SPEED-GRADIENT ALGORITHMS FOR UNDERACTUATED NONLINEAR SYSTEMS

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Abstract: A new method for control of underactuated nonlinear systems is proposed, based on introducing artificial invariants and using speed-gradient algorithms. General statement concerning achievement of the control goal is formulated and proven. Application of the proposed approach is illustrated by an example: stabilization of cart-pendulum oscillations around the upper equilibrium.

1. INTRODUCTION

In a variety of control problems the number of regulated variable outputs exceeds the number of control inputs. Such problems are important in control of mechanical, electromechanical and mechatronic systems termed "underactuated" systems, see, e.g. (Jager and Nijmeijer (Eds.), 2000). We will use the term "underactuated" for general nonlinear systems with the number of regulated outputs exceeding the number of inputs.

Typical examples of underactuated systems are "cart-pendulum" and "Furuta pendulum" systems which have become recently benchmark systems for nonlinear control (Furuta *et al.*, 1994; Fradkov *et al.*, 1995; Acosta *et al.*, 2001; Aracil *et al.*, 2002).

In this paper a new method for control of underactuated nonlinear systems is proposed based on creating invariant surfaces for the system by means of feedback and the application of the speed-gradient method for control design. Creating an auxiliary invariant surface and controlling the system to that surface as an intermediate step for control design has been used in many approaches, e.g. sliding modes (Utkin, 1992), synergetic control (Kolesnikov, 1987), backstepping (Krstic *et al.*, 1995), to mention a few. The novelty of the proposed approach is the combination of the created invariants for the fully actuated part of the problem with other existing invariants into a single goal function and the usage of the speed-gradient method for control design. In Sec. 2 the problem statement and control design method are presented. In Sec. 3 a general result providing analytical conditions for successful design (Theorem 1) is formulated and proven. In Sec.4 the method is illustrated by a novel version of the benchmark "cart-pendulum" example: stabilization of the pendulum oscillations around the upper equilibrium.

2. PROBLEM STATEMENT AND CONTROL DESIGN

The class of systems considered in this paper is

$$\dot{x}_1 = f_1(x_1, x_2) + g_1(x_1, x_2)u, \quad (1)$$
$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)u, \quad (2)$$

where $x_1 \in R^{n_1}, x_2 \in R^{n_2}, u \in R^m$ and f_i, g_i, are smooth functions of corresponding dimensions.

Let the control goal be stabilization of output function $y = h_1(x_1) \in R^m$ at zero level: $y(t) \to 0$ when $t \to \infty$. Besides, boundedness of the trajectories of the closed loop system is required. Although no restrictions on the structure of system (1)–(2) are imposed, the variables x_1 and x_2 play different roles (usually x_2 represents the state of the driving system) and will be treated in a different way with respect to the goal by means of the introduction of additional assumptions.

The first step of the proposed solution is to perform a feedback transformation that makes the goal function an invariant of the transformed system. To this end the auxiliary goal

$$\dot{y}(t) \equiv 0, \quad (3)$$

and the feedback transformation

$$u = u^c(x_1, x_2) + v, \quad (4)$$

are introduced, where $u^c(x_1, x_2)$ is a *conservative* feedback –i.e., it makes the system conservative in the sense that the goal (3) is achieved. Since

$$\dot{y} = \left(\frac{\partial h_1}{\partial x_1}\right)\dot{x}_1 = \left(\frac{\partial h_1}{\partial x_1}\right) f_1 + \left(\frac{\partial h_1}{\partial x_1}\right) g_1 u, \quad (5)$$

fulfillment of (3) is ensured if the $m \times m$-matrix $(\frac{\partial h_1}{\partial x_1})g_1$ is nonsingular and the conservative control is determined as follows

$$u^c(x_1, x_2) = -\left[\left(\frac{\partial h_1}{\partial x_1}\right) g_1\right]^{-1}\left(\frac{\partial h_1}{\partial x_1}\right) f_1. \quad (6)$$

However, nonsingularity of $(\frac{\partial h_1}{\partial x_1})g_1$ is not strictly necessary (see example in Sec. 4). It suffices that a continuous function $u^c = u^c(x_1, x_2)$ exists satisfying the identity

$$\left(\frac{\partial h_1}{\partial x_1}\right) f_1 + \left(\frac{\partial h_1}{\partial x_1}\right) g_1 u^c = 0. \quad (7)$$

The transformed system (1)–(2) takes the form

$$\dot{x}_1 = \tilde{f}_1(x_1, x_2) + g_1(x_1, x_2)v, \quad (8)$$
$$\dot{x}_2 = \tilde{f}_2(x_1, x_2) + g_2(x_1, x_2)v, \quad (9)$$

where $\tilde{f}_1 = f_1 + g_1 u^c$, $\tilde{f}_2 = f_2 + g_2 u^c$, and $v \in R^m$ is a new control to be determined.

The next step of the design is the introduction of an additional set of invariants for the free ($v = 0$) transformed system (8)–(9); $z = h_2(x_1, x_2) \in R^{n_2}$. The invariance of z means validity of identities

$$\left(\frac{\partial h_2}{\partial x_1}\right)\tilde{f}_1 + \left(\frac{\partial h_2}{\partial x_2}\right)\tilde{f}_2 = 0, \quad (10)$$

which are partial differential equations (PDE) for a function $h_2(x_1, x_2)$. In what follows we assume that PDE (10) are solvable.

Introduction of the additional invariant is an important step, aimed at introducing the additional control goal

$$z(t) \equiv 0. \quad (11)$$

The aim of this new goal is to link variables x_2 to variables x_1 in order to assure boundedness. It should have a physical meaning and allow to meet additional requirements to the system. For example, in the cart-pendulum it might include restrictions on the motion of the cart.

In order to design the control $v(t)$ ensuring the control goals (3) and (11), the speed-gradient (SG) method (Fradkov, 1996; Fradkov and Pogromsky, 1998) is employed. Introduce the goal function

$$Q(x_1, x_2) = \frac{1}{2}||y||^2 + \frac{1}{2}z^T P z, \quad (12)$$

where $P = P^T > 0$ is a positive-definite symmetric $n_2 \times n_2$-matrix to be chosen later. Evaluating the speed of changing $Q(x_1, x_2)$ along trajectories of the control system (8)–(9):

$$\dot{Q} = y^T\left(\frac{\partial y}{\partial x_1}\right)\left(\tilde{f}_1 + g_1 v\right) + z^T P \dot{z}, \quad (13)$$

and using invariance of y,z, we obtain

$$\dot{Q} = \left[y^T\left(\frac{\partial y}{\partial x_1}\right) g_1 + z^T P\left(\frac{\partial h_2}{\partial x_1} g_1 + \frac{\partial h_2}{\partial x_2} g_2\right)\right] v. \quad (14)$$

Calculating the gradient of (14) with respect to v, we arrive at the speed-gradient algorithm

$$v = -\Gamma\left[g_1^T\left(\frac{\partial y}{\partial x_1}\right)^T y + \left(g_1^T\left(\frac{\partial h_2}{\partial x_1}\right)^T + g_2^T\left(\frac{\partial h_2}{\partial x_2}\right)^T\right) P z\right] \quad (15)$$

where $\Gamma = \Gamma^T > 0$ is a positive-definite gain matrix. Applying control (15) ensures that $Q_t = Q(x_1(t), x_2(t))$ decreases monotonically, which is important for achievement of the control goal $Q_t \to 0$ as $t \to \infty$, combining both goals $y \to 0$ and $z \to 0$. The conditions ensuring achievement of the goal are given in the next section.

3. CONDITIONS OF THE CONTROL GOAL ACHIEVEMENT

Let us formulate the result establishing conditions for the control goal achievement.

Theorem 1. Let the equations (7) have solutions for a continuous $u^c(x_1, x_2)$ and the PDE (10) have solutions for a smooth $h_2(x_1, x_2)$ for all $(x_1, x_2) \in$

Ω_0, where $\Omega_0 = \{(x_1, x_2) : Q(x_1, x_2) \leq Q_0\}$, for some $Q_0 > 0$

A1. The solutions of the system (1), (2), (4), (7), (10) and (15) with initial conditions in Ω_0 are well defined for all $t \geq 0$.

Then the functions $y(t)$, $z(t)$ are bounded for $t \geq 0$ and $v(t) \to 0$ as $t \to \infty$ for $(x_1(0), x_2(0)) \in \Omega_0$.

Let the additional assumptions hold:

A2. $y(t) \equiv 0$, $z(t) \equiv 0$ for all $(x_1(0), x_2(0)) \in R_0 \cap \Omega_0$, where R_0 is the maximal invariant set of the free system (8)–(9) (with $v = 0$), contained in the set $R \cap \Omega_0$, where $R = \{(x_1, x_2) : \dot{Q} = 0\}$.

A3. The set Ω_0 is bounded.

Then the goals $y(t) \to 0$, $z(t) \to 0$ are achieved in the system (1), (2), (7), (10), (15) for all initial conditions $x_1(0)$,$x_2(0)$ from Ω_0.

Proof Relations $Q(x_1(0), x_2(0)) \leq Q_0$ and $\dot{Q} \leq 0$, (the later follows from the design of the control algorithm) imply that $(x_1(t), x_2(t)) \in \Omega_0$ for all $t \geq 0$ (note that $x_1(t)$, $x_2(t)$ are well defined by assumption of the theorem). Therefore, functions $y(t)$ and $z(t)$ are bounded.

Since $\dot{Q}_t = -\Gamma \mid v \mid^2 (t)$, we have $\int_0^\infty \mid v \mid^2 dt < \infty$. The function $v(t)$ is uniformly continuous in t, due to its continuous dependence of $y(t)$,$z(t)$. Now we obtain the first part of the theorem: $v(t) \to 0$ as $t \to \infty$ from the Barbalat lemma (see, e.g.(Fradkov *et al.*, 1999)).

To prove the second part, note that since Q_t is monotonically decreasing there exists $\lim_{t\to\infty} Q_t = Q_\infty$. If $Q_\infty = 0$, the theorem is proven. Let $Q_\infty > 0$. Then, from A3, the trajectory $(x_1(t), x_2(t))$ possesses a limit point (x_1^*, x_2^*) and $Q(x_1^*, x_2^*) = Q_\infty > 0$, i.e. $(x_1^*, x_2^*) \in R_0 \cap \Omega_0$. Consider a new trajectory with initial conditions (x_1^*, x_2^*). Then, $Q(x_1^*, x_2^*) = Q_\infty$, which contradicts assumption A2.

Hence $Q_t \to 0$ and $y(t) \to 0$, $z(t) \to 0$ as $t \to \infty$. The theorem is proven.

Remark 1. Assumption A1 is weaker than the standard *forward completeness* condition, since it only requires that the trajectories starting from the set Ω_0 are well defined. To verify A1 the following sufficient condition is useful:

A1'. The right-hand sides of the equations (1), (2), (4), (7), (10) and (15) are bounded in Ω_0.

It is easy to see that A1' implies A1. Indeed, for any solution of the system (1), (2), (4), (7), (10), (15) cannot leave Ω_0 owing to the inequality $\dot{Q} \leq 0$ which holds as far as the solution is well defined. Therefore, the right-hand sides are bounded by A1'. On the other hand, a solution of the differential system with bounded right-hand sides cannot grow faster than linearly and, therefore, cannot escape in a finite time. The condition A1' has natural physical interpretation: all the forces are bounded when the energy-like function Q is bounded.

Remark 2. To verify assumption A3, minimum-phase-like properties of the system can be used. Namely, A3 might be removed if the system has so called BOBS-property with respect to output (y, z). BOBS (bounded-output-bounded-state) means that for bounded output functions $y(t)$, $z(t)$ the state $(x_1(t), x_2(t))$ is also bounded.

4. EXAMPLE

In this section, an example that illustrates the possibilities of the method stated in the previous sections is included. The example chosen is the well-known inverted pendulum on a cart. The objective is to achieve an oscillatory movement of the pendulum around the upper vertical. This case was previously studied in (Aracil *et al.*, 2002) where a solution was proposed but not fully justified.

The normalized equations of the pendulum, after partial linearization, are given by

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \sin\xi_1 - \cos\xi_1 u \\ \dot{\xi}_3 &= u \end{aligned} \tag{16}$$

where ξ_1 and ξ_2 are, respectively, the angular position and velocity of the pendulum ($\xi_1 = 0$ corresponds to the up-right position) and ξ_3 is the velocity of the cart. This system is an example of a system affine in control of the type (1)-(2) with $x_1 = [\xi_1 \;\; \xi_2]^\top$ and $x_2 = \xi_3$.

First, we consider the x_1 subsystem, that is, the single pendulum only, whose equations are given by the two first ones of (16). The goal is that this subsystem oscillates through a closed curve $y(\xi_1, \xi_2) = 0$. From energetic considerations we take

$$y(\xi_1, \xi_2) = \frac{\xi_2^2}{2} - \cos\xi_1 + 1 - \mu,$$

with $\mu > 0$ (μ is the tuning parameter that determines the amplitude of the oscillations). This curve can be seen as a level curve of the virtual energy of the system, considering this virtual energy as the one that the system has if we invert the gravity force.

Equation (7) is in this case $2\xi_2 \sin(\xi_1) = \xi_2 \cos(\xi_1) u^c$, which yields the conservative control law

$$u^c = 2\tan\xi_1. \tag{17}$$

Remark 3. Notice that, in this case, $(\frac{\partial h_1}{\partial \xi_1}) g_1(\xi_1, \xi_2)$ is singular for $\xi_2 = 0$, but the control law (17) is well-defined for $\cos\xi_1 \neq 0$ and ensures the equality $\dot{y} = 0$. Therefore, in this case, the singularity of $(\frac{\partial h_1}{\partial \xi_1}) g_1(\xi_1, \xi_2)$ does not affect the results (see the choice of the set of valid initial conditions Ω below).

The PDE (10) takes the form

$$\xi_2 \frac{\partial z}{\partial \xi_1} - \sin\xi_1 \frac{\partial z}{\partial \xi_2} + 2\tan\xi_1 \frac{\partial z}{\partial \xi_3} = 0. \qquad (18)$$

A solution of this PDE for the region defined by $2\cos\xi_1 - \xi_2^2 > 0$ is

$$z = \xi_3 + \frac{4\Psi(\xi_1, \xi_2)}{\sqrt{2\cos\xi_1 - \xi_2^2}},$$

with

$$\Psi(\xi_1, \xi_2) = \arctan \frac{\xi_2}{\sqrt{2\cos\xi_1 - \xi_2^2}}.$$

Applying SG method with $Q = y^2/2 + \alpha z^2/2$ we have

$$\dot{Q} = y\dot{y} + \alpha z \dot{z}.$$

Thus,

$$\begin{aligned}\dot{Q} &= y(\xi_2\dot{\xi}_2 + \sin\xi_1\dot{\xi}_1) + \alpha z \frac{\partial z}{\partial x_1}\dot{x_1} + \alpha z \frac{\partial z}{\partial x_2}\dot{x_2} \\ &= -y\xi_2\cos\xi_1 v + \alpha z \left(-\frac{\partial z}{\partial \xi_2}\cos\xi_1 + \frac{\partial z}{\partial \xi_3}\right) v\end{aligned}$$

On the other hand, we have

$$\frac{\partial z}{\partial \xi_2} = -\frac{4\left(-2\cos\xi_1 + \xi_2^2 - \Psi(\xi_1,\xi_2)\xi_2\sqrt{2\cos\xi_1 - \xi_2^2}\right)}{4\cos^2\xi_1 - 4\cos\xi_1\xi_2^2 + \xi_2^4},$$

and

$$\frac{\partial z}{\partial \xi_3} = 1.$$

Therefore, and renaming parameter Γ as γ since in our case it is scalar, the control law results

$$u = 2\tan\xi_1 + \gamma\left(y\xi_2\cos\xi_1 - \alpha z\left(\frac{\partial z}{\partial \xi_2}\cos\xi_1 + 1\right)\right). \qquad (19)$$

Let us check the conditions of the Theorem. The set of initial conditions Ω_0 should be within the set $D = \{(\xi_1, \xi_2, \xi_3) : \cos\xi_1 > 0,\ 2\cos\xi_1 - \xi_2^2 > 0\}$. Within this set the right-hand sides are well defined and bounded and assumption A1 holds too in view of Remark 1. To fit the relation $\Omega_0 \subset D$ it suffices to choose the value Q_0 satisfying inequality $2Q_0 < (1-\mu)^2$ which is always possible for $\mu < 1$. In this case variables ξ_1, ξ_2 are bounded and z is bounded. Hence, ξ_3 is bounded too and the assumption A3 holds.

To verify condition A2 note that in the set R_0 (i.e. for $v = 0, u = u^c$) plant equations (17) read $\dot{\xi}_1 = \dot{\xi}_2, \dot{\xi}_2 = -\sin\xi_1, \dot{\xi}_3 = 2\tan\xi_1$. Therefore, $\xi_2(t)2/2 - \cos\xi_1(t) = \text{const}$ and $y(t) = \text{const} = y_*$. Then $\dot{Q} = 0$ implies $Q(t) = \text{const}$ and $z(t) = \text{const} = z_*$. Let $\xi_2(t)2/2 - \cos\xi_1(t) = \nu$. Since $\dot{Q} = -\gamma v^2$, the relation $\dot{Q} = 0$ also implies

$$\begin{aligned}&2y_*\xi_2 + 8\alpha z_*\Bigg(\frac{2}{\cos\xi_1} \\ &\quad + \frac{\sqrt{-2\nu} - \xi_2\arctan(\xi_2\sqrt{-2\nu}))}{\sqrt{(-2\nu)^3}}\Bigg) = 0. \qquad (20)\end{aligned}$$

It is easy to see that $\xi_2(t)$ and the function in the square brackets are linearly independent in R_0. Hence, the identity (20) implies identities $y_* = 0, z_* = 0$, i.e. condition A2 is verified.

Remark 4. In order to weaken the restrictions for the set of initial conditions Ω_0, it is reasonable to combine algorithm (15) with another algorithm used at the first stage as a preliminary algorithm, ensuring driving $(\xi_1(t), \xi_2(t), \xi_3(t))$ into the set Ω_0. E.g. the standard SG-algorithm of energy control for "lower" oscillation can be used:

$$u = \gamma_2(H - H^*)\xi_2\cos\xi_1, \qquad (21)$$

where H is a kind of energy function and H^* is the desired energy level. As soon as the condition $(\xi_1(t), \xi_2(t), \xi_3(t)) \in \Omega_0$ is fulfilled, the control should be switched to the algorithm (19).

Theorem 1 establishes convergence $y(t) \to 0, z(t) \to 0$. In order to evaluate transient processes and performance of the closed loop system, a number of computer simulations have been performed. Figures 1 and 2 show the results of two of such simulations. The parameter values are $\mu = 0.1$, $\gamma = 10$ and $\alpha = .01$. The initial conditions in Fig. 1 are $(\xi_1, \xi_2, \xi_3) = (.01, 0, 0)$ while in Fig. 2 are $(1, 0, 0)$. In each figure the first graph presents the projection of the trajectory in the $\xi_1 - \xi_2$ plane. The second and third graphs show, respectively, the time evolution of ξ_1 and ξ_3. It can be seen that, in both simulations, the goal is achieved: the desired oscillation is obtained for ξ_1 and ξ_2 (that is, $y \to 0$) and ξ_3 is bounded. The last graph in both figures shows the time evolution of the goal function Q (amplified) and its components y and z. It can be seen that these three functions go to zero and, besides, Q is not increasing.

Figure 3 shows another simulation with initial conditions outside Ω_0 (the pendulum starts near the downward position). Therefore, control law (19) is not valid and other strategy has to be applied. Following the ideas exposed in Remark 4, control law (21) has been used outside Ω_0 with H the physical energy of the pendulum subsystem $H = \xi_2^2/2 + \cos\xi_1 - 1$ (the parameter values are $H^* = 0$, $\gamma_2 = 1$ and $\mu = 0$), while the controller changes to (19) once the state of the

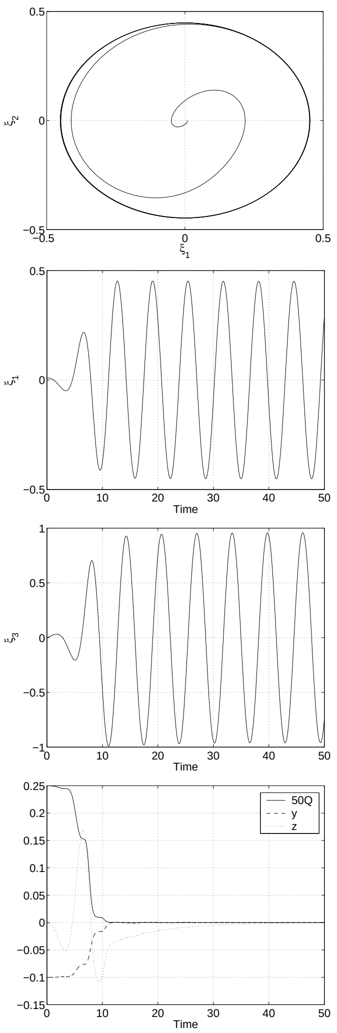

Fig. 1. Results of a simulation with small initial conditions

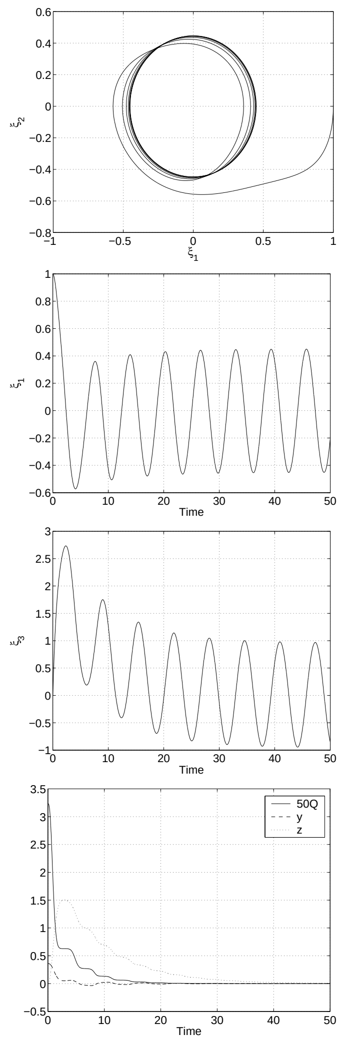

Fig. 2. Results of a simulation with large initial conditions

system enters Ω_0. Furthermore, in order to make the simulation more realistic, the control signal has been saturated ($|u| \leq 1.5$). The graphs in this figure have the same meaning than in the previous figures except the last one that presents the evolution of the control signal u (notice the effect of the saturation). Besides, the commutation curve (the frontier of the set D) has been represented (dashed curve) in the top graph together with the evolution of the state.

In order to study the robustness of the obtained law with respect to changes of the plant parameters a new simulation is presented in Fig. 4. This simulation corresponds to the same control law (with the same parameters) controlling a system in which the second Eq. of (16) is changed to

$$\dot{\xi}_2 = \frac{1}{1.2}(0.8 \sin \xi_1 - 0.8 \cos \xi_1 u).$$

This model represents a pendulum with different parameters values (after partial linearization) from that used in the computation of the control law. In the figure the dashed curves represent the "nominal" behavior (same as Fig. 3), while the solid curves represent the actual evolution. It can be seen that the changes are not severe.

5. CONCLUSIONS

A new method for control of underactuated nonlinear systems is proposed, based on introducing artificial invariants and using speed-gradient algorithms. General statement concerning achievement of the control goal is formulated and proven. The proposed approach is methodologically simple. Application of the proposed approach to stabilization of cart-pendulum oscillations around upper equilibrium confirms satisfactory transient behavior of the closed loop system and good robustness properties.

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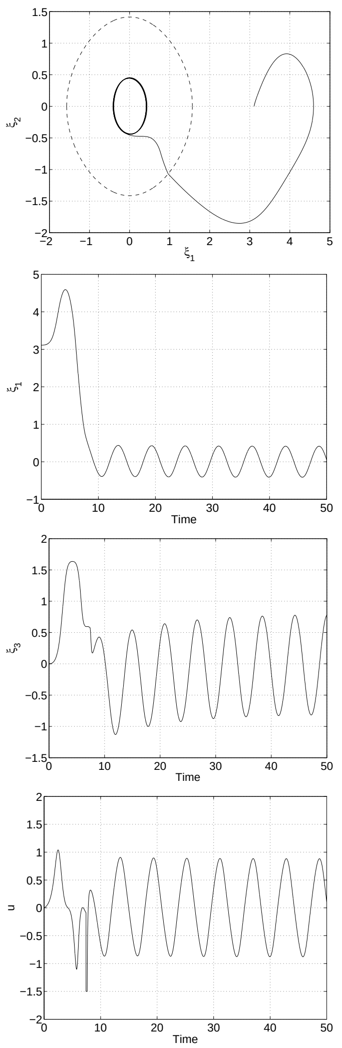

Fig. 3. Results of a simulation with initial conditions close to the downward position. Switching strategy.

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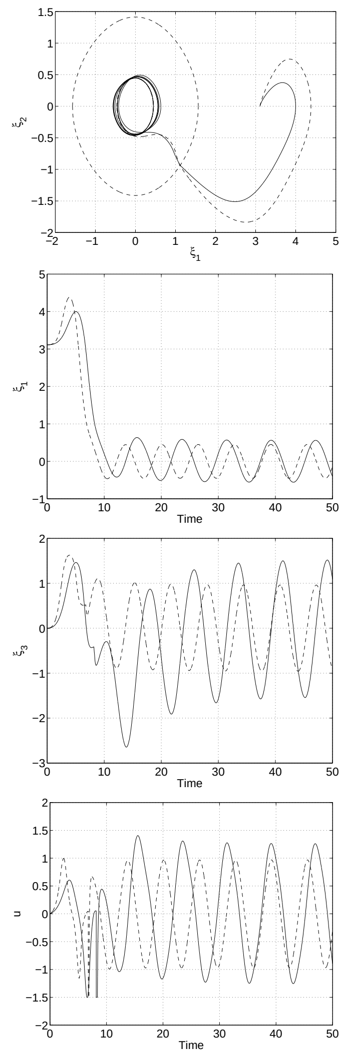

Fig. 4. Behavior of the closed-loop system with model mismatch.

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