A DECOMPOSITION APPROACH FOR SOLVING KYP-SDPS [1]

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Abstract: Semidefinite programs originating from the Kalman-Yakubovich-Popov lemma are convex optimization problems and there exist polynomial time algorithms that solve them. However, the number of variables is often very large making the computational time extremely long. Algorithms more efficient than general purpose solvers are thus needed. In this paper a generalized Benders decomposition algorithm is applied to the problem to improve efficiency.

Keywords: Optimization, Decomposition Methods, Robust Control

1. INTRODUCTION

Semidefinite programs derived from the Kalman-Yakubovich-Popov lemma (KYP-SDPs) are convex optimization problems and have the following form

$$v_{\text{opt}} = \inf_{x,P} \; c^T x + \text{Tr}(CP) \quad \text{subj. to}$$
$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + M_0 + \sum_{k=1}^{p} x_k M_k > 0 \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $P \in \mathbb{S}^n$ and $M_k \in \mathbb{S}^{n+m}$, $k = 0, 1, \ldots, p$. Notation $\text{Tr}(\cdot)$ denotes the trace of the given matrix and $\mathbb{S}^n$ is the set of symmetric $n \times n$ matrices. We assume that the pair (A, B) is stabilizable and that A is Hurwitz. The second assumption can be relaxed; details are given in Section 5. The matrix C is negative semidefinite which is the usual case in applications. There can be several constraints of the above type, but for simplicity we only treat the case with one constraint. A generalization is straightforward. Note that a standard LMI

$$F = F_0 + \sum_{k+1}^{p} x_k F_k > 0$$

is a special case of the constraint (1) with $n = 0$. Thus, in general, we can handle a mixture of KYP constraints and standard LMIs.

KYP-SDPs are quite common in control and signal processing applications. The size of the SDP often gets very large, making it hard or even impossible to solve with general purpose software. Some interior point algorithms utilizing structure are presented in (Hansson and Vandenberghe, 2001; Wallin *et al.*, 2003; Vandenberghe *et al.*, 2005; Gillberg and Hansson, 2003). In this paper we apply a generalized Benders decomposition to the KYP-SDP problem. The generalized Benders decomposition is a popular choice when the problem at hand is hard to solve, e.g. due to nonconvexity, but decomposes into easier problems if some of the variables are fixed (Geoffrion, 1972). One example of such a case is mixed integer linear programming (Benders, 1962). However, the KYP-SDP is already a convex problem and is thus considered easy to solve. Our motivation for using a generalized Benders decomposition is instead to lower the computational complexity.

[1] Research supported by the Swedish Research Council under contract Nr. 62-2004-4691, the Göran Gustafson Foundation, Sweden and MRIO, Melbourne University, Australia.

The generalized Benders decomposition can be considered as a cutting plane algorithm and work related to what is presented in this paper can be found in (Kao *et al.*, 2004) and (Parrilo, 2001). It may seem that KYP-SDPs are problems of a special form, but they appear in numerous applications in control and signal processing. A far from complete list of applications includes linear system design and analysis (Boyd and Barratt, 1991; Hindi *et al.*, 1998), robust control analysis using integral quadratic constraints (Megretski and Rantzer, 1997; Jönsson, 1996; Balakrishnan and Wang, 1999), quadratic Lyapunov function search (Boyd *et al.*, 1994), and filter design (Alkire and Vandenberghe, 2002).

2. A GENERALIZED BENDERS DECOMPOSITION

We have to make some assumptions about the KYP-SDP in order to use a generalized Benders decomposition. The assumptions are that there exists a feasible, bounded, optimal solution $(x_{\text{opt}}, P_{\text{opt}})$, that $x \in \mathcal{D} \subset \mathbb{R}^p$, where $\mathcal{D}$ is a nonempty compact set and that x_{opt} lies in the interior of $\mathcal{D}$.

2.1 Problem Reformulation

Let us first define the matrix $M(x)$ and its partitioning as

$$M(x) = \begin{bmatrix} Q_x & S_x \\ S_x^T & R_x \end{bmatrix} = M_0 + \sum_{k=1}^{p} x_k M_k$$

the operator $\mathcal{F}$ as

$$\mathcal{F}(P) = \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix}$$

and the set $\mathcal{X}$ as

$$\mathcal{X} = \{x \in \mathcal{D} : \exists P \in S^n : \mathcal{F}(P) + M(x) > 0\}$$

This is the set of all x for which there exists a feasible solution to (1). If we then separate the optimization in x and P and for all $x \in \mathcal{X}$ define the function

$$h(x) = c^T x + \inf_P \operatorname{Tr}(CP) \quad \text{s.t. } \mathcal{F}(P) + M(x) > 0$$

we can write (1) as

$$v_p = \inf_{x \in \mathcal{X}} h(x) \tag{2}$$

It is clear that $v_{\text{opt}} = v_p$ and that the optimal solution x is equal for (1) and (2). Strong duality holds as we are dealing with a semidefinite program and have assumed that (1) is strictly feasible. Hence, $h(x)$ can be rewritten as

$$h(x) = c^T x - \sup_Z \operatorname{Tr}(ZM(x)), \quad \text{subj. to}$$
$$AZ_{11} + Z_{11}A^T + BZ_{12}^T + Z_{12}B^T = C$$
$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} \geq 0$$

Let us define the operator $\mathcal{F}^*$ as

$$\mathcal{F}^*(X) = AX_{11} + X_{11}A^T + BX_{12}^T + X_{12}B^T$$

and the sets $\mathcal{Z}$ and $\mathcal{Y}$ as

$$\mathcal{Z} = \{Z : \mathcal{F}^*(Z) = C, Z \geq 0\}$$
$$\mathcal{Y} = \{Z : \mathcal{F}^*(Z) = 0, Z \geq 0, Z \neq 0\}$$

According to a theorem of alternatives (Balakrishnan and Vandenberghe, 2002) we can express the feasible set as

$$\mathcal{X} = \{x \in \mathcal{D} : \operatorname{Tr}(ZM(x)) > 0, \forall Z \in \mathcal{Y}\} \tag{3}$$

The epigraph formulation of (2) may now be written as

$$v_p = \inf_{x,q} q, \quad \text{subj. to}$$
$$\begin{cases} q - c^T x + \operatorname{Tr}(ZM(x)) \geq 0, & \forall Z \in \mathcal{Z} \\ \operatorname{Tr}(ZM(x)) > 0, & \forall Z \in \mathcal{Y} \\ x \in \mathcal{D} \end{cases} \tag{4}$$

This problem is equivalent to (1). If $\mathcal{D}$ is chosen to be a polyhedron every $x \in \mathcal{D}$ has to fulfil

$$b_k - a_k^T x \geq 0, \qquad k = 1, 2, \ldots, r$$

for some a_k and b_k. Then, (4) is a linear program, albeit with an infinite number of constraints. The idea behind the generalized Benders decomposition is to alternate between solving a master problem and solving a subproblem until an ϵ-optimal solution is found. The master problem has a finite number of constraints and is an approximation of (4). The subproblem adds new constraints to the master problem resulting in a better and better approximation. The master problem and the subproblem are described in Section 2.2 and Section 2.3 respectively.

2.2 The Master Problem

The master problem at stage N is a linear problem

$$\inf_{x,q} q \quad \text{subj. to}$$
$$\begin{cases} q - c^T x + \operatorname{Tr}(Z_k M(x)) \geq 0, \ Z_k \in \mathcal{Z}, & k \in V_N \\ \operatorname{Tr}(Z_k M(x)) > 0, \ Z_k \in \mathcal{Y}, & k \in F_N \\ b_k - a_k^T x \geq 0, & k \in D_N \end{cases} \tag{5}$$

where V_N, F_N, D_N are mutually exclusive, and $V_N \cup F_N \cup D_N = \{1, 2, \ldots, N\}$. Because (4) is approximated with a finite number of constraints the optimal value of the master problem is a lower bound on v_p. The output of this linear program is a new trial point, $\bar{x}$. As the master problem accumulates information it will produce a monotonically improving sequence of solutions and bounds.

The first type of constraints in (5) are called value cuts and the second type of constraints are called feasibility cuts. The subproblem generates a new value cut whenever the master problem produces a trial point $\bar{x} \in \mathcal{X}$ and new feasibility cuts whenever the master problem produces a trial point $\bar{x} \notin \mathcal{X}$. The master problem is initialized with the constraints describing the polytope. We emphasize that the master problem is a linear program however of growing complexity as N increases.

2.3 *The Auxiliary Problems*

The purpose of the subproblem is to determine if a trial point $\bar{x}$ is in $\mathcal{X}$ and to generate cuts improving the approximation of (4). For the KYP-SDP the subproblem itself can be divided into subproblems. First, feasibility is checked. If the trial point $x \notin \mathcal{X}$ feasibility cuts are generated and added to the master problem. Otherwise, a value cut is added to the master problem.

2.3.1. Requirements for $x \in \mathcal{X}$ We would like to have some simple tests to see if a given x is feasible. To this end, let us introduce the quadratic matrix

$$\mathcal{Q}(P,x) = A^TP + PA + Q_x - (PB + S_x)R_x^{-1}(PB + S_x)^T$$

Using Schur complement on the constraint in (1) we conclude that

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} + \begin{bmatrix} Q_x & S_x \\ S_x^T & R_x \end{bmatrix} > 0 \iff \mathcal{Q}(P,x) > 0 \quad \text{and} \quad R_x > 0 \tag{6}$$

A first requirement on a fixed, given $x \in \mathcal{D}$ to be feasible is thus that $R(x)$ is positive definite. When $R(x) > 0$ and A has no eigenvalues on the imaginary axis $\bar{x} \in \mathcal{X}$ if and only if the Hamiltonian matrix

$$H(x) = \begin{bmatrix} A - BR_x^{-1}S_x^T & BR_x^{-1}B^T \\ Q_x - S_xR_x^{-1}S_x^T & -(A - BR_x^{-1}S_x^T)^T \end{bmatrix}$$

has no eigenvalues on the imaginary axis. See (Balakrishnan and Vandenberghe, 2002). Hence, we can check feasibility of x by computing the eigenvalues of R_x and $H(x)$.

2.3.2. Generating feasibility cuts To check if the matrix $R(x)$ is positive definite or not we make an eigenvalue decomposition of it. For every eigenvector, v_k, corresponding to a nonpositive eigenvalue, λ_k, construct

$$Z_k = \begin{bmatrix} 0 \\ v_k \end{bmatrix} \begin{bmatrix} 0 & v_k^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & v_kv_k^T \end{bmatrix}$$

Obviously, Z_k is in $\mathcal{Y}$ as

$$\begin{aligned} &AZ_{k_{11}} + Z_{k_{11}}A^T + BZ_{k_{12}}^T + Z_{k_{12}}B^T = 0 \\ &Z_k \geq 0 \\ &Z_k \neq 0 \end{aligned}$$

If the Hamiltonian matrix has an eigenvalue on the imaginary axis, $\lambda = i\omega$ we can construct a Z_k from the eigenvectors, v_k, corresponding to the nonpositive eigenvalues, λ_k, of the matrix

$$G(x,\omega) = \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}^* M(x) \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}$$

This is similar to what is done in the proof of Proposition 12 in (Balakrishnan and Vandenberghe, 2002). Introduce

$$u_k = (i\omega I - A)^{-1}Bv_k \tag{7}$$

and

$$U = \begin{bmatrix} \text{Re}(u_k) & \text{Im}(u_k) \end{bmatrix}, \quad V = \begin{bmatrix} \text{Re}(v_k) & \text{Im}(v_k) \end{bmatrix}$$

From (7) we get

$$\begin{aligned} BV &= \begin{bmatrix} B\text{Re}(v_k) & B\text{Im}(v_k) \end{bmatrix} \\ &= \begin{bmatrix} -\omega\text{Im}(u_k) - A\text{Re}(u_k) & \omega\text{Re}(u_k) - A\text{Im}(u_k) \end{bmatrix} \end{aligned} \tag{8}$$

We will show that

$$Z_k = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U^T & V^T \end{bmatrix} \in \mathcal{Y}$$

Using (8) we get

$$\begin{aligned} &\mathcal{L}(Z_k) = AZ_{k_{11}} + BZ_{k_{12}} = (AU + BV)U^T \\ &= (\begin{bmatrix} A\text{Re}(u_k) & A\text{Im}(u_k) \end{bmatrix} \\ &+ \begin{bmatrix} -\omega\text{Im}(u_k) - A\text{Re}(u_k) & \omega\text{Re}(u_k) - A\text{Im}(u_k) \end{bmatrix})U^T \\ &= -\omega\text{Im}(u_k)\text{Re}(u_k)^T + \omega\text{Re}(u_k)\text{Im}(u_k)^T \end{aligned}$$

This yields

$$\begin{aligned} \mathcal{F}^*(Z_k) &= \mathcal{L}(Z_k) + \mathcal{L}(Z_k)^T \\ &= -\omega\text{Im}(u_k)\text{Re}(u_k)^T + \omega\text{Re}(u_k)\text{Im}(u_k)^T \\ &\quad - \omega\text{Re}(u_k)\text{Im}(u_k)^T + \omega\text{Im}(u_k)\text{Re}(u_k)^T \\ &= 0 \end{aligned}$$

The matrix Z_k is nonzero and positive semidefinite by construction.

2.3.3. Generating value cuts When x is fixed to $\bar{x} \in \mathcal{X}$ the KYP-SDP can be rewritten as

$$\begin{aligned} &\inf_P \ \text{Tr}(CP) \quad \text{subj. to} \\ &\begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} + M(\bar{x}) > 0 \end{aligned} \tag{9}$$

To generate a value cut in the generalized Benders decomposition we need the solution to the dual problem of (9)

$$\begin{aligned} &\sup_Z \ -\text{Tr}(ZM(\bar{x})) \quad \text{subj. to} \\ &AZ_{11} + Z_{11}A^T + BZ_{12}^T + Z_{12}B^T = C \\ &Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} \geq 0 \end{aligned} \tag{10}$$

Solving this semidefinite program as it stands is not a very good idea. It would be even more costly than solving the SDP (1) as the number of variables is larger. However, there is another, less computationally heavy, way to obtain the solution. An optimal solution to (9) is given by the maximal solution of the algebraic Riccati equation (note, $\bar{x}$ is fixed)

$$\mathcal{Q}(P,\bar{x}) = 0$$

The maximal solution, P_r has the following properties

$$\begin{aligned} &P_r > P \quad \forall\ P \text{ such that } \mathcal{Q}(P,\bar{x}) > 0 \\ &A_r = A + BF_r \text{ is Hurwitz, where} \\ &F_r = -R_{\bar{x}}^{-1}(P_rB + S_{\bar{x}})^T \end{aligned}$$

This, together with the fact that the product of two semidefinite matrices has nonnegative eigenvalues proves that the maximal solution will minimize $\mathrm{Tr}(CP)$.

$$\mathrm{Tr}(C(P_r - P)) = -\mathrm{Tr}(-C(P_r - P)) \leq 0$$

Here the above inequality follows from $-C \geq 0$ and $P_r - P > 0$. This implies that $\mathrm{Tr}(CP_r) \leq \mathrm{Tr}(CP)$ for any P such that $\mathcal{Q}(P) > 0$. An optimal solution to (10) is

$$Z_k = \begin{bmatrix} I \\ F_r \end{bmatrix} Z_{k_{11}} \begin{bmatrix} I & F_r^T \end{bmatrix}$$

where $Z_{k_{11}}$ solves the Lyapunov equation

$$A_r Z_{k_{11}} + Z_{k_{11}} A_r^T = C \leq 0$$

The matrix $Z_{k_{11}}$ is positive semidefinite as A_r is Hurwitz. That $Z_{k_{11}}$ is positive semidefinite in turn means that $Z_k \geq 0$. We also have that

$$\begin{aligned} &A Z_{k_{11}} + Z_{k_{11}} A^T + B Z_{k_{12}}^T + Z_{k_{12}} B^T \\ &= A_r Z_{k_{11}} + Z_{k_{11}} A_r^T = C \end{aligned}$$

Hence, Z_k is feasible. As we have assumed that the primal is bounded from below and strictly feasible the dual is optimal if and only if complementary slackness holds (Ben-Tal and Nemirovski, 2001). This is true as

$$Z_k \begin{bmatrix} A^T P_r + P_r A + Q_{\bar{x}} & P_r B + S_{\bar{x}} \\ B^T P_r + S_{\bar{x}}^T & R_{\bar{x}} \end{bmatrix} = \begin{bmatrix} I \\ F_r \end{bmatrix} Z_{k_{11}} \mathcal{M}$$

where

$$\begin{aligned} \mathcal{M} &= \begin{bmatrix} I & F_r^T \end{bmatrix} \begin{bmatrix} A^T P_r + P_r A + Q_{\bar{x}} & P_r B + S_{\bar{x}} \\ B^T P_r + S_{\bar{x}}^T & R_{\bar{x}} \end{bmatrix} \\ &= \begin{bmatrix} A^T P_r + P_r A + Q_{\bar{x}} + F_r^T (B^T P_r + S_{\bar{x}}^T) \\ P_r B + S_{\bar{x}} + F_r^T R_{\bar{x}} \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{Q}(P_r, \bar{x}) \\ P_r B + S_{\bar{x}} - P_r B - S_{\bar{x}} \end{bmatrix} = 0 \end{aligned}$$

The cost for solving the Riccati equation is $O(n^3)$ compared to the cost for solving the dual problem (10) which is at least $O(n^6)$. Solving the Lyapunov equation requires $O(n^3)$ operations. Any feasible $\bar{x} \in \mathcal{X}$ with a corresponding P_r will give us an upper bound on v_p. Thus, every time we find an $\bar{x} \in \mathcal{X}$ we have to update the upper bound according to

$$\begin{aligned} &\text{new upper bound} \\ &= \min(\text{old upper bound}, c^T \bar{x} + \mathrm{Tr}(CP_r)) \end{aligned}$$

The algorithm is stopped when the difference between the upper and lower bound is sufficiently small.

3. CENTERING AND OTHER IMPROVEMENTS

A proof for ϵ-convergence in a finite number of steps, when a generalized Benders decomposition is applied to a convex problem, is given in (Geoffrion, 1972). However, in practice the convergence is too slow. Measures can be taken to accelerate the convergence though. In this section, we propose some possible alternatives to improve the computational efficiency.

3.1 Interpretation as a Kelley's cutting plane algorithm

The master problem which the Benders decomposition algorithm solves in each iteration has the form (5). The linear program (5) can be equivalently formulated as

$$\begin{aligned} &\inf_x \max_{k \in V_N} \{c^T x - \mathrm{Tr}(Z_k M(x))\}, \text{ subj. to} \\ &\begin{cases} \mathrm{Tr}(Z_k M(x)) > 0, \; Z_k \in \mathcal{Y}, & k \in F_N \\ b_k - a_k^T x \geq 0, & k \in D_N \end{cases} \end{aligned} \tag{11}$$

Comparing formulation (11) and the original problem (2), one can see that the piece-wise linear function

$$\bar{q}_k(x) := \max_{k \in V_N} \{c^T x - \mathrm{Tr}(Z_k M(x))\}$$

serves as an approximation of $h(x)$ which supports $h(x)$ from below, and the constraints in (11) defines a polyhedron which approximates $\mathcal{X}$. Hence, the Benders decomposition algorithm proposed in Section 2.1 is indeed another formulation of the Kelley's cutting plane algorithm. (See, for instance, (Boyd and Barratt, 1991)).

In the Kelley's cutting plane algorithm, the minimizer of $\bar{q}(x)$ over the polyhedral approximation of the $\mathcal{X}$ is chosen as a candidate of the optimal solution of the original problem. This method works well and converges rapidly if (11) approximates (2) well. However, convergence is often quite poor if this is not the case. An alternative to the Kelley's cutting plane algorithm is the centering method.

3.2 Centering

In the centering method, a certain kind of center of the polyhedral approximation of the feasible set is computed in each iteration to serve as a candidate for the optimal solution. To achieve good speed of convergence, one should choose a center which not only allows the algorithm to reduce the polyhedral approximation significantly but also can be computed efficiently. Among all possible centers, the analytical center is a good choice for that purpose. Let $\mathcal{P}_k$ be the polyhedron approximation of feasible set at the k^{th} iteration, which is defined as

$$\mathcal{P}_k = \{x \mid d_i - c_i^T x \geq 0, \; i = 1, \cdots, N_k\}$$

The analytical center of $\mathcal{P}_k$ is the unique minimizer of

$$\mathcal{L}_k(x) = -\sum_{i=1}^{N_k} \log(d_i - c_i^T x)$$

over the interior of $\mathcal{P}_k$.

The constraints in (5) can all be written on the above form. Thus, instead of solving the master problem (5) we solve the analytic center problem corresponding to (5)

$$\begin{aligned} &\inf_x \; \mathcal{L}_k(x), \quad \text{subj. to} \\ &x \in \mathrm{Interior}(\mathcal{P}_k) \end{aligned} \tag{12}$$

The analytic center problem can be solved efficiently with, for example, methods described in (Ye, 1997).

3.3 Generating Feasibility and Value Cuts

Let x^k be the solution (or an approximation of the solution) of (12). Then checking whether $x^k \in \mathcal{X}$, and generating a feasibility cut when $x^k \notin \mathcal{X}$ are done exactly as proposed in Sections 2.3.1 and 2.3.2.

On the other hand, suppose that $x^k \in \mathcal{X}$. Then an alternative value cut can be produced as follows. We note that $h(x)$ is convex and differentiable on $\mathcal{X}$. Let $\nabla h(x^k)$ be the gradient of $h(x)$ at x^k. Then we have

$$h(x) \geq h(x^k) + \nabla h(x^k)^T(x - x^k),\ \forall\, x \in \mathcal{X} \quad (13)$$

This immediately gives a value cut: for those x in the half plane $\{x \mid \nabla h(x^k)^T(x - x^k) > 0\}$, $h(x) \geq h(x^k) + \nabla h(x^k)^T(x - x^k) > h(x^k)$. Our task is to minimize $h(x)$. Hence, the half plane $\{x \mid \nabla h(x^j)^T(x - x^j) > 0\}$ can be ruled out, and the polyhedron approximation for $\mathcal{X}$ is replaced by $\mathcal{P}_{k+1} := \mathcal{P}_k \cap \{x \mid \nabla h(x^k)^T(x - x^k) \leq 0\}$.

The gradient of $h(x)$ at x^k can be computed using the Z_k in Section 2.3.3. More precisely, let $\nabla_i h(x^k)$ be the i^{th} element of $\nabla h(x^k)$. Then

$$\nabla_i h(x^k) = c_i - \mathrm{Tr}(Z_k M_i), \quad i = 1, \cdots, p \quad (14)$$

To see this, recall that in Section 2.3.3, it is shown that Z_k is dual optimal. Therefore, with zero duality gap, we have

$$c^T x^k - \mathrm{Tr}(Z_k M(x^k)) = c^T x^k + \inf_{x^k \in \mathcal{X}} \mathrm{Tr}(CP)$$
$$= h(x^k)$$

This shows that the hyperplane

$$y = c^T x - \sum_{i=1}^{p} \mathrm{Tr}(Z_k M_i) x_i - \mathrm{Tr}(Z_k M_0) \quad (15)$$

passes through $(x^k, h(x^k))$. This supports $h(x)$ from below and since $h(x)$ is convex and differentiable, there is only one such hyperplane at each x which is the tangent plane. Hence, (14) follows immediately from comparing the right hand sides of (13) and (15).

3.4 Upper and Lower Bounds

Every feasible solution found give an upper bound for the optimal solution of (2). Hence the smallest upper bound is obtained by finding the minimum over the set of h values evaluated at the available feasible solutions.

Lower bounds of (2) can be found, as we have seen previously, by solving (5). However, a more efficient way is to solve the dual of (5). The optimal objective of the dual is equal to the optimal objective of (5), and more importantly, any suboptimal feasible solution to the dual problem will provide a lower bound. Thus, we do not have to solve the dual problem exactly.

3.5 Further Unimplemented Improvements and Numerical Issues

In (Goffin and Vial, 1999) other ways to accelerate convergence of analytic center cutting plane algorithms are described. Suggestions include

- Making duplicates of the value cuts. In the original master problem this has no effect but for the analytic center problem this results in an importance weight on the value cuts. The suggested number of duplicates is $p + 1$.
- Warm start with the old analytic center as an initial guess of the new analytic center. Algorithms for this can be found in (Ye, 1997).

If $R_{\bar{x}}$ is close to being singular it is not recommended to check if the Hamiltonian matrix has eigenvalues on the imaginary axis. A much better choice, from a numerical point of view, is to compute the finite generalized eigenvalues of the pencil

$$\begin{bmatrix} 0 & A & B \\ A^T & Q_{\bar{x}} & S_{\bar{x}} \\ B^T & S_{\bar{x}}^T & R_{\bar{x}} \end{bmatrix} + \lambda \begin{bmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which are the same as the eigenvalues of the Hamiltonian matrix. Also, the solution to the algebraic Riccati equation can be obtained by computing a basis for the stable deflating subspace of this pencil (Van Dooren, 1981). None of this is yet implemented.

4. NUMERICAL EXAMPLE

SeDuMi (Sturm, 1999) is considered one of the best general purpose solvers for semidefinite programs. To see how well the analytic center cutting plane solver, implemented in Matlab, performs we compare that solvers computational time to the time used by SeDuMi interfaced through YALMIP (Löfberg, 2004). We chose to vary the number of states,n, and a constant number of inputs and x-variables, $m = 5$ and $p = 5$. The matrices A, B C, M_0, M_k and the vector c are randomly generated and fulfil the assumptions. The platform used is a SunBlade 100 workstation. Ten problems of each size are generated and the results are given in Table 1. We see that for a small number of states SeDuMi has a slightly faster computational time but for larger systems there is much to gain by using the analytic center cutting plane method. The iterations were terminated when the difference between the upper and lower bound was less than 10^{-6}.

5. RELAXING ASSUMPTIONS

The assumption that A is Hurwitz is only used to assure that A has no eigenvalues on the imaginary axis. Suppose A is not Hurwitz. As the pair (A, B) is stabilizable we know that there exists an F such that $\bar{A} = A + BF$ is Hurwitz. Define the full rank matrix

$$T = \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}$$

# of states	mean time ACCPM [s]	mean time SeDuMi [s]
30	30.06	22.58
40	44.12	96.43
50	48.68	323.77
60	98.79	903.68

Table 1. The mean computational time for the analytic center cutting plane method, ACCPM, and SeDuMi for varying number of states.

Premultiplying the constraint (1) with T^T and postmultiply with T yields

$$\begin{aligned} &\inf_{x,P} c^T x + \mathrm{Tr}(CP) \\ &\text{s.t. } \begin{bmatrix} \bar{A}^T P + P\bar{A} & PB \\ B^T P & 0 \end{bmatrix} + \bar{M}_0 + \sum_{k=1}^{p} x_k \bar{M}_k > 0 \end{aligned}$$

where $\bar{A} = A + BF$ is Hurwitz and

$$\bar{M}_k = T^T M_k T, \qquad k = 0, 1, \ldots, p$$

The solution $(x_{\text{opt}}, P_{\text{opt}})$ is the same as for the original problem.

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