AUTO-TUNING OF FRACTIONAL LEAD-LAG COMPENSATORS

C. A. Monje*, B. M. Vinagre*, A. J. Calderón*, V. Feliu and Y. Q. Chen*****

**Industrial Engineering School, University of Extremadura, Badajoz, Spain,*
email: {cmonje,bvinagre,ajcalde}@unex.es
***High Technical School of Industrial Engineering, University of Castilla-La Mancha, Ciudad Real, Spain,*
email: Vicente.Feliu@uclm.es
****CSOIS, Utah State University, Logan, Utah, USA,*
email: yqchen@ece.usu.edu

Abstract: In this paper, a method for auto-tuning of a fractional order lead-lag compensator using relay feedback tests is proposed. A design method for this kind of compensators is discussed, based on the magnitude and phase measurement of the plant to be controlled from relay feedback tests at a frequency of interest. Simple relationships among the parameters of this fractional controller are established and specifications such as the static error constant (k_{ss}), phase margin (φ_m) and gain crossover frequency (ω_c) can be fulfilled, with a robustness argument by inspecting the flatness of phase Bode plot of the compensator. The auto-tuning method proposed can be taken as a first step for a latter generalization of these lead-lag compensators to the fractional $PI^\lambda D^\mu$ controllers.

Keywords: Controller automatic tuning, relay feedback test, fractional lead compensator, fractional $PI^\lambda D^\mu$ controller, robust control.

1. INTRODUCTION

The basic concepts on auto-tuning control design were born over 1950 with the fundamental theoretical tools and formulations established in 1960. However, it was in the next decade, the 1970s, when a key method for auto-tuning was approached by K. J. Åström and Hägglund (see Åström and Wittenmark, 1973). This method consists of the use of a relay feedback test. During relay feedback tests, processes dynamics typically encountered in process control (e.g., delays) will usually generate sustaining limit cycle oscillations and subsequently, the auto-tuner can identify one point on the Nyquist curve of the process from this experiment.

In this work, the relay tests will be used for the auto-tuning of a fractional order lead-lag compensator. Fractional order calculus and its potentials in many areas of science and engineering have started to get appreciated. In particular, many research efforts related to the applications of fractional controllers have touched various aspects of control analysis and synthesis. However, in practical industrial setting, a similar auto-tuning procedure of fractional order controllers is rarely found and in strong demand. Therefore, the theme of our work is to develop auto-tuning techniques for non-integer order controllers. Our ultimate goal is develop a method to auto-tune a fractional $PI^{\lambda}D^{\mu}$ controller, formulated as:

$$C(s) = k_p + \frac{k_i}{s^{\lambda}} + k_d s^{\mu}. \qquad (1)$$

However, in this work, as a preliminary step to our research and development theme mentioned above, we consider the following fractional order compensator, a generalization of the commonly used lead-lag compensator:

$$C(s) = k_c \left(\frac{s + 1/\lambda}{s + 1/x\lambda} \right)^{\alpha} = \qquad (2)$$

$$= k_c x^{\alpha} \left(\frac{\lambda s + 1}{x \lambda s + 1} \right)^{\alpha}, \qquad 0 < x < 1,$$

where $1/\lambda = \omega_{zero}$ is the zero frequency and $1/x\lambda = \omega_{pole}$ is the pole frequency (when $\alpha > 0$). As it can be observed, this compensator corresponds to a fractional lead compensator when $\alpha > 0$ and $0 < x < 1$, and to a fractional lag compensator when $\alpha < 0$ and $0 < x < 1$. The condition $0 < x < 1$ is maintained in both cases.

In brief, in this paper a method is proposed for the auto-tuning of the fractional lead-lag compensator in equation (2), using relay tests. This paper is organized as follows. In section 2, the basic idea of relay tests is described. In section 3, a design method for the fractional order lead-lag compensator is studied and an auto-tuning method is proposed. In section 4, an illustrative example is given in details. Finally, some conclusions are drawn and future works are remarked in section 5.

2. RELAY TEST FOR AUTO-TUNING

The relay auto-tuning scheme is shown in figure 1. Let us assume that there is a limit cycle with period T_u ($\omega_u = \frac{2\pi}{T_u}$) so that the relay output is periodically symmetric. If the relay output is d, a simple Fourier series expansion of the relay output for $\epsilon = 0$ (no hysteresis) gives the first harmonic with amplitude $4d/\pi$. If it is further assumed that the process dynamic has low-pass character and that the contribution of the first harmonic dominates the output, then the output signal has the amplitude:

$$a = \frac{4d}{\pi} \left| G(j\frac{2\pi}{T_u}) \right|. \qquad (3)$$

The condition for oscillation is thus:

$$\arg \left(G(j\frac{2\pi}{T_u}) \right) = -\pi, \qquad (4)$$

$$\left| G(j\frac{2\pi}{T_u}) \right| = \frac{\pi a}{4d} = \frac{1}{N(a)}, \qquad (5)$$

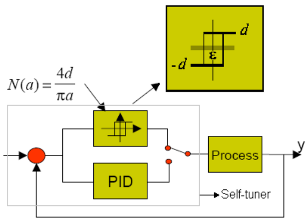

Fig. 1. Relay auto-tuning scheme

where $N(a)$ is the equivalent relay gain. Therefore, one point on the Nyquist curve of the plant can be obtained, at the frequency $\omega_u = \frac{2\pi}{T_u}$. However, this relay test has the drawback that only one frequency response point is obtainable, and it may be insufficient for describing some processes or for designing model based controllers. In order to solve this problem, an alternative scheme for the relay test can be used by introducing an artificial time delay θ_a in the loop (after the relay) to change the oscillation frequency due to the relay feedback (see Chen et al., 2003). For each value of θ_a a different point on the Nyquist curve of the plant is obtained. Therefore, a point on the Nyquist curve of the plant at a particular desired frequency ω_c can be identified, for example, at the gain crossover frequency required for the controlled system. Let us denote that the phase and magnitude at this frequency point are given by:

$$\arg (G(j\omega_c)) = -\pi + \omega_c \theta_a, \qquad (6)$$

$$|G(j\omega_c)| = \frac{\pi a}{4d} = \frac{1}{N(a)}. \qquad (7)$$

The problem then would be how to select the right value of θ_a which corresponds to a specific frequency ω_c. An iterative method can be used to solve this problem as presented in (Chen et al., 2003). The artificial time delay parameter can be updated using a simple interpolation/extrapolation scheme, as follows:

$$\theta_n = \frac{\omega_c - \omega_{n-1}}{\omega_{n-1} - \omega_{n-2}} (\theta_{n-1} - \theta_{n-2}) + \theta_{n-1},$$

where n represents the current iteration number. With the new θ_n, after the relay test, the corresponding frequency ω_n can be recorded and compared with the frequency ω_c so that the iteration can continue or stop. Two initial values of the delay (θ_{-1} and θ_0) and their corresponding frequencies (ω_{-1} and ω_0) are needed to start the iteration. The pair (θ_{-1}, θ_0) can also be easily and automatically estimated by using a scheme similar to the one for θ_n.

In this work, a fractional lead-lag compensator is used which can be easily tuned from this relay test experiment and whose parameters relations are simple and easy to determine. Besides, a robustness performance of the controlled system is achieved with the design method proposed for this kind of compensators, without increasing the complexity of these parameter relationships.

In the next section the fractional lead-lag compensator and its tuning method are explained in detail.

3. THE FRACTIONAL LEAD-LAG CONTROLLER

In this section, we focus on the fractional order lead-lag compensator (FOLLC) and introduce a new tuning parameter, α, the fractional order. An analytical method is proposed for its design, based on the lead-lag regions defined for the compensator in the complex plane, depending on the value of α. This method allows a flexible and direct selection of the parameters of the fractional structure through the knowledge of the magnitude and phase of the plant at the frequency of interest, obtained with the relay tests. Specifications of error constant, k_{ss}, gain crossover frequency, ω_c, and phase margin, φ_m, can be fulfilled, following a robustness argument based on the flatness of the phase Bode plot of the compensator.

The proposed FOLLC has the form in equation (2). This transmittance corresponds to a frequency bounded fractional derivator/integrator which is at the very origin of the CRONE control (see Oustaloup, 1995; Oustaloup, 1988). An infinite-dimensional state-space representation for this kind of controllers has been studied in (Raynaud and Zergaïnoh, 2000). It has been also used on the modeling and the feedback control laws for the stability of viscoelastic control systems (see Skaar et al., 1988).

The frequency characteristics of this fractional compensator when $\alpha > 0$ (lead compensator) are shown in figure 2. For values of $\alpha < 0$ (lag compensator) the slope of the magnitude curve is negative and the compensator introduces a phase lag. As it can be seen in the figure, the value of x sets the distance between the fractional zero (ω_{zero}) and pole (ω_{pole}) and the value of λ sets their position in the frequency axis. These two values depend on the value of α. It is observed that for a fixed pair (x, λ), the higher the absolute value of α, the higher the slope of the magnitude of $C(s)$ and the higher the maximum phase ϕ_m that the compensator can give.

As in the case of an integer lead-lag compensator, the frequency ω_m is the geometric mean of the corner frequencies ω_{zero} and ω_{pole}, and its expression is given by $\omega_m = 1/\lambda\sqrt{x}$. At this frequency the characteristics of the compensator $C(s)$ are:

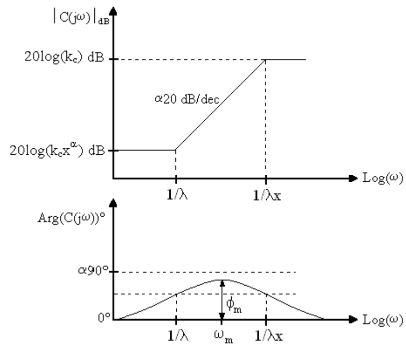

Fig. 2. Bode plots of the transmittance $C(s)$ when $\alpha > 0$

$$\left|\frac{C(s)}{k_c x^\alpha}\right|_{\omega=\omega_m} = \left|C'(s)\right|_{\omega=\omega_m} = \tag{8}$$
$$= \left(\sqrt{\frac{(\lambda\omega_m)^2+1}{(x\lambda\omega_m)^2+1}}\right)^\alpha = \left(\frac{1}{\sqrt{x}}\right)^\alpha,$$

$$\arg\left(C'(s)\right)_{\omega=\omega_m} = \phi_m = \alpha \sin^{-1}\left(\frac{1-x}{1+x}\right). \tag{9}$$

Let us give some remarks on the contribution of the parameter α. For a fixed set $(x_n, \lambda_n, \alpha_n)$ and considering variations in the parameter α_n, it is observed that the lower the value of α is (in comparison with its nominal value α_n), the longer the distance between the zero and pole must be, and vice versa, in order to compensate the variation of phase produced by the variation of α. This fact makes the controller more flexible and allows considerations of robustness in the design. This point will be explained in more detail next.

3.1 Method of Design

Let us imagine that the fractional lead-lag compensator in equation (2) has to be tuned for a general unknown process:

$$G(s) = \frac{k\prod_i(\tau_i s+1)}{s^n\prod_j(\tau_j s+1)}. \tag{10}$$

The static error constant $k_{ss} = \lim_{s\to 0} s^n G(s)$ can be measured and it is assumed to be known (in type 0 systems, $n = 0$, the static error constant is equal to the static gain of the system $G(0)$). Therefore, the compensator gain $k' = k_c x^\alpha$ can be set in order to fulfill an error constant specification for the compensated system, being both related by the expression $k_{ss} = k'k$.

For a specified phase margin (φ_m) and gain crossover frequency (ω_c), the following relationship for the open loop system can be given in the complex plane:

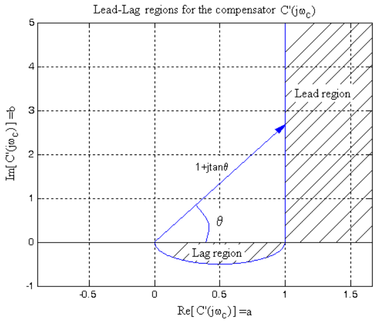

Fig. 3. Lead and lag regions for the integer order compensator

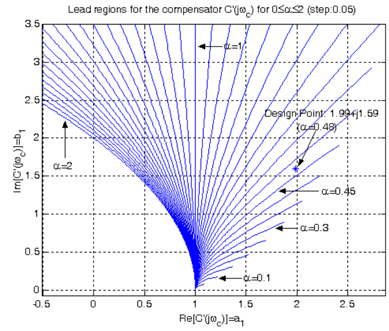

Fig. 4. Lead regions for the fractional compensator for $0 \leq \alpha \leq 2$

$$G(j\omega_c) \cdot k' \left(\frac{j\lambda\omega_c + 1}{jx\lambda\omega_c + 1} \right)^{\alpha} = e^{j(\pi + \varphi_m)} \Rightarrow \quad (11)$$

$$\Rightarrow \left(\frac{j\lambda\omega_c + 1}{jx\lambda\omega_c + 1} \right)^{\alpha} = \frac{e^{j(\pi + \varphi_m)}}{k' G(j\omega_c)} = a_1 + jb_1 \Rightarrow$$

$$\Rightarrow \left(\frac{j\lambda\omega_c + 1}{jx\lambda\omega_c + 1} \right) = (a_1 + jb_1)^{1/\alpha} = a + jb,$$

where $G(s)$ is the plant to be controlled and (a_1, b_1) is called in this paper the "*design point*". After some simple calculations, the expressions for x and λ can be given by:

$$x = \frac{a - 1}{a(a-1) + b^2}, \quad (12)$$

$$\lambda = \frac{a(a-1) + b^2}{b\omega_c}.$$

Studying the conditions for a and b to find a solution, it can be concluded that a lead compensator is obtained when $a > 1$ and $b > 0$, and a lag compensator when $\frac{1-\sqrt{1-4b^2}}{2} < a < \frac{1+\sqrt{1-4b^2}}{2}$ and $-1/2 < b < 0$. Figure 3 shows these lead and lag regions in the complex plane for the integer order compensator.

Let us focus firstly on the lead compensation. It is clear that for the conventional lead compensator ($\alpha = 1$) the vector $a + jb = a_1 + jb_1$ is perfectly known through the knowledge of the plant (relay test) and the specifications of phase margin and gain crossover frequency required for the system, as it can be seen in (11). Knowing the pair (a, b), the values of x and λ are directly obtainable by (12), and the compensator is then designed.

As shown in figure 3, the vector $1 + j\tan\theta$ defines the borderline of the lead region. Using the polar form of this vector:

$$\sqrt{1 + \tan^2\theta}\, e^{j\theta} = \frac{1}{\cos\theta} e^{j\theta}, \quad (13)$$

and expressing the vector $(a_1 + jb_1)^{1/\alpha}$ in its polar form:

$$\left(\sqrt{a_1^2 + b_1^2} \right)^{1/\alpha} e^{j \frac{\tan^{-1}(b_1/a_1)}{\alpha}} = \rho^{1/\alpha} e^{j\frac{\delta}{\alpha}}, \quad (14)$$

where $\rho = \left(\sqrt{a_1^2 + b_1^2} \right)$ and $\delta = \tan^{-1}(b_1/a_1)$, the following relationships can be established from (11):

$$\delta = \theta\alpha, \quad (15)$$

$$\rho^{1/\alpha} = \frac{1}{\cos\theta} \Rightarrow 1 = \rho \left[\cos\left(\frac{\delta}{\alpha} \right) \right]^{\alpha}.$$

Then, solving numerically the function $1 = \rho \left[\cos\left(\frac{\delta}{\alpha} \right) \right]^{\alpha}$, the lead compensation regions in the complex plane for different positive values of α are obtained, as shown in figure 4. The zone to the right of each curve is the lead region, and any design point in this zone can be fulfilled with a fractional compensator having a value of α equal or bigger than the one defining the curve which passes through the design point ($\alpha_{\min}$). For instance, for the design point in figure 4, the value of $\alpha_{\min}$ is 0.48. By choosing the minimum value $\alpha_{\min}$, the distance between the zero and the pole of the compensator will be the maximum possible (minimum value of parameter x). In this case, the phase curve of the compensator is the most flat possible and variations in a frequency range centered at ω_c will not produce a significant phase change as in other cases, improving the robustness of the system.

Figure 5 shows the pairs (x, λ) obtained for each value of α in the range $\alpha_{\min} \leq \alpha \leq 2$, with $\alpha_{\min} = 0.48$ (compensation of the design point in figure 4). It is observed that the minimum value of x is obtained for $\alpha_{\min}$ (maximum robustness). Therefore, through the curves in Figs 4 and 5 the selection of the parameters of the compensator is flexible and direct.

This method proposed for the fractional lead compensator (Monje et al., 2004) can be used here for the design of a fractional lag compensator with some modifications that are explained next.

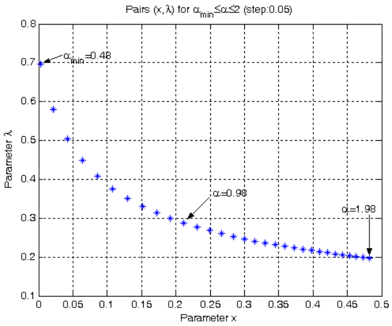

Fig. 5. Pairs (x, λ) for $\alpha_{\min} \leq \alpha \leq 2$

First of all, to determine whether a lead or lag compensator is required to fulfill the specification of phase margin, a simple computation has to be done:

1) If $\pi + \arg(G(j\omega_c)) < \varphi_m \rightarrow$ Lead Compensator
2) If $\pi + \arg(G(j\omega_c)) > \varphi_m \rightarrow$ Lag Compensator

In case a phase lag (φ_{lag}) is required for the system, the compensator will be designed as a lead one giving a phase $\varphi_{lead} = -\varphi_{lead}$, and then the sign of α is changed. Therefore, in order to keep the specification of phase margin, the phase of the compensator (φ_{lag}) will now be given by:

$$\varphi_{lag} = -\pi + 2[\pi + \arg(G(j\omega_c))] - \varphi_m - \arg(G(j\omega_c)). \quad (16)$$

Let us remember that in the case of a lead compensation the phase of the compensator (φ_{lead}) is given by $\varphi_{lead} = -\pi + \varphi_m - \arg(G(j\omega_c))$. Besides, it has to be taken into account that the fact of changing the sign of α for the lag compensation also changes the magnitude of the compensator designed (makes it the inverse). So, in order to keep the gain unchanged (fulfilling already the specification of crossover frequency), the lag compensator should be multiplied by a gain $k_{lag} = 1/(k' \cdot |G(j\omega_c)|)^2$. Therefore, the fractional lag compensator will be given by:

$$C_{lag}(s) = k_{lag} k' \left(\frac{\lambda s + 1}{x \lambda s + 1} \right)^{-\alpha}, \quad (17)$$

with α a positive real number.

4. AN ILLUSTRATIVE EXAMPLE

In this section the auto-tuning method proposed will be illustrated. First, with the relay test the value of the magnitude and phase of the plant to control will be obtained at the crossover frequency, ω_c. With these values and the value of the desired phase margin, φ_m, the "design point"

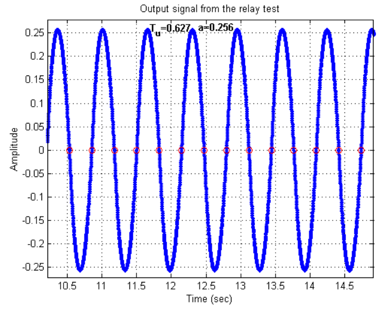

Fig. 6. Output signal from the relay test

$a_1 + jb_1$ is defined, and the parameters of the compensator are then obtained by simple calculations, following the robustness feature explained in the previous section. In this case the plant to be controlled is a position servo given by:

$$G(s) = \frac{2}{s(0.5s + 1)}.$$

The design specifications considered here are just taken as an example of application. In our case, the gain crossover frequency is specified as $\omega_c = 10 rad/\sec$. The relay will have an output amplitude of $d = 5$, without hysteresis, $\epsilon = 0$. The two initial values (θ_{-1} and θ_0) of the delay used to reach the frequency specified are 0.05 sec and 0.02 sec, respectively. The iterative process explained before is done and finally it is obtained the output signal shown in figure 6.

The value of the delay θ_a obtained for the selection of the frequency specified is $\theta_a = 0.0180 \sec$, and the corresponding frequency is $\omega_u = 10.0088 rad/\sec$. The amplitude and period of this oscillatory signal are $a = 0.256$ and $T_u = 0.627 \sec$, respectively. Therefore, the magnitude and phase of the plant estimated through the relay experiment at the frequency $\omega_u = 10.0088 rad/\sec$ are $|G(j\omega_u)|_{dB} = -28.1188 dB$ and $\arg(G(j\omega_u)) = -169.6534°$, respectively. At this frequency, the plant has a real magnitude of $-28.1441 dB$ and a phase of $-168.6997°$. So, only a slight error is committed in the estimation.

Next, a fractional compensator will be designed with the proposed tuning method to obtain a velocity error constant $k_v = 20$ and a phase margin of $\varphi_m = 50°$ at the gain crossover frequency $\omega_c = 10 rad/\sec$. As it can be observed, a lead compensator is needed in this case. Using the method proposed in the previous section, the resulting compensator is:

$$C(s) = 10 \left(\frac{0.6404s + 1}{0.0032s + 1} \right)^{0.5},$$

with $k' = 10$, $x = 0.0050$, $\lambda = 0.6404$ and $\alpha = 0.5$. The Bode plots of this compensator are shown in figure 7. At the crossover frequency

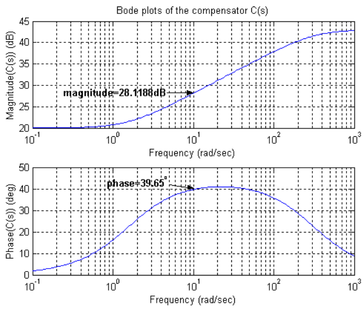

Fig. 7. Bode plots of the compensator C(s)

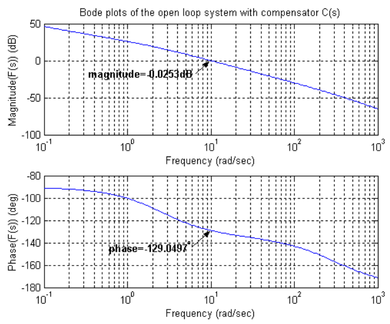

Fig. 8. Bode plots of the open loop system with compensator $C(s)$

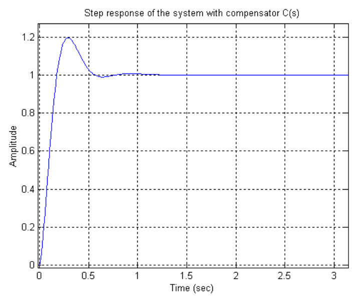

Fig. 9. Step response of the system with compensator $C(s)$

$\omega_c = 10 rad/\sec$, the compensator has a magnitude of $28.1188 dB$ and a phase of $39.65°$. At that frequency the magnitude of the open loop system is $-0.0253 dB$, and the phase margin obtained is $50.9503°$ (see figure 8). So, the specifications are fulfilled with only a slight error.

For the sake of implementation a discrete version of the controller must be used. Several discretization methods can be considered, for instance the Tustin method (see Vinagre et al., 2003). However, for simulation, this compensator has been realized by using a frequency identification method (Matlab function *invfreqs*), with a 4^{th} order numerator and denominator, resulting the step response of the closed-loop system shown in figure 9.

5. CONCLUDING REMARKS AND FUTURE WORKS

In this paper an auto-tuning method for the fractional lead-lag compensator using the relay test has been proposed. This method allows a flexible and direct selection of the parameters of the compensator through the knowledge of the magnitude and phase of the plant at the frequency of interest, obtained with the relay tests. Specifications of error constant, k_{ss}, gain crossover frequency, ω_c, and phase margin, φ_m, can be fulfilled with a robustness property based on the flatness of the phase curve of the compensator. The simulation results illustrates the effectiveness of the control tuning method proposed.

We are currently working on the generalization of this auto-tuning method to the fractional PID controller ($PI^{\lambda}D^{\mu}$). Besides, good experimental results have been obtained from the implementation of the auto-tuning method proposed here.

6. REFERENCES

Åström, K. J. and B. Wittenmark (1973). On Self-tuning Regulators. *Automatica,* **9**, 185-199.

Chen, Y. Q. and K. L. Moore (2005). Relay Feedback Tuning of Robust PID Controllers with Iso-Damping Property. *IEEE Transactions on Systems, Man, and Cybernetics, Part-B,* **35** (1), 23-31.

Monje, C. A., A. J. Calderón, B. M. Vinagre, and V. Feliu (2004). The Fractional Order Lead Compensator. In *Proceedings of the IEEE International Conference on Computational Cybernetics,* pp. 347-352. Vienna, Austria.

Oustaloup, A (1998). From Fractality to Non Integer Derivation Through Recursivity, a Property Common to these two Concepts: A Fundamental Idea from a new Process Control Strategy. In *Proceedings of the 12th IMACS World Congress*, pp. 203-208. Paris, France.

Oustaloup, A (1995). *La Dérivation Non Entière.* Hermès, Paris.

Raynaud, H. F., and A. Zergaïnoh (2000). State-Space Representation for Fractional Order Controllers. *Automatica*, **36**, 1017-1021.

Skaar, S. B., A. N. Michel, and R. K. Miller (1998). Stability of Viscoelastic Control Systems. *IEEE Transactions on Automatic Control*, **33** (4), 348-357.

Vinagre, B. M., Y. Q Chen and I. Petráš (2003). Two direct Tustin Discretization Methods for the Fractional-Order Differentiator/Integrator. *Journal of The Franklin Institute,* **340,** 349-362.