40, 119991 Moscow GSP-1, Russia

Abstract: The author investigates bifurcation of equilibrium price in price formation model of Walras type with delays, finds the stability limits for equilibrium price and gives their economic interpretation. The Hopf's bifurcation theorem for the class of differential equations with delay is proved. It is shown that in the case of nonessential commodity market a periodic orbit emerges or dies after loss of stability. The results of the investigation allow using this model as a block of a macroeconomic model for describing endogenous crises inception.

Keywords: system analysis economic systems differential equations stability domains stability limits

1. INTRODUCTION

The problem of modeling economic crises is well known in mathematical economy. However no satisfactory model describing the emergence of such crises has been offered. To analyze the reason of this fact consider known methods of economy description. A description of economic phenomenon begins with selection of characteristic times of processes. While constructing economic models, it is usually considered that all processes are divided into slow (production reorganization, modification of consumer's preferences) and fast (price formation processes), according to Marshall's hypothesis of times separation. Slow-time processes are studied by macroeconomic theory. The modification of industrial capacities, production technologies and consumer's preferences are macroeconomic processes which change the structure of demand and supply. Thus, the form of demand and supply functions changes in slow time.

While constructing macroeconomic models it is usually considered that fast processes are in equilibrium state. Properties and stability of this equilibrium are considered in microeconomic theory. Usually macro and micro models are not connected with each other.

Microeconomic equilibrium models in fast time describe the results of interactions between demand and supply in the market on characteristic times, when preferences of consumers, production capacities and technologies are considered to be a constant. Therefore, in fast processes supply and demand functions are known and depend on time implicitly. Under these conditions demand for the goods and their supply are adjusted to demand and supply prices only.

Let economic processes be separated under their characteristic times. Let $\mathbf{x}$ be a vector of slow variables (macro variables) and $\mathbf{y}$ be vector of fast variables (micro variables). Then the schematic representation of economic system is the following Tychonov's system:

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = f(\mathbf{x},\mathbf{y}) \\ \varepsilon \dfrac{d\mathbf{y}}{dt} = g(\mathbf{x},\mathbf{y}), \end{cases} \qquad (1)$$

where ε is a small parameter. The first equation (1) describes slow processes, and the second - fast processes (Petrov, et al., 1996). Let the left part of the second equation of (1) be equal to zero. Then (1) turns to the following system:

$$\begin{cases} \dfrac{d\mathbf{x}}{dt} = f(\mathbf{x},\mathbf{y}) \\ g(\mathbf{x},\mathbf{y}) = 0. \end{cases} \tag{2}$$

According to the Tychonov's theorem, the solutions of (2) approximate solutions of (1) when the stationary solution $\mathbf{y}(\mathbf{x})$ of $\varepsilon\frac{d\mathbf{y}}{dt} = g(\mathbf{x},\mathbf{y})$ is asymptotically stable. Substituting the stationary solution $\mathbf{y}(\mathbf{x})$ to the first equation in (2) makes it clear that the economic process is described by the equation

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x},\mathbf{y}(\mathbf{x})). \tag{3}$$

The model (3) represents the general scheme of classical macroeconomic model.

The idea of modeling of emerging crises consists in construction of macroeconomic model based on micro descriptions. The model should describe stability loss of fast relaxation processes by modification of slow parameters. As a result a complicated price formation dynamic is obtained so that economic agents can't forecast their activity. Thus, the obtained model of emerging economic crises reflects actual processes of economic development more authentically. The problem in this case is to construct a price formation model, in which equilibrium price loses stability due to the change of slow parameters. The majority of macroeconomic models describe dynamic of gross domestic product. Thus the consistent price formation model should describe scalar quantity - price index.

Grandmont (1985) and Shananin (1991) suggested price formation models in discrete time. Shananin (1991) considered the discrete analog of elementary price formation model of Walras type $p_n = \frac{p_{n-1}C(p_{n-1})}{g(p_{n-1})}$, where p is a price of commodity, $C(p)$ is a demand function, $g(p)$ is a supply function. Shananin (1991) analyzed the equilibrium price stability and gave economic interpretation of stability loss for this model. Namely, the equilibrium price loses stability when industrial capacity exceeds some critical value. Also it was shown that a sequence of doubling period bifurcations appears in the system after stability loss. So when capacities are large enough the behavior of the system becomes chaotic and economic agents cannot forecast their activity reliably (overproduction crisis).

However, discrete models have essential defect. The time step in discrete models is not interpreted in any way. Besides, it is difficult to use discrete price formation models as a part of a macroeconomic model described in continuous time. Therefore, it is necessary to build the model in continuous time, which inherits the properties of discrete model. The simplest model of Walras type, which is a direct analog of discrete price formation model, is not appropriate here, because under usual assumptions about supply and demand functions (i.e. demand decreases with growth of the price, and supply increases with growth of the price) the equilibrium price in this model is always stable. The elementary model of Walras type has the following form

$$\frac{dp}{dt} = \chi\frac{C(p)-g(p)}{g(p)}p, \tag{4}$$

where parameter $\chi > 0$ has dimension $\frac{1}{\text{time}}$ and characterizes the speed of market reaction to price change.

The reason of instability appearance in discrete model is the implicit inertia in consumer's and producer's reaction to price changes. There are two ways of inertia modeling in continuous models. By the first way, suggested by Lorenz (1989), the inertia is modeled as follows: supply doesn't directly depend upon price, but supply's derivative does. Thus, in this model price formation process is described by a system of differential equations. However, parameters have unsatisfactory economic interpretation.

In this paper another way is used. Delays in consumer's and producer's reaction are added to the model of Walras type. They reflect real market processes and have good economic interpretation. For example, producer's delay may be interpreted as characteristic time of production cycle.

2. MODEL DESCRIPTION

Consider homogeneous commodity market. According to the traditions, which date back to Marshall and Walras, we shall consider the following assumptions to be true:

a) at every moment of time the commodity is sold at the single price p;

b) consumer's and producer's behavior is described by demand function $C(p)$ and supply function $g(p)$ respectively. Moreover, typical time of demand and supply functions changing is considerably larger than typical time of price changing. It means that these functions do not explicitly depend on time.

Under these assumptions the model of Walras type with delays has the following form:

$$\frac{dp}{dt} = \chi \left[\frac{C(p(t-\tau_1)) - g(p(t-\tau_2))}{g(p(t-\tau_2))} \right] p(t), \qquad (5)$$

where coefficient $\chi > 0$ has dimension $\frac{1}{\text{time}}$ and characterizes the speed of market reaction to price change, $\tau_1 \geq 0$ and $\tau_2 \geq 0$ are constant consumer's and producer's delays respectively. Assume that

1) the equilibrium price exists in the model, i.e. equation $C(p) = g(p)$ has a solution $p = p^*$,
2) $C(p)$ and $g(p)$ are n-time differentiable ($n \geq 9$) in a neighborhood of p^*,
3) demand function $C(p)$ is steadily decreasing and supply function $g(p)$ is steadily increasing,
4) functions $C(p)$ and $g(p)$ are bounded for $p \geq 0$.

3. INVESTIGATION OF EQUILIBRIUM PRICE STABILITY LIMITS

Stability limits for the price formation model (5) are found in terms of supply elasticity $\sigma_g = \left[\frac{p}{g(p)} \frac{dg}{dp} \right]_{p=p^*}$ and demand elasticity $\sigma_c = -\left[\frac{p}{C(p)} \frac{dC}{dp} \right]_{p=p^*}$ calculated at equilibrium price p^*. These parameters are primary characteristics of production technology and consumer's demand structure.

The stability analysis is made with the help of Lyapunov method of system linearization. Change of variables $\theta_1 = \tau_1 \chi,\ \theta_2 = \tau_2 \chi,\ x(t) = \ln\left(\frac{p(t)}{p^*} \right)$, time scaling and linearization in the neighborhood of $x = 0$ in (5) leads to the following linear equation:

$$\frac{dx(t)}{dt} = -\sigma_c x(t-\theta_1) - \sigma_g x(t-\theta_2), \qquad (6)$$

where $\sigma_c, \sigma_g \geq 0$.

There are several problems in applying the Lyapunov method for equations with delays. In this case the characteristic equation is an analytic function and the investigation of real part sign of its root is not simple problem. This investigation is done in this paper for three cases: producer's delay only, consumer's delay only and equal delays of both producer and consumer. In order to analyze the sign of real part of the characteristic equation roots the following lemma is proved.

Lemma 1. Quasipolynomial $\varphi(z) = z + ae^{-z\theta}$ *with* $a, \theta > 0$ *has no roots in the right half-plane* $\operatorname{Re} z > 0$ *if and only if* $0 < a\theta < \pi/2$.

With the help of *Lemma 1* the following proposition is proved.

Proposition 1. In the case of equal consumer's and producer's delays, i.e. $\tau_1 = \tau_2 = \tau$, *the equilibrium price* $p*$ *is asymptotically stable if* $\sigma_g < \sigma_g^*$, *and unstable if* $\sigma_g > \sigma_g^*$, *where* $\sigma_g^* = \frac{\pi}{2\tau\chi} - \sigma_c$.

Analysis of the linearization (6) allows proving the following results.

Proposition 2. Consider the case of producer's delay only, i.e. $\tau_1 = 0, \tau_2 > 0$.

1) In the case $0 < \frac{\sigma_c}{\sigma_g} < 1$ *the equilibrium price is asymptotically stable, if* $\sigma_g \tau_2 \chi \sqrt{1 - \frac{\sigma_c^2}{\sigma_g^2}} < arccos\left(-\frac{\sigma_c}{\sigma_g} \right)$, *and unstable if* $\sigma_g \tau_2 \chi \sqrt{1 - \frac{\sigma_c^2}{\sigma_g^2}} > arccos\left(-\frac{\sigma_c}{\sigma_g} \right)$.

2) In the case $\frac{\sigma_c}{\sigma_g} > 1$ *the equilibrium price is asymptotically stable.*

Proposition 3. Consider the case of consumer's delay only, i.e. $\tau_1 > 0, \tau_2 = 0$.

1) In the case $0 < \frac{\sigma_g}{\sigma_c} < 1$ *the equilibrium price is asymptotically stable if* $\sigma_c \tau_1 \chi \sqrt{1 - \frac{\sigma_g^2}{\sigma_c^2}} < arccos\left(-\frac{\sigma_g}{\sigma_c} \right)$, *and unstable if* $\sigma_c \tau_1 \chi \sqrt{1 - \frac{\sigma_g^2}{\sigma_c^2}} > arccos\left(-\frac{\sigma_g}{\sigma_c} \right)$.

2) In the case $\frac{\sigma_g}{\sigma_c} > 1$ *the equilibrium price is asymptotically stable.*

The stability limits obtained in *Propositions 1-3* have an explicit parametric form on the plane (v, u), where $v = \Theta \sigma_g$, $u = \Theta \sigma_c$, and

$$\Theta = \begin{cases} \tau_1 \chi & if \ \ \tau_2 = 0, \\ \tau_2 \chi & if \ \ \tau_1 = 0, \\ \tau \chi & if \ \ \tau_1 = \tau_2 = \tau. \end{cases} \qquad (7)$$

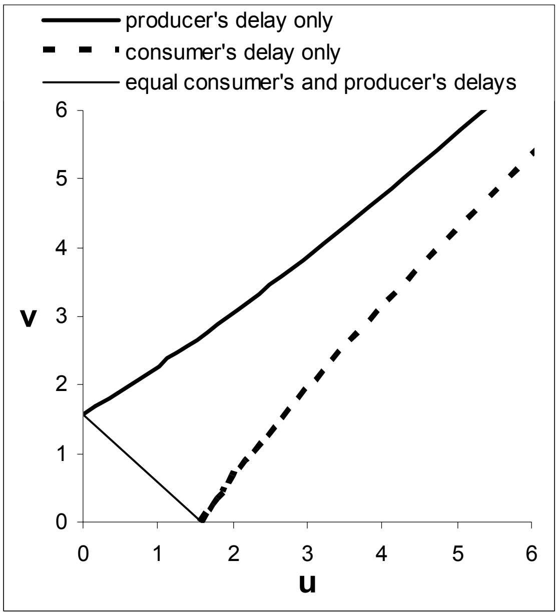

Fig. 1. Stability limits of equilibrium price in the price formation model (5).

The schematic form of stability limits is shown on Figure 1. In cases of producer's delay only and equal producer's and consumer's delays the stability domain is under the corresponding stability bound. In the case of consumer's delay only the stability domain is above the corresponding stability bound.

Analysis of stability limits shows that the equilibrium loses stability when the supply elasticity σ_g exceeds the critical value σ_g^* in the cases of equal delays and producer's delay only. In the case of consumer's delay only there exists constraint from below on supply elasticity σ_g. The advantage of the price formation model (5) in comparison with the discrete models is the possibility to investigate the dependence between stability areas and delays values. According to economist's apprehension, increasing of delay value leads to stability areas decreasing.

4. ECONOMIC INTERPRETATION OF STABILITY LIMITS

Consider an example that allows giving an economic interpretation of the obtained results. Let the supply and demand functions have an explicit form that shows their dependence from slow macroeconomic parameters. In the following example the functions of form suggested by Shananin (1991) is used. Supply function is based on the Houthakker-Johansen model of manufacture and has the form

$$g(p) = M\left(1-\left(\frac{s\nu}{p}\right)^{\alpha}\right), \qquad (8)$$

where M is gross industrial capacity, $\alpha = 1+\frac{\gamma}{\mu}$, where $\gamma \geq 0$ is capacity growth rate, $\mu \geq 0$ is the rate of capacity retirement. Parameter $s\nu$ characterizes the industry, moreover $p > s\nu$ (the condition of the best industry technology not to be unprofitable).

The consumer's demand is described by the model function

$$C(p) = C\left(\frac{p}{s\nu}\right)^{\beta-1}, \qquad (9)$$

where constant C is demand characteristic, $\beta \leq 1$ is a degree of commodity necessity.

For supply and demand functions (8), (9), the supply elasticity calculated at equilibrium price is $\sigma_g = \alpha(1-\Delta)$, where $\Delta = \dfrac{M}{C(p^*)}$ and demand elasticity is $\sigma_c = 1-\beta$. Parameter Δ characterizes ratio between industrial capacities and equilibrium consumer's demand. In economic theory, economic crises inception is usually connected with change of the parameter Δ.It follows from *Propositions* 1-2 that in the cases of both equal delays of producer and consumer and producer's delay only the equilibrium price loses stability when industrial capacity (parameter Δ) exceeds some critical value. This fact accords to economists' opinion about causes of overproduction crises inception. In the case of consumers delay only (*Proposition 3*) the constraint from below on Δ value appears. In this case, the stability loss is the result of demand's increasing, for example, in consequence of investment activity increasing. The analysis of stability limits shows that increasing of the industry growth rate leads to decreasing of stability funds ("economy overheating").

Dependence of the stability areas from consumer's elasticity is obtained. In the example above, the consumer's elasticity σ_c decreases when commodity necessity degree β increases. The stability area grows when σ_c increases in the case of producer's delay only and declines in cases of consumer's delay only and equal producer's and consumer's delays.

Thus, the stability limits dependence on slow macroeconomic parameters and their economic interpretation are obtained.

5. ANALYSIS OF EQUILIBRIUM PRICE BIFURCATION TYPE

In order to analyze the equilibrium price bifurcation in the model (5), the Hopf's bifurcation theorem for the stationary solution $x(t) \equiv 0$ is proved in class of differential equations with delay

$$\frac{dx}{dt} = H\left(x(t), x(t-\tau)\right), \ \ \tau > 0. \qquad (10)$$

Let the function $H(x, y)$ fulfil the following conditions:

1) $H(x, y)$ is n-time differentiable ($n \geq 9$);

2) for either continuous differentiable function $y(t)$, $t \in [0, \tau]$ the equation (10) with $x(t-\tau) = y(t)$ has a bounded solution on $[0, \tau]$;

3) $u = -\tau \left.\frac{\partial H(x, y)}{\partial x}\right|_{(0,0)} \geq 0$ and $v = -\tau \left.\frac{\partial H(x, y)}{\partial y}\right|_{(0,0)} \geq 0$.

The linearization of (10) agrees with the linearization (6) in the case of $\theta_1 = 0$ with accuracy to denotations and time scale. Thus, it follows from *Proposition 1* that in the case of $u = 0$ the solution $x = 0$ of (10) is asymptotically stable if $v < v^*$ and unstable if $v > v^*$, where

$$v^* = \frac{\pi}{2}. \qquad (11)$$

In the case of $u > 0$ it follows from *Proposition 2* that the stability limit v^* of stationary solution $x = 0$ can be found from the equation

$$v^* \sqrt{1 - \frac{u^2}{v^{*2}}} = arccos\left(-\frac{u}{v^*}\right). \qquad (12)$$

Theorem 1. Let $u > 0$ *and* $u \neq \frac{2\pi}{3\sqrt{3}}$. *When the stationary solution* $x = 0$ *of (10) loses stability (i.e. parameter v increases and crosses through the stability limit* v^* *found from (12)) a periodic orbit emerges or dies in the system (10).*

The proof of *Theorem 1* consists of four main steps.

- For bifurcation analysis with the help of steps method (Bellman and Cooke, 1963) the initial equation (10) is reduced to discrete dynamic system $x_{n+1}(t) = \Psi_\mu(x_n(t))$ in functional space $C^1[0,1]$. The mapping $\Psi_\mu : C^1[0,1] \to C^1[0,1]$ belongs to C^p - class ($p \geq 9$) in zero neighborhood of the space $C^1[0,1]$. Parameter $\mu = v - v^*$. In this case the fixed point $x = 0$ of the dynamic system corresponds to the stationary solution $x = 0$ of (10). Now investigate the bifurcation by μ in the case of fixed u.
- On the second step, it is proved that the central manifold theorem conditions (Lanford, 1972) are true for the infinity-dimensional mapping Ψ_μ. The proof of this fact needs the linearization of Ψ_μ in zero neighborhood with the help of implicit function theorem and spectrum analysis of obtained linear operator A_μ. The analysis of operator A_μ shows that the central manifold theorem can be applied to the mapping $(\Psi_\mu, \mu) : C^1[0,1] \times R \to C^1[0,1] \times R$. With the help of the mentioned theorem the initial infinity-dimensional problem is reduced to two dimensional.
- On the third step, the Hopf's theorem (Marsden and McCracken, 1976) is applied to the obtained two-dimensional mapping. From the Hopf's theorem follows that two time continuously differentiable one-dimensional manifold emerges or dies in the system when parameter μ increases and crosses through zero.
- The final step is to prove that a periodic orbit accords to the obtained one-dimensional manifold. To prove this result the circle diffeomorphisms theory (Denjoy's theorem and rotation number definition) (Nitecki, 1971) is used.

The same method allows analyzing the type of bifurcation in the case of $u = 0$. It is proved that in this case the strong resonance is observed in the system (10) when parameter v increases and crosses through the stability limit v^* (found from (11)).

With the help of the *Theorem 1* the stability loss type of equilibrium price in the price formation model (5) is analyzed.

Theorem 2. Let $\tau_1 = 0, \tau_2 > 0$, $\sigma_c > 0$ *and* $\sigma_c \neq \frac{2\pi}{3\sqrt{3}\chi\tau_2}$. *A periodic orbit emerges or dies in the system (5) when the equilibrium price* p^* *loses stability.*

The analogous result is proved in the case of $\tau_1 > 0, \tau_2 = 0$.

Computational experiments show that in the case of $\tau_1 = 0, \tau_2 > 0$ a periodic orbit emerges in the model.

It is proved that in the case of equal producer's and consumer's delays, there is the strong resonance in the system (5) when equilibrium price loses stability. In this case computational experiments also show that a periodic orbit emerges in the system (5).

6. CONCLUSION

At the first glance the obtained results do not solve the problem of economic crises inception modeling. It should seem that economic agents may predict self-behavior taking into account the periodic market behavior. The change of supply description makes this prediction possible for the producers. But empirical data, for example the well known "pig cycle" in Great Britain, shows that economic agents can't follow periodic price fluctuations (Coase and Fowler, 1937). This fact gives apology to producer's description in the model (5).

The results of the present work allow using the price formation model of Walras type with delays as a block of a macroeconomic model for describing endogenous crises inception.

ACKNOWLEGMENTS

This work was financially supported by the Russian Science Support Fund, by the program for State Support of Leading Russian Scientific Schools (project no. NSc-1843.2003.1) and by the Russian Foundation for Basic Research (project nos. 05-01-00942, 04-01-00600).

REFERENCES

Bellman, R. and K.L. Cooke (1963). *Differential-Difference Equations*. Academic Press, London.

Coase, R.H. and Fowler R.F. (1937). The Pig-Cycle in Great Britain: An Explanation. *Economica*, **4**, **13**, 55-82.

Grandmont, J.-M. (1985). On endogenous competitive business cycles. *Econometrica* ,**53**, **5,** 995-1045.

Lanford, O.-E. (1972). Bifurcation of periodic solutions into invariant tori: The work of Ruelle and Takens. *Lecture Notes in Mathematics*, **322**, 159-192.

Lorenz, H.-W. (1989). *Nonlinear Dynamical Economics and Chaotic Motion.* Springer-Verlag, New York-Berlin.

Marsden, J.E., M. McCracken (1976). *The Hopf bifurcation and its applications. With contributions by P. Chernoff, G. Childs, S. Chow, J. R. Dorroh, J. Guckenheimer, L. Howard, N. Kopell, O. Lanford, J. Mallet-Paret, G. Oster, O. Ruiz, S. Schecter, D. Schmidt, and S. Smale*. Springer-Verlag, New York-Berlin.

Nitecki, Z. (1971). *Differentiable Dynamics: an Introduction to the Orbit Structure of Diffeomorphisms*. The MIT Press, Cambridge.

Petrov, A.A., I.G. Pospelov and A.A. Shananin (1996). *Opit matematicheskogo modelirovaniya economiki.* Energoatomisdat, Moscow.

Shananin, A.A. (1991). Ob ustojchivosti rinochnich mechanismov. *Matematicheskoje modelirovanije*, **3**, **2**, 42-62.