MATRIX INEQUALITY CONDITIONS FOR DISSIPATIVITY OF CONTINUOUS-TIME DESCRIPTOR SYSTEMS AND ITS APPLICATION TO SYNTHESIS OF CONTROL GAINS

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Abstract: This paper is concerned with a KYP-type result for descriptor systems. A matrix inequality is shown that provides a necessary and sufficient condition of dissipativity of descriptor systems, without any additional restriction on the realization unlike previous results. Further, a dual matrix inequality condition is derived and applied to synthesis of control gains to attain dissipativity by solving LMIs.

Keywords: Descriptor systems, dissipativity, LMIs.

1. INTRODUCTION

It has been well understood that the descriptor form provides system representations that are more natural and general than state-space systems (See e.g., Lewis (1986)). The descriptor form is useful to represent such as mechanical systems, electric circuits, interconnected systems, parameter-varying systems, and so on. Among considerable number of basic notions of dynamical systems generalized to descriptor systems, dissipativity, including positive and bounded realness, is one of the most important properties and plays crucial roles in various problems of analysis and synthesis of control systems.

For linear time-invariant systems, Kalman-Popov-Yakuvobich (KYP) Lemma and related results give characterization of positive or bounded realness in terms of the state space realization (Anderson, 1967; Willems, 1971; Rantzer, 1996). Those results have been generalized to descriptor systems, providing matrix equations and inequalities (Takaba et al., 1994; Masubuchi et al., 1997; Wang et al., 1998; Rehm, & Allgöwer, 2000; Zhang et al., 2002). However, for continuous-time systems, most of the existing results require a certain assumption or restriction on the realization of descriptor systems, while KYP Lemma for state-space systems is valid independently of the choice of the realization. Recently, a modified matrix inequality condition is proposed that is necessary and sufficient for dissipativity of descriptor systems with any realization (Masubuchi, 2004).

One of the merits of matrix inequality conditions for dissipativity is the fact that they can yield LMI conditions for synthesis of a controller that attains closed-loop dissipativity in the control system. Several standard techniques are developed to derive LMIs for synthesis. However, the new dissipativity condition possesses structure to which such methodology is not obviously applicable. In this paper, we show some variants of the dissipativity inequality and derive LMI conditions for synthesis of control gains to attain dissipativity and admissibility of the control system.

Notation. For a matrix X, we denote by X^{-1}, X^{T}, $X^{-\mathsf{T}}$ and X^* the inverse, the transpose, the inverse of the transpose and conjugate transpose of X, respectively. $\mathbf{He}X$ stands for $X + X^\mathsf{T}$. For a symmetric matrix represented blockwise, such as $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^\mathsf{T} & X_{22} \end{bmatrix}$, offdiagonal blocks can be abbreviated with '$*$', as $X = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}$.

2. PRELIMINARIES

2.1 Dissipativity of descriptor systems

Consider the following descriptor system:

$$\begin{cases} E\dot{x} = Ax + Bw, \\ z = Cx + Dw, \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$ is the descriptor variable, $w \in \mathbf{R}^m$ is the input and $z \in \mathbf{R}^p$ is the output of the system. Let $E \in \mathbf{R}^{n\times n}$ and $\mathrm{rank}E = r$.

Next, let $S = S^\mathsf{T} \in \mathbf{R}^{(m+p)\times(m+p)}$ and consider the following quadratic form of (u, y):

$$s(u, y) = \begin{bmatrix} u \\ y \end{bmatrix}^\mathsf{T} S \begin{bmatrix} u \\ y \end{bmatrix}, \tag{2}$$

which defines a *supply rate*.

Definition 1. The descriptor system (1) is said to be *dissipative* with respect to the supply rate $s(\cdot,\cdot)$ if the pencil $sE - A$ is regular, the descriptor system (1) have no impulsive modes and for any $u(t)$ it holds that

$$\int_0^T s(u(t), y(t))\mathrm{d}t \le 0, \quad \forall T \ge 0 \tag{3}$$

provided $x(0) = 0$.

Suppose that $u \in L_2[0,\infty)$. Then the condition (3) is equivalent to the following frequency-domain condition:

$$\begin{bmatrix} I \\ G(j\omega) \end{bmatrix}^* S \begin{bmatrix} I \\ G(j\omega) \end{bmatrix} \le 0, \ \forall \omega \in \mathbf{R} \cup \{\infty\}, \tag{4}$$

where $G(s) = C(sE - A)^{-1}B + D$. By setting

$$M = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\mathsf{T} S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}, \tag{5}$$

the inequality (4) is written as

$$\begin{bmatrix} (j\omega E - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega E - A)^{-1}B \\ I \end{bmatrix} \le 0, \tag{6}$$

which we consider in the next subsection.

2.2 KYP-type Lemma for descriptor systems

Consider the inequality (6), without assuming that M has the structure of (5).

Theorem 2. (Masubuchi (2004)). Suppose that the following assumptions (7)–(9) hold.

$$\det(j\omega E - A) \ne 0, \quad \forall \omega \in \mathbf{R} \tag{7}$$

$$\lim_{\omega\to\infty} (j\omega E - A)^{-1} \text{ exists.} \tag{8}$$

$$(E, A, B) \text{ is finite-dynamics controllable.} \tag{9}$$

Then the following two conditions are equivalent.

(i) For any $\omega \in \mathbf{R} \cup \{\infty\}$, it holds that

$$\begin{bmatrix} (j\omega E - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega E - A)^{-1}B \\ I \end{bmatrix} \le 0. \tag{10}$$

(ii) There exist matrices $X \in \mathbf{R}^{n\times n}$ and $W \in \mathbf{R}^{n\times m}$ that satisfy the following matrix equations and inequalities.

$$\begin{cases} E^\mathsf{T}X = X^\mathsf{T}E, \quad E^\mathsf{T}W = 0, \\ M + \mathbf{He}\begin{bmatrix} X^\mathsf{T} \\ W^\mathsf{T} \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \le 0. \end{cases} \tag{11}$$

Proof. See Appendix.

Remark 3. If $E = I$, we have $W = 0$, $X = X^\mathsf{T}$ and the inequality condition (11) reduces to that for state-space systems (Rantzer, 1996).

Remark 4. The result of Theorem 2 gives a necessary and sufficient condition for the dissipativity inequality (10) to hold for $\omega \in \mathbf{R} \cup \{\infty\}$. A generalization of the KYP lemma for descriptor systems with finite regions of ω has been derived (Iwasaki & Hara, 2003).

Below are corollaries of Theorem 2.

Corollary 5. Suppose that the assumptions (7) and (8) hold. Then the following two conditions are equivalent.

(i) For any $\omega \in \mathbf{R} \cup \{\infty\}$, it holds that

$$\begin{bmatrix} (j\omega E - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega E - A)^{-1}B \\ I \end{bmatrix} < 0. \tag{12}$$

(ii) There exist matrices $X \in \mathbf{R}^{n\times n}$ and $W \in \mathbf{R}^{n\times m}$ satisfying

$$\begin{cases} E^\mathsf{T}X = X^\mathsf{T}E, \quad E^\mathsf{T}W = 0, \\ M + \mathbf{He}\begin{bmatrix} X^\mathsf{T} \\ W^\mathsf{T} \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} < 0. \end{cases} \tag{13}$$

Definition 6. The descriptor system (1) is said to be *admissible* if the pencil $sE - A$ is regular and has no impulsive modes and no unstable exponential modes.

The following corollary gives a matrix inequality condition for dissipativity with admissibility.

Corollary 7. Consider partition of M as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{\mathsf{T}} & M_{22} \end{bmatrix}, \quad M_{11} \in \mathbf{R}^{n \times n}$$

and suppose that $M_{11} \geq 0$. Then the descriptor system (1) is admissible and satisfies (12) if and only if the matrix inequality (13) holds as well as $E^{\mathsf{T}}X \geq 0$.

The $(1,1)$-block of (13) is $A^{\mathsf{T}}X + X^{\mathsf{T}}A + M_{11} < 0$. Hence the inequality (13) implies that X is regular.

2.3 LMI conditions for strict positive realness and bounded realness

Here we show LMI conditions for strict positive realness and bounded realness (H_∞ norm condition) of descriptor systems, by setting matrix S appropriately.

Corollary 8. The descriptor system (1) is admissible and strictly positive real if and only if there exist matrices $X \in \mathbf{R}^{n \times n}$ and $W \in \mathbf{R}^{n \times m}$ satisfying the following LMI condition.

$$\begin{cases} E^{\mathsf{T}}X = X^{\mathsf{T}}E \geq 0, \quad E^{\mathsf{T}}W = 0, \\ \begin{bmatrix} A^{\mathsf{T}}X + X^{\mathsf{T}}A & A^{\mathsf{T}}W + X^{\mathsf{T}}B - C^{\mathsf{T}} \\ * & W^{\mathsf{T}}B + B^{\mathsf{T}}W - D - D^{\mathsf{T}} \end{bmatrix} < 0. \end{cases}$$

Corollary 9. The descriptor system (1) is admissible and H_∞ norm from w to z is less than γ if and only if there exist matrices $X \in \mathbf{R}^{n \times n}$ and $W \in \mathbf{R}^{n \times m}$ that satisfy

$$\begin{cases} E^{\mathsf{T}}X = X^{\mathsf{T}}E \geq 0, \quad E^{\mathsf{T}}W = 0, \\ \begin{bmatrix} \mathbf{He}\begin{bmatrix} X^{\mathsf{T}} \\ W^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\gamma^2 I \end{bmatrix} & * \\ \begin{bmatrix} C & D \end{bmatrix} & -I \end{bmatrix} < 0. \end{cases}$$

In Zhang et al. (2002), a necessary and sufficient condition has been proposed for strict positive realness under the assumption $D + D^{\mathsf{T}} > 0$. Matrix inequalities for bounded realness have been shown (Takaba et al., 1994; Wang et al., 1998) with $\|D\| < \gamma$ and in Masubuchi et al. (1997) with $D = 0$. The LMI condition for dissipativity presented in Rehm, & Allgöwer (2000) is not a necessary condition unless $S_{12} = 0$ and $D = 0$. Unlike these results, Theorem 2 provides a necessary and sufficient condition for every realization of a descriptor system and for every supply rate. Roughly speaking, previous results are retrieved by setting $W = 0$. As shown in Masubuchi (2004), this restriction can bring conservatism.

3. A PSEUDO-DUAL MATRIX INEQUALITY

In this section, assuming that the matrix M has the form of (5), we consider a certain 'dual' of the matrix inequality condition stated in Corollary 7. It will play an important role in the following section. Let us denote S in the following partitioned form

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathsf{T}} & S_{22} \end{bmatrix}, \quad S_{11} \in \mathbf{R}^{m \times m}$$

according to the sizes of u, y. Further, we assume that $S_{22} \geq 0$. Substituting (5) to (11) and simple manipulations yield

$$\begin{cases} E^{\mathsf{T}}X = X^{\mathsf{T}}E \geq 0, \quad E^{\mathsf{T}}W = 0, \\ \mathbf{He}\left(\begin{bmatrix} X^{\mathsf{T}} & 0 \\ W^{\mathsf{T}} & I \end{bmatrix}\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ S_{12} \end{bmatrix}\begin{bmatrix} C & D \end{bmatrix}\right) \\ \quad + \begin{bmatrix} 0 & 0 \\ 0 & S_{11} \end{bmatrix} + \begin{bmatrix} C^{\mathsf{T}} \\ D^{\mathsf{T}} \end{bmatrix} S_{22} \begin{bmatrix} C & D \end{bmatrix} < 0. \end{cases} \tag{14}$$

Since $S_{22} \geq 0$, this matrix inequality is equivalent to an LMI of decision variables (X, W), which is affine also with respect to coefficient matrices $(E; A, B, C, D)$. Further, positive semidefiniteness of S_{22} implies that X is regular, whereby we define

$$Y = X^{-\mathsf{T}}, \quad Z = -W^{\mathsf{T}}X^{-\mathsf{T}}. \tag{15}$$

From the first and second items of (14), we derive $Y^{-1}E = E^{\mathsf{T}}Y^{-\mathsf{T}}, \quad 0 = -ZY^{-1}E$ and immediately

$$EY^{\mathsf{T}} = YE^{\mathsf{T}} \geq 0, \quad ZE^{\mathsf{T}} = 0. \tag{16}$$

Multiplying

$$\begin{bmatrix} X^{\mathsf{T}} & 0 \\ W^{\mathsf{T}} & I \end{bmatrix}^{-1} = \begin{bmatrix} Y & 0 \\ Z & I \end{bmatrix} \tag{17}$$

to (14) from the left and the transpose of (17) from the right, respectively, we obtain

$$\begin{cases} EY^{\mathsf{T}} = YE^{\mathsf{T}} \geq 0, \quad EZ^{\mathsf{T}} = 0, \\ \mathbf{He}\left(\begin{bmatrix} A & B \\ S_{12}C & S_{12}D \end{bmatrix}\begin{bmatrix} Y^{\mathsf{T}} & Z^{\mathsf{T}} \\ 0 & I \end{bmatrix}\right) + \begin{bmatrix} 0 & 0 \\ 0 & S_{11} \end{bmatrix} \\ \quad + \begin{bmatrix} Y & 0 \\ Z & I \end{bmatrix}\begin{bmatrix} C^{\mathsf{T}} \\ D^{\mathsf{T}} \end{bmatrix} S_{22} \begin{bmatrix} C & D \end{bmatrix}\begin{bmatrix} Y^{\mathsf{T}} & Z^{\mathsf{T}} \\ 0 & I \end{bmatrix} < 0. \end{cases} \tag{18}$$

Proposition 10. The descriptor system (1) satisfies dissipativity as well as admissibility if and only if the matrix inequality condition (18) holds for some $Y \in \mathbf{R}^{n\times n}, Z \in \mathbf{R}^{m\times n}$.

Remark 11. The LMI condition (18) is utilized in the next section to obtain LMIs to compute control gains. If $E = I$, the inequality (18) is simplified to

$$\begin{cases} Y = Y^{\mathsf{T}} > 0, \\ \begin{bmatrix} AY + YA^{\mathsf{T}} & * \\ B^{\mathsf{T}} + S_{12}C & S_{11} + S_{12}D + D^{\mathsf{T}}S_{12}^{\mathsf{T}} \end{bmatrix} \\ \quad + \begin{bmatrix} YC^{\mathsf{T}} \\ D^{\mathsf{T}} \end{bmatrix} S_{22} \begin{bmatrix} CY^{\mathsf{T}} & D \end{bmatrix} < 0. \end{cases} \quad (19)$$

If $S_{12} = 0 \in \mathbf{R}^{m\times p}$ or $S_{12} = \pm I \in \mathbf{R}^{m\times m}$ with $m = p$ for the latter, the inequality condition (19) gives a necessary and sufficient condition for dissipativity and internal stability of the dual state-space system, with $G^{\mathsf{T}}(s) = B^{\mathsf{T}}(sI - A^{\mathsf{T}})^{-1}C^{\mathsf{T}} + D^{\mathsf{T}}$. In the general case, the inequality condition (18) may not be obtained by applying the results of the previous sections to the dual descriptor system $G^{\mathsf{T}}(s)$.

4. SYNTHESIS OF CONTROL GAINS

Based on the results on dissipativity analysis of descriptor systems in the previous sections, we consider synthesis of a control gain to attain dissipativity of the closed-loop descriptor system. Let us represent the plant as follows:

$$\begin{cases} E\dot{x} = Ax + B_1w + B_2u, \\ \quad z = C_1x + D_{11}w + D_{12}u, \end{cases} \quad (20)$$

where $x \in \mathbf{R}^n$ is the descriptor variable, $w \in \mathbf{R}^{m_1}$ is the external input, $u \in \mathbf{R}^{m_2}$ is the control input and $z \in \mathbf{R}^{p_1}$ is the controlled output. We consider two different control laws: (i) constant-gain feedback of the dynamic part of the descriptor variable and (ii) constant-gain feedback of the descriptor variable and feedforward of the external input.

First, let $K \in \mathbf{R}^{m_2\times n}$ and consider the following control input:

$$u = KEx, \quad (21)$$

by which all of the dynamic part of the descriptor variable is available to compute u. Applying this input to the plant (20) yields the closed-loop system as follows:

$$\begin{cases} E\dot{x} = (A + B_2KE)x + B_1w, \\ \quad z = (C_1 + D_{12}KE)x + D_{11}w. \end{cases} \quad (22)$$

Proposition 12. There exists a gain K for which the closed-loop system (22) is admissible and satisfies dissipativity if and only if the following LMIs hold for $Y \in \mathbf{R}^{n\times n}$, $Z \in \mathbf{R}^{m\times n}$ and $\tilde{K} \in \mathbf{R}^{m_2\times n}$:

$$\begin{cases} EY^{\mathsf{T}} = YE^{\mathsf{T}} \geq 0, \quad EZ^{\mathsf{T}} = 0, \\ \begin{bmatrix} R_{11} & * \\ R_{21} & -I \end{bmatrix} < 0, \end{cases} \quad (23)$$

where

$$R_{11} = \mathbf{He}\left(\begin{bmatrix} A & B_1 \\ S_{12}C_1 & S_{12}D_{11} \end{bmatrix} \begin{bmatrix} Y^{\mathsf{T}} & Z^{\mathsf{T}} \\ 0 & I \end{bmatrix} + \begin{bmatrix} B_2 \\ S_{12}D_{12} \end{bmatrix} \begin{bmatrix} \tilde{K}E^{\mathsf{T}} & 0 \end{bmatrix} \right) + \begin{bmatrix} 0 & 0 \\ 0 & S_{11} \end{bmatrix},$$

$$R_{21} = T_{22}^{\mathsf{T}} \begin{bmatrix} C_1 & D_{11} \end{bmatrix} \begin{bmatrix} Y^{\mathsf{T}} & Z^{\mathsf{T}} \\ 0 & I \end{bmatrix} + T_{22}^{\mathsf{T}}D_{12} \begin{bmatrix} \tilde{K}E^{\mathsf{T}} & 0 \end{bmatrix}$$

and T_{22} is any decomposition of $S_{22} \geq 0$ as $S_{22} = T_{22}T_{22}^{\mathsf{T}}$. If the LMIs (23) have a solution, without loss of generality the matrix Y can be assumed to be regular [1]. One of the gains satisfying the admissibility and dissipativity of the closed-loop system is given by $K = \tilde{K}Y^{-\mathsf{T}}$.

Proof. (Necessity) Substitute the expression of the closed-loop system (22) to the matrix inequality (18). Then we see a bilinear terms of KEY^{T} and KEZ^{T}. The latter vanishes since $EZ^{\mathsf{T}} = 0$ and by $EY^{\mathsf{T}} = YE^{\mathsf{T}}$ the former reduces to $\tilde{K}E^{\mathsf{T}}$, where $\tilde{K} = KY^{\mathsf{T}}$. Then performing Schur complement completes the proof. *(Sufficiency)* The proof of the sufficiency follows easily.

By virtue of the structure of the matrix inequality (18), the standard technique of linearizing change of variables for state-feedback gains is applicable to remove bilinear terms. This is not obvious for dissipativity inequalities of descriptor systems other than (18).

Next, consider the following control input

$$u = Fx + Gw \quad (24)$$

with $F \in \mathbf{R}^{m_2\times n}$ and $G \in \mathbf{R}^{m_2\times m_1}$. The closed-loop system is given by

$$\begin{cases} E\dot{x} = (A + B_2F)x + (B_1 + B_2G)w, \\ \quad z = (C_1 + D_{12}F)x + (D_{11} + D_{12}G)w. \end{cases} \quad (25)$$

Proposition 13. There exists a pair of gains (F, G) satisfying the admissibility and dissipativity of the closed-loop system if and only if there exist

[1] See e.g., Masubuchi et al. (1997).

matrices $Y \in \mathbf{R}^{n\times n}$, $Z \in \mathbf{R}^{m\times n}$, $\tilde{F} \in \mathbf{R}^{m_2\times n}$ and $\tilde{G} \in \mathbf{R}^{m_2\times m_1}$ such that the following LMIs hold:

$$\begin{cases} EY^{\mathsf{T}} = YE^{\mathsf{T}} \geq 0, \quad EZ^{\mathsf{T}} = 0, \\ \begin{bmatrix} R'_{11} & * \\ R'_{21} & -I \end{bmatrix} < 0, \end{cases} \tag{26}$$

where

$$R'_{11} = \mathbf{He}\left(\begin{bmatrix} A & B_1 \\ S_{12}C_1 & S_{12}D_{11} \end{bmatrix}\begin{bmatrix} Y^{\mathsf{T}} & Z^{\mathsf{T}} \\ 0 & I \end{bmatrix} + \begin{bmatrix} B_2 \\ S_{12}D_{12} \end{bmatrix}\begin{bmatrix} \tilde{F} & \tilde{G} \end{bmatrix}\right) + \begin{bmatrix} 0 & 0 \\ 0 & S_{11} \end{bmatrix},$$

$$R'_{21} = T_{22}^{\mathsf{T}}\begin{bmatrix} C_1 & D_{11} \end{bmatrix}\begin{bmatrix} Y^{\mathsf{T}} & Z^{\mathsf{T}} \\ 0 & I \end{bmatrix} + T_{22}^{\mathsf{T}}D_{12}\begin{bmatrix} \tilde{F} & \tilde{G} \end{bmatrix}.$$

If the LMIs (26) is solvable, the matrix Y can be assumed to be regular without loss of generality and one of the gains satisfying the admissibility and dissipativity of the closed-loop system is given by $F = \tilde{K}Y^{-1}$, $G = \tilde{G} - \tilde{F}Y^{-\mathsf{T}}Z^{\mathsf{T}}$.

Proof. Straightforward.

Remark 14. Sometimes it is pointed out that non-strict LMI condition $K(\xi) \geq 0$, where $K(\xi)$ is a symmetric-matrix-valued affine function of ξ, is involved with numerical problems. This is true if there exist no relatively interior point solutions to $K(\xi) \geq 0$. All the nonstrict inequalities in this paper has the form of $E^{\mathsf{T}}X = X^{\mathsf{T}}E \geq 0$ and the other inequalities are strict. The set of relatively interior point solutions to $E^{\mathsf{T}}X = X^{\mathsf{T}}E \geq 0$ is given by

$$\left\{X = L\begin{bmatrix} X_{11} & 0 \\ X_{21} & X22 \end{bmatrix}R^{-1} : X_{11} = X_{11}^{\mathsf{T}} > 0\right\},$$

where L and R are regular matrices such that $L^{\mathsf{T}}ER = \mathrm{diag}\{I_{r\times r}, 0\}$. This set is nonempty.

5. NUMERICAL EXAMPLES

Consider the following coefficient matrices for the descriptor system (1):

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 3 & 1-\kappa \\ 0 & 0 & 0 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} \kappa \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Note that this descriptor representation gives the same system for any κ.

First, we solved the LMI (23) resulting optimal $\gamma = 2.247$ for every κ. The solution $(Y, Z, \tilde{K})$ to (23) for, e.g., $\kappa = -10$ is

$$Y = \begin{bmatrix} 0.0456 & -0.001127 & -0.003379 \\ -0.001127 & 0.1079 & -0.03235 \\ 0 & 0 & 0.00727 \end{bmatrix},$$
$$Z = \begin{bmatrix} 0 & 0 & 0.9178 \end{bmatrix},$$
$$\tilde{K} = \begin{bmatrix} 0 & -0.111 & 0 \end{bmatrix}$$

and the control input is given by

$$u = \begin{bmatrix} -0.0253 & -1.0307 & 0 \end{bmatrix} x.$$

The second LMI (26) also resulted the same optimal value $\gamma = 2.247$, for every κ. For $\kappa = -10$, the solution $(Y, Z, \tilde{F}, \tilde{G})$ to (26) is derived as

$$Y = \begin{bmatrix} 0.0456 & -0.001127 & -0.00423 \\ -0.001127 & 0.1079 & -0.04325 \\ 0 & 0 & 0.00882 \end{bmatrix},$$
$$Z = \begin{bmatrix} 0 & 0 & 0.9127 \end{bmatrix},$$
$$\tilde{F} = \begin{bmatrix} 0.001641 & 0.0887 & -0.00286 \end{bmatrix},$$
$$\tilde{G} = 0.00260.$$

The corresponding control law is

$$u = \begin{bmatrix} -0.0175 & -0.9524 & -0.3244 \end{bmatrix} x + 0.2987w.$$

6. CONCLUSIONS

In this paper, we have shown matrix inequalities that provide a necessary and sufficient condition for dissipativity of descriptor systems, without additional restriction. Based on this result, we have proposed an LMI condition to synthesize a control gain of two types to satisfy dissipativity of the closed-loop system. An important extension of the results of this paper is to develop an output feedback control synthesis method, to which the solution to the full information problem of the paper can be a first step.

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Appendix A. PROOF OF THEOREM 2

Proof. Let L and R be regular matrices for which

$$L^{\mathsf{T}}(sE-A)R = \begin{bmatrix} sI - A_1 & 0 \\ 0 & s\Lambda - I \end{bmatrix}, \quad \text{(A.1)}$$

where $A_1 \in \mathbf{R}^{\bar{r}\times\bar{r}}$, and $\Lambda \in \mathbf{R}^{(n-\bar{r})\times(n-\bar{r})}$ is a nilpotent matrix. Since $\lim_{\omega\to\infty}(j\omega E - A)^{-1}$ exists, $\Lambda = 0$ and $\bar{r} = r$. Denoting $L^{\mathsf{T}}B = \begin{bmatrix} B_1^{\mathsf{T}} & B_2^{\mathsf{T}} \end{bmatrix}^{\mathsf{T}}$, we have

$$(j\omega E - A)^{-1}B = R\begin{bmatrix} (j\omega I - A_1)^{-1}B_1 \\ -B_2 \end{bmatrix} \quad \text{(A.2)}$$

and that (A_1, B_1) is controllable. Let

$$\tilde{M} = \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} M \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}.$$

Then the inequality (10) is expressed as

$$\begin{bmatrix} (j\omega I - A_1)^{-1}B_1 \\ I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \tilde{M} \begin{bmatrix} I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} (j\omega I - A_1)^{-1}B_1 \\ I \end{bmatrix} \le 0$$

From the KYP-Lemma for state space systems (Rantzer, 1996), the above inequality holds for all $\omega \in \mathbf{R} \cup \{\infty\}$ if and only if there exist a matrix $P = P^{\mathsf{T}} \in \mathbf{R}^{r\times r}$ such that

$$\begin{bmatrix} I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \tilde{M} \begin{bmatrix} I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix} + \begin{bmatrix} PA_1 + A_1^{\mathsf{T}}P & PB_1 \\ B_1^{\mathsf{T}}P & 0 \end{bmatrix} \le 0 \quad \text{(A.3)}$$

holds. By the elimination lemma, (A.3) is equivalent that for some matrices F, G, H.

$$\tilde{M} + \begin{bmatrix} PA_1 + A_1^{\mathsf{T}}P & 0 & PB_1 \\ 0 & 0 & 0 \\ B_1^{\mathsf{T}}P & 0 & 0 \end{bmatrix} + \mathbf{He}\begin{bmatrix} F \\ G \\ H \end{bmatrix}\begin{bmatrix} 0 & I & B_2 \end{bmatrix} \le 0$$

holds. This inequality is rewritten as

$$\tilde{M} + \mathbf{He}\begin{bmatrix} P & F \\ 0 & G \\ 0 & H \end{bmatrix}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & I & B_2 \end{bmatrix} \le 0,$$

which implies the inequality in (11). This is seen by setting

$$\begin{bmatrix} X^{\mathsf{T}} \\ W^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} R^{-\mathsf{T}} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} P & F \\ 0 & G \\ 0 & H \end{bmatrix} L^{\mathsf{T}} \quad \text{(A.4)}$$

and performing congruent transformation. Also it is easy to see that the equality conditions in (11) hold if and only if X and W have the form of (A.4).