INTEGRATED OPTIMIZATION OF STRUCTURES AND LQR CONTROL SYSTEMS FOR REDUCED ORDER MODELS

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Abstract: In this study, integrated optimum design of structures and control systems is studied by using reduced order models. The structures and controllers are optimized simultaneously and successively. Since the degree-of-freedom (DOF) for structures is very large in practice, model order reduction techniques have to be employed at every controller design iteration during optimization in integrated optimum design approaches, that increase the CPU time and involves modeling errors. In this study, Subspace Based Identification (SBI) method is used as a model order reduction technique in frequency domain. It is shown that simultaneous optimization of structures and controllers by using LQR formulations can be achieved by an equivalent decoupled optimization problem where structures are optimized by shaping the structural singular values and following any control law of interest can be designed. Decoupled optimization of structures and controllers has certain advantages, especially for structures having large DOF.

Keywords: Structural optimization, optimum control, singular value decomposition, finite element method, model order reduction.

1. INTRODUCTION

In this study, linear quadratic regulator (LQR) formulation is applied to the integrated optimum design of structures and control systems problem. If the degree-of-freedom (DOF) for structures were very large, model order reduction techniques would have to be employed at every LQR design iteration during optimization to be able to design a control law in integrated optimum design approaches, that increases the CPU time and involves modeling errors. It is shown that LQR formulations can be approximated by a decoupled optimization problem for a structure and its controller in which the structure is optimized by shaping the structural singular values and then the controller can be designed in any desired way; this approach has advantages such as yielding optimized structures faster, allowing decoupled design of the structure and its controller, eliminating the modeling errors due to employment of model order reduction techniques to design controllers, and being feasible for structures having large DOF. The relations between LQR formulations and total strain and kinetic energy of a structure are derived, which has also links with a structural singular value shaping problem. The outline of this study is as follows: Subspace-Based Identification Method is defined in Section 2. LQR problem is defined in Section 3. Following, the relations for optimum design of controlled structures are presented in Section 4. Numerical examples are given in Section 5 and conclusions are drawn in Section 6.

2. SUBSPACE-BASED IDENTIFICATION METHOD

SBI method estimate the parameters of state space representation of the transfer function. In this study, parameters of discrete time state space form are estimated and after that, by using the parameter, frequency and step response of discrete system are obtained. To do this, in algorithm, the row size for the square Hankel matrix is suggested to be at most half of the number of sampled continuous time frequency response data and entered by the user. Also, the incremental step size for the model order and maximum model order are entered by the user (e.g., McKelley and Akçay , 1996).

2.1. Problem Formulation

Assume that G is a stable, multi-input, multi-output (MIMO), linear time-invariant, discrete-time system with input-output properties characterized by the impulse response coefficients g_k through the equation

$$y(t) = \sum_{k=0}^{\infty} g_k u(t-k) \tag{1}$$

where $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ and $g_k \in \mathbb{R}^{p\,x\,m}$. We also assume that the system is of finite order n and can thus be described by a state-space model

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{2}$$

where $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ and $x(t) \in \mathbb{R}^n$. The state space model (2) has the impulse response

$$g_k = \begin{cases} D, & k = 0 \\ CA^{k-1}B, & k > 0. \end{cases} \tag{3}$$

The frequency response of (2) is calculated as

$$G(e^{j\omega}) = \sum_{k=0}^{\infty} g_k e^{-j\omega k}, \quad 0 \le \omega \le \pi \tag{4}$$

which for the state space model (2) can be written

$$G(e^{j\omega}) = C(e^{j\omega}I - A)^{-1}B + D. \tag{5}$$

In (4), $j = \sqrt{-1}$ is the imaginary unit.

2.2. Uniformly Spaced Data

This section is devoted to the case of uniformly spaced data. Assume that M + 1 frequency response data G_k on a set of uniformly spaced frequencies

$$\omega_k = \frac{\pi k}{M}, \qquad k = 0, \ldots, M \tag{6}$$

are given. If the impulse response coefficients (3) are given, well-known realization algorithms can be used to obtain a state space realization. The algorithm presented in this section is closely related to these results, but uses the coefficients of the inverse discrete Fourier transform (IDFT) from samples of the frequency response function. Since G is a transfer with a real valued impulse response (1), frequency response data on $[0, \ \pi]$ can be extended to $[\pi, \ 2\pi]$ by taking the complex conjugate of the given data G_k which forms the first step of the identification algorithm.

Algorithm:

1) Extend the transfer function samples to the full unit circle

$$G_{M+k} := G^{*}_{M-k}, \qquad k = 0, \ldots, M \tag{7}$$

where $(.)^{*}$ denotes complex conjugate.

2) Let $\hat{h}_i$ be defined by the 2M-point IDFT

$$\hat{h}_i := \frac{1}{2M} \sum_{k=0}^{2M-1} G_k e^{j2\pi ik/2M}, \quad i = 0, ,2M\text{-}1. \tag{8}$$

3) Let the block Hankel matrix $\hat{H}$ be defined as

$$\hat{H} := \begin{bmatrix} \hat{h}_1 & \hat{h}_2 & \cdots & \hat{h}_r \\ \hat{h}_2 & \hat{h}_3 & \cdots & \hat{h}_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{h}_q & \hat{h}_{q+1} & \cdots & \hat{h}_{q+t-1} \end{bmatrix} \in \mathbb{R}^{q\,p\,x\,r} \tag{9}$$

With number of block rows q > n and block columns r≥ n. The dimension of $\hat{H}$ is bounded by q + r ≤ 2M.

4) Calculate the singular value decomposition (SVD) of the Hankel matrix

$$\hat{H} = \hat{U}\hat{\Sigma}\hat{V}^T.$$

5) Determine the system order n by inspecting the singular values and partition the SVD such that $\hat{\Sigma}_s$ contains the n largest singular values

$$\hat{H} = \begin{bmatrix} \hat{U}_s & \hat{U}_o \end{bmatrix} \begin{bmatrix} \hat{\Sigma}_s & 0 \\ 0 & \hat{\Sigma}_o \end{bmatrix} \begin{bmatrix} \hat{V}^T \\ \hat{V}_o^T \end{bmatrix} \tag{10}$$

6) Determine the system matrices $\hat{A}$ and $\hat{C}$ as

$$\hat{A} = (J_1 \hat{U}_s)^{\dagger} J_2 \hat{U}_s \tag{11}$$

$$\hat{C} = J_3 \hat{U}_s \tag{12}$$

where

$$J_1 = \begin{bmatrix} I_{(q-1)p} & 0_{(q-1)pxp} \end{bmatrix} \tag{13}$$

$$J_2 = \begin{bmatrix} 0_{(q-1)pxp} & I_{(q-1)p} \end{bmatrix} \tag{14}$$

$$J_3 = \begin{bmatrix} I_p & 0_{px(q-1)p} \end{bmatrix} \tag{15}$$

and I_i denotes the i x i identify matrix, $0_{i\,x\,j}$ the i x j zero matrix, and $X^{\dagger} = (X^T X)^{-1} X^T$ the Moore-Penrose pseudoinverse of the full column rank matrix X.

7) Solve a least squares problem to determine $\hat{B}$ and $\hat{D}$

$$\hat{B}, \hat{D} = \arg \min_{\substack{B \in \mathbb{R}^{n\,x\,m} \\ D \in \mathbb{R}^{p\,x\,m}}} \sum_{k=0}^{M} \left\| G_k - D - \hat{C}(e^{j\omega_k} I - \hat{A})^{-1} B \right\|_F^2 \tag{16}$$

where $\|X\|_F = \sum_k \sum_s |x_{ks}|^2$ denotes the Frobenius norm.

8) The estimated transfer function is defined as

$$\hat{G}^M(z) = \hat{D} + \hat{C}(zI - \hat{A})^{-1} \hat{B}. \tag{17}$$

Notice that B and D appear linearly in the transfer function for fixed A and C. Hence, the optimization (16) has an analytical solution.

3. LQR DESIGN METHOD

Let a linear time-invariant structure be described by the following standard form (Zhou, Doyle and Glover, 1996).

$$\dot{x} = Ax + B_2 u, \tag{18}$$

$x(0) = x_0$ given but arbitrary

$$z = C_1 x + D_{12} u \tag{19}$$

Then, standard LQR design problem is defined by Problem definition: Find an optimal control law $u \in L_2[0,\infty)$ such that the performance criterion $\|z\|_2^2$ is minimized (Anderson and Moore, 1990). On the other hand, extended LQR problem is determined as follow:

Problem definition: Find an optimal control law $u \in L_2[0,\infty)$ such that the system is internally stable, i.e., $x \in L_2[0,\infty)$ and the following performance criterion minimized (Anderson and Moore, 1990)

$$J = \int_0^t \left(x^T Q x + u^T R u\right) dt \tag{20}$$

where $Q = Q^T \geq 0$ and $R = R^T > 0$ are the penalty matrices for the states *x* and inputs *u*, respectively. Optimal gain matrix K is calculated such that the state-feedback law $u = -Kx$ minimizes the cost function *J*.

4. OPTIMUM DESIGN OF CONTROLLED STRUCTURES

Structural and controller parameters should be optimized simultaneously or successively by using iterative optimization algorithms to minimize the cost function *J* in (20) where the penalty matrices *Q* and *R* are given a priori by the designer.

4.1 Successive Optimization of Structures and LQR Design Problem

It is shown below that integrated optimum design of a structure and LQR can be decoupled and cast as a structural singular value shaping problem. Consider the following LQR cost function for the infinite horizon problem (Anderson and Moore, 1990)

$$\min_X J = \min_X \int_0^\infty (x^T Q x + u^T R u) dt \tag{21}$$

where *X* is the design parameter vector. On the other hand, suppose that finite element methods are employed to obtain structural equations and semidiscrete equations of the structure are given by

$$M\ddot{z} + C\dot{z} + Kz = f \tag{22}$$

Note that the equations of the structure given by (22) can easily be cast into the state space form (1) by defining the state vector as follows

$$x = \begin{bmatrix} z & \dot{z} \end{bmatrix} \tag{23}$$

Since, the strain energy *U* and kinetic energy *T* of the structure can be written as follows (Haug and Choi, 1986).

$$U = \frac{1}{2} z^T K z \tag{24}$$

$$T = \frac{1}{2} \dot{z}^T M \dot{z} \tag{25}$$

Then, total energy *E* of the system is given by

$$E = U + T = \begin{bmatrix} z & \dot{z} \end{bmatrix} \begin{bmatrix} \frac{1}{2}K & 0_{nxn} \\ 0_{nxn} & \frac{1}{2}M \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix} \tag{26}$$

where 0_{nxn} is the zero matrix of size *n* by *n,* and *n* is the degree-of-freedom (DOF) of the associated structure. Note that the second term in (22) is the damping term and is responsible for dissipated energy. Lets assume that the state penalty matrix *Q* in (21) is equal to the following

$$Q = \begin{bmatrix} \frac{1}{2}K & 0_{nxn} \\ 0_{nxn} & \frac{1}{2}M \end{bmatrix} \tag{27}$$

where the corresponding states of the structure are defined by (23); subsequently, LQR formulation means that minimization of total structural energy, because minimization of *J* defined by (21) requires minimization of each term in (21) and the first term yields total structural energy for this definition of *Q*. In general, for an arbitrary *Q* matrix, a state weighting matrix W_s can be found (Horn and Johnson, 1995) such that

$$Q = W_s^T \begin{bmatrix} \frac{1}{2}K & 0 \\ 0 & \frac{1}{2}M \end{bmatrix} W_s \tag{28}$$

that corresponds to a structure whose states are defined by

$$x' = W_s \begin{bmatrix} z \\ \dot{z} \end{bmatrix} \tag{29}$$

Since the state penalty matrix *Q* is usually chosen as a diagonal matrix, the weighting matrix W_s can be chosen as follows

$$W_s = \begin{bmatrix} X_k & 0 \\ 0 & X_m \end{bmatrix} \tag{30}$$

where the columns of matrices X_k and X_m are the eigenvectors of matrices *K* and *M,* respectively. If the matrix *Q* is not chosen to be diagonal, then the matrix equation (28) should be solved for W_s for the given *Q* matrix. In sum, for an arbitrary state penalty matrix *Q*, minimization of *J* defined by (21) corresponds to minimization of total energy of the structure whose states are defined by (29); that is, total structural energy is minimized in a weighted sense where states are weighted as (29).

Meanwhile, due to (Postlethwaite and Edmunds, 1981), for a deterministic aperiodic input vector *u(t),* the energy-density ratio is bounded by the squares of the maximum and minimum singular values as follows

$$\sigma_n^2(\omega) \le \frac{\|\hat{y}(j\omega)\|_2^2}{\|\hat{u}(j\omega)\|_2^2} \le \sigma_1^2(\omega) \tag{31}$$

where $y(t)$ is the output vector in response to the input vector $u(t)$, and $\hat{y}(j\omega)$ and $\hat{u}(j\omega)$ are respectively the *Fourier transforms* of $y(t)$ and $u(t)$ defined by

$$\hat{y}_i(j\omega) = \int_{-\infty}^{+\infty} y_i(t) e^{-j\omega t} dt \tag{32}$$

$$\hat{u}_i(j\omega) = \int_{-\infty}^{+\infty} u_i(t) e^{-j\omega t} dt \tag{33}$$

Hence, for an arbitrary penalty matrix Q, minimization of total structural is equivalent to minimization of σ_1 and σ_n of the structure whose states are defined by (29).

In parallel to the above conclusion, the objective functions used for decoupled optimization of structures are related to total structural energy as well. Recall that G and P denote transfer matrices for disturbance to output and reference input to output, respectively. Then, minimization of $\sigma_1(G(j\omega))$ in decoupled structural optimization problems is equivalent to minimization of total structural energy E_d in response to the disturbance input d. Similarly, maximization of $\sigma_1(P(j\omega))$ and $\sigma_n(P(j\omega))$ in decoupled structural optimization problems is equivalent to maximization of total structural energy E_u in response to the control input u. Consequently, (45) is equivalent to the following energy ratio

$$\underset{\omega}{Minimize} \frac{E_d(\omega)}{E_u(\omega)} \tag{34}$$

or equivalently

$$\underset{\omega}{Minimize}\, E_d(\omega) \text{ while } \underset{\omega}{Maximize}\, E_u(\omega) \tag{35}$$

Now, the second term in (21) will be studied. Let $\tilde{u}$ denote the magnitude of input vector (i.e., $u = u(t) = \tilde{u} Sin\, \omega t$). Then, the following holds for a structure whose input-output transfer matrix is $T(s)$ (Postlethwaite and Edmunds, 1981)

$$\sigma_n^2(\omega) \le \frac{SMSVO(\omega)}{SMSVI} \le \sigma_1^2(\omega) \tag{36}$$

where $\sigma_i(\omega)$ is the ith singular value of $T(j\omega)$ (namely, structural singular values), $SMSVO(\omega)$ denotes "sum of the mean-squared values of the m steady-state outputs y_{ss} over one period" given by

$$SMSVO(\omega) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} y_{ss}^T(t) y_{ss}(t) dt = \frac{1}{2} \tilde{u}^T S(\omega) \tilde{u} \tag{37}$$

$SMSVI$ denotes the "sum of the mean-squared values of the inputs over one period" given by

$$SMSVI = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} u^T(t) u(t) dt = \frac{1}{2} \tilde{u}^T \tilde{u} \tag{38}$$

And

$$S(\omega) = \frac{1}{2}\left[T^*(j\omega)T(j\omega) + \overline{T^*(j\omega)T(j\omega)}\right] \tag{39}$$

where a superposed asterisk denotes conjugated transpose and over line denotes conjugation. Assuming that the control input penalty matrix R is decomposed into the following form

$$R = W_R^T \Lambda W_R \tag{40}$$

where W_R is orthogonal and Λ is diagonal (Horn and Johnson, 1995). By setting

$$\tilde{u} = \Lambda^{1/2} W_R u \tag{41}$$

Then, (21) becomes

$$SMSVI = \frac{1}{2}\tilde{u}^T\tilde{u} = \frac{1}{2} u^T W_R^T \Lambda^{1/2} \Lambda^{1/2} W_R u = \frac{1}{2} u^T R u \tag{42}$$

which means that $SMSVI$ of the system for the weighted input given by (41) is equal to the last term. Since both (31) and (43) involve only σ_1 and σ_n, selective eigenvalue solvers can be used that reduces the computational cost of the associated optimization problem (Muğan A, 2002).

In brief, decoupled LQR formulation for optimum design of structures and controllers are equivalent to shaping of structural singular values in frequency domain.

$$\frac{1}{\sigma_1^2(\omega)} \le \frac{SMSVI}{SMSVO(\omega)} \le \frac{1}{\sigma_n^2(\omega)} \tag{43}$$

It is noteworthy that while $SMSVI$ is the function of the input u, structural singular values σ_1 and σ_n depend only on structural parameters, and $SMSVO$ is the function of both input u and structural parameters.

Note that the first and second terms in the objective function (21) are respectively equivalent to conflicting structural singular value shaping problems expressed by (31) and (43). In fact, there is a trade off in the objective function J defined by (21) due to the contributions of the first and second terms, which is balanced by using the penalty matrices Q and R. Subsequently, The following objective function is used in numerical solutions

$$\underset{\omega}{Minimize}\; A(\sigma_1(\omega) + \sigma_n(\omega)) + B\left(\frac{1}{\sigma_1(\omega)} + \frac{1}{\sigma_n(\omega)}\right) \tag{44}$$

which is a decoupled optimization problem for the structure. The coefficients A and B can be selected independently to penalize total structural energy and *SMSVI*, respectively. Accordingly, Thus, minimization of J means that minimization of *SMSVI* for the weighted input.

5. NUMERICAL EXAMPLES

In this study, a truss system having 46 DOF is reduced to 7 DOF by using FEM and SBI method. Then, for the truss system and its controller described, the associated simultaneous and successive optimum design problems are solved by using LQR formulations. The reduced order model is obtained by the SBI method, whose frequency response is given in Figure 2. The truss system formed of seven truss elements is shown in Figure 1 and structural parameters are as follows: elasticity modulus is $E = 2.1x10^9 N/m^2$, truss element lengths are $l = 1$ *m*, material density is $\rho = 7850 kg/m^3$. Note that unlike robust control problems, existence of disturbance load is not important in designing LQR; hence, it is ignored. Sequential Quadratic Programming method is employed for optimization, and the optimization problems are solved as a constrained optimization problem in which the maximum stress $(\tau_{ij})_{\max}$ in the truss system is bounded by a priori safety stress τ_{safety} = 6000 N/cm^2

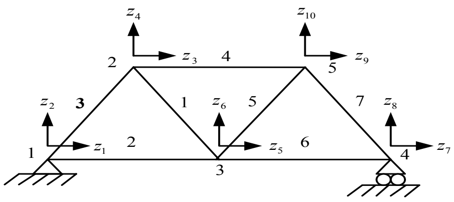

Fig. 1. Truss system.

for St37 steel material for safety factor is chosen to be four. Design parameters for the truss system are the cross-sectional areas h_i of truss elements whose lower and upper bounds are respectively set to be $LB = 1.4x10^{-4} m^2$ and $UB = 12x10^{-4} m^2$ in the optimization algorithm. Mass of the structure is lumped at nodes. Rayleigh damping of the form C=αK is assumed in numerical simulations where α=0.01. In order to find the global minimum, initial conditions for the structural parameters are changed during numerical solutions.

State feedback matrix *K* for the control input *u*=-*Kx* computed by LQR formulations. For the state penalty matrix *Q* and control penalty matrix *R* in (21), identity matrices are selected in all design studies for simplicity. Arbitrary *Q* and *R* matrices could also be selected as elaborated in Section 3.1.

For successive optimization, the following cost function is minimized by choosing *A*=0 *and B*=1 in (44)

$$\underset{\omega}{Minimize} \frac{1}{\sigma_1(P(j\omega))} + \frac{1}{\sigma_n(P(j\omega))} \qquad (45)$$

that yields almost the same results obtained by simultaneous optimization of structures and controllers. Note that since $\sigma_n(P(j\omega))$ is very small (in the order of 10^{-5}) and $\sigma_1(P(j\omega))$ is in the order of 10^2, the term $1/\sigma_n(P(j\omega))$ is dominant and subsequently the cost function of $1/\sigma_n(P(j\omega))$ yields the same results as those of (45). The values of the cost function *J* in response to an impulse, the corresponding H_2 norms of the transfer matrix and values of cost function (45) are given in Figures 3 and 4 for respectively simultaneous and successive optimum design of the truss system and LQR as the total cross-sectional area of bars varies in the solution sets. It is the designer's task to choose the best solution among the solution.

Observe that the optimum solution obtained by successive optimization (e.g., see Table 1) is better than that of simultaneous optimization (e.g., see Table 2) approach.

6. CONCLUSION

It is pointed out that using LQR in simultaneous optimization problems has certain disadvantages such as necessity of employing model order reduction techniques for the structure at every design iteration that is a slow process involving significant approximation errors especially for structures having large DOF. It is noteworthy that optimum design of controlled structures is a multi-objective optimization problem. Simultaneous optimization of a structure and its controller can be approximated by a decoupled (successive) optimum design approach as follows: beforehand, the structure is optimized by shaping the structural singular values, then the controller can be designed by any method of interest. The objective functions for structural optimization in successive optimization approach are expressed in terms of structural singular values. Computational cost of the associated singular value shaping problem is very low even though the DOF of FEM model is large, since it is only necessary to compute the largest and smallest singular values that can be computed by using selective eigenvalue solvers and the other singular values are not needed. It is observed that solutions of simultaneous LQR optimization problem can be obtained by the successive optimization approach as well.

Model order reduction techniques should be employed at every design iteration for simultaneous optimization approach in order to design LQR laws; In successive optimization approach, the controller is designed after the structure is optimized; hence, it needs much less CPU times than simultaneous optimization approach. In the case that no model order reduction technique is employed, overall CPU times of successive optimization approach lower than those simultaneous optimization approach.

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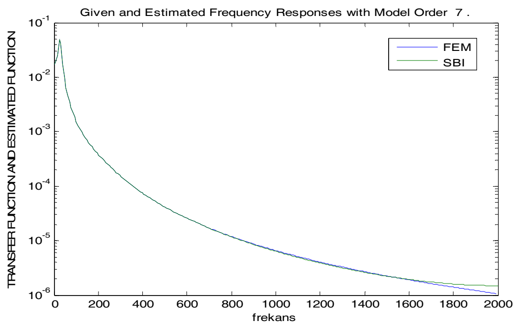

Fig. 2. Frequency response of the reduced order model between 0 to 2000 rad/sec.

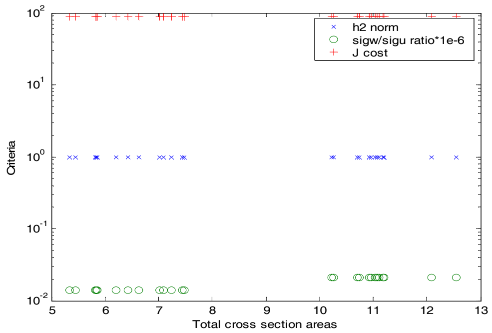

Fig. 3: Convergence for simultaneous optimum design of the truss system by using the objective function (45).

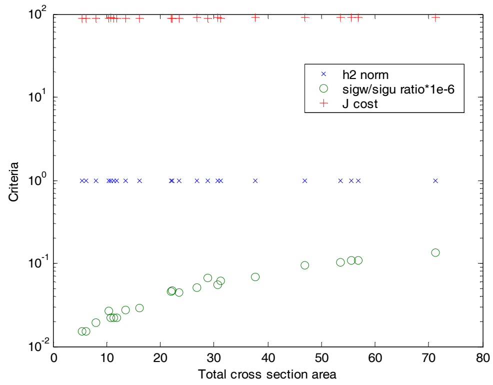

Fig. 4. Convergence for successive optimum design of the truss system by using the objective function (45).

Table 1: Some solutions obtained by simultaneous LQR optimization problem.

Solution number	Optimum cross sectional areas (mm^2)	Total area (mm^2)	J	$\frac{1}{\sigma_1}+\frac{1}{\sigma_n}$	CPU Time in sec.
1	$h_{optimal}$ = [1.52, 1.4, 1.88, 1.84, 1.4, 1.4, 1.4]	10.84	89.79	23380	4.96
2	$h_{optimal}$ = [3.99, 4.99, 2.23, 1.4, 4.56, 4.65, 4.91]	26.77	89.84	53470	9.42
3	$h_{optimal}$ = [8.08, 10.5, 6.23, 1.4, 10.5, 10.53, 10.37]	57.56	90.37	109620	14.82

Table 2: Some solutions obtained by successive LQR optimization approach.

Solution number	Optimum cross sectional areas (mm^2)	Total area (mm^2)	J	$\frac{1}{\sigma_1}+\frac{1}{\sigma_n}$	CPU Time in sec.
1	$h_{optimal}$ = [1.41, 1.41, 1.82, 1.4, 1.4, 1.4, 1.4]	10.25	89.67	20990	18.03
2	$h_{optimal}$ = [1.74, 1.69, 1.81, 1.4, 1.4, 1.74, 1.4]	11.20	89.50	20990	28.45
3	$h_{optimal}$ = [1.88, 1.43, 1.82, 1.4, 1.4, 1.4, 1.4]	10.73	89.69	20990	67.51

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