OBSERVER-BASED CONTROLLER FOR DELAYED STATE LINEAR SINGULAR SYSTEMS

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Abstract: Based on Lyapunov-Krasovskii functional, this paper concerns an observer based stabilization problem for delayed state linear singular systems. The linear quadratic control is used and the state variables are provided by an observer. An observer based controller, in which the influence of the time-delays is considered, the design of the controller and observer are separated. LMIs Delay-dependent sufficient conditions for stabilizability is established. Some numerical examples are provided to show the usefulness of the proposed results.

Keywords: singular systems, observers, feedback control, linear matrix inequality.

1. INTRODUCTION

The problem of stabilization of state-space systems is of both practical and theoretical importance and has attracted the attention of many researchers in the past years, a number of significant result on this issue have been reported and different approaches have been proposed in the literature (see, (Soh, 1985), (Trofino-Neto, 1993)). However, as for singular systems (also known as descriptor systems, implicit systems, generalized state-space systems, differential- algebraic systems, semi-state systems), there are only a few papers dealing with the stabilization problem and results on this topic are far fewer than those on state-space systems((Dai, 1989),(Pandolfi, 1980) and (A, 1995)). On the other hand, control of singular systems has been extensively studied in the past years due to the fact that singular system better describe physical systems than regular ones. Very recently, much attention has been paid to singular systems with time delay. For the continuous case, numerical methods for such systems were discussed in (see, (Ascher and Petzold, 1995), (Campbell, 1980), (S. Xu, 2000) and (S. Xu and Lam, 2002)).
However, it is not easy to measure the state, so it is difficult to realize state feedback. Therefore, the problem of designing an observer based feedback controller for a linear plant to make the closed-loop system stable has been discussed in many papers during the two decades (Zhang M. and Y., 1998) , (Zidong W. and Unbehauen, 2001) and (Su H. and J., 1998). At present the observer design of linear time delay systems have mainly two methods. One does include no delay information in the observer (see, (Zhang M. and Y., 1998) , (Zidong W. and Unbehauen, 2001) and (Su H. and J., 1998)). The design of this observer is quite simple, but this observer can't reflect the message of the system itself completely and the design of the controller and observer are not separated. The second method takes account of delay information in the observer (see, (Su H. and J., 1998), (C. and C., 96)). This observer can reflects completely the message of the system itself and the design of the controller and that of the observer are separated.

In this paper, we address the problem of stabilization by state feedback control laws provided by an observer. The sufficient conditions are developed for checked by an iterative algorithm if this class of sin-

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gular time-delay system is regular, impulse free and stable. The paper is organized as follows. In section 2, the problem is stated and the required assumptions are formulated. Section 3 presents the main results obtained for the class of systems under study. Section 4 presents some numerical examples to show the usefulness of the proposed results.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following uncertain singular systems with multiple delays :

$$E\dot{x}(t) = Ax(t) + A_d x(t-h) + Bu(t) \tag{1}$$

where $x(t)$ is the state vector, in $\mathbb{R}^n$, h is the delay of the system and the matrices A and A_d are of appropriate dimension.

Definition 2.1. (Dai, 1989)

(1) The pair (E, A) is said to be regular if $det(sE - A)$ is not identically zero.
(2) The pair (E, A) is said to be impulse free if $deg(det(sE - A)) = rankE$.

The singular delay system (1) may have an impulsive solution, however the regularity and the absence of impulses of the pair (E, A) ensure the existence and uniqueness of an impulse free solution to this system, which is shown in following lemma.

Lemma 2.1. Suppose the pair (E, A) is regular and impulse free, then the solution to (1) exists and is impulse free and unique on $[0, \infty)$

In view of this, we introduce the following definition for singular delay system (1).

Definition 2.2. (S. Xu and Lam, 2002)

- The singular delay system (1) is said to be regular and impulse free if the pair (E, A) is regular and impulse free.
- The singular delay system (1) is said to be stable if for any $\varepsilon > 0$ there exists a scalar $\delta(\varepsilon) > 0$ such that, for any compatible initial conditions $\phi(t)$ satisfying $\sup_{-\tau \le t \le 0} \parallel \phi(t) \parallel \le \delta(\varepsilon)$, the solution $x(t)$ of system (1) satisfies $\parallel x(t) \parallel \le \varepsilon$ for $t \ge 0$. Furthermore

$$x(t) \to 0 \; t \to \infty$$

The following lemma is very useful for our development in this paper.

Lemma 2.2. For any $z, y \in \mathbb{R}^n$ and for any symmetric positive-definite matrix $X \in \mathbb{R}^{n \times n}$:

$$-2z^\top y \le z^\top X^{-1} z + y^\top X y$$

Lemma 2.3. Consider the function $\varphi : \mathbb{R}^+ \to \mathbb{R}$. if $\dot{\varphi}$ is bounded on $[0, \infty)$, that is, there exists a scalar $\alpha > 0$ such that $\mid \dot{\varphi}(t) \mid \le \alpha$ for all $t \in [0, \infty)$, then φ is uniformly continuous on $[0, \infty)$.

Lemma 2.4. Barbalat's Lemma: Consider the function $\varphi : \mathbb{R}^+ \to \mathbb{R}$. if φ is uniformly continuous and $\int_0^\infty \varphi(s) ds < \infty$, then

$$\lim_{t \to \infty} \varphi(t) = 0$$

In the rest of this paper the notation is standard unless it is specified otherwise. $L > 0$ ($L < 0$) means that the matrix L is symmetric and positive-definite matrix (symmetric and negative-definite).

Remark 2.1. : Notation

- In the sequel $Sym\{\centerdot\}$ is defined as

$$Sym\{X\} = X + X^\top$$

for any matrix X
- for example, The matrix $M_{e,x} < 0$ is equivalent to

$$M_x < 0 \quad \text{and} \quad M_e < 0$$

3. OBSERVER BASED CONTROLLER SYNTHESIS

The goal of this section consists of establishing what will be the sufficient conditions that can be used to check whether or not the class of systems under study is stable with an observer based controller. Consider the system given by the following dynamics:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-h) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2}$$

The observer based controller is given as

$$u(t) = Kz(t) \tag{3}$$

with K is the gain of the controller and the vector $z(t)$ is the state of the observer whose dynamics are given by

$$E\dot{z}(t) = Az(t) + A_d z(t-h) + Bu(t) + L\,(y(t) - Cz(t)) \tag{4}$$

where L is the gain of the observer. Theorem 3.1 states the stability conditions of the system feedback by an observer based controller.

Theorem 3.1. If there exist the matrices $P_x > 0$, $Q_x > 0$, $W_x > 0$, Y_x, Z_x, $P_e > 0$, $Q_e > 0$, $W_e > 0$, Y_e, Z_e, $S_1 > 0$ and $S_2 > 0$ such that the following hold:

$$\begin{cases} E^\top P_e = P_e^\top E \ge 0 \\ E^\top P_x = P_x^\top E \ge 0 \end{cases} \tag{5}$$

$$
\begin{cases}
\mathbb{T}_e = \begin{bmatrix} Z_e & Y_e \\ Y_e^\top & E^\top W_e E \end{bmatrix} \geq 0 \\
\mathbb{T}_x = \begin{bmatrix} Z_x & Y_x \\ Y_x^\top & E^\top W_x E \end{bmatrix} \geq 0
\end{cases} \tag{6}
$$

$$
\mathbb{M}_e = \begin{bmatrix} (\mathbb{M}_e)_{11} & (\mathbb{M}_e)_{12} & hA_0^\top W_e \\ (\mathbb{M}_e)_{12}^\top & -\Psi_{x_2} & hA_d^\top W_e \\ hW_eA_0 & hW_eA_d & -hW_e \end{bmatrix} < 0
$$

$$
\mathbb{M}_x = \begin{bmatrix} (\mathbb{M}_x)_{11} & (\mathbb{M}_x)_{12} & (\mathbb{M}_x)_{31}^\top \\ (\mathbb{M}_x)_{12}^\top & -\bar{\Psi}_{x_2} & P_x^{-1}A_d^\top \\ (\mathbb{M}_x)_{31} & A_dP_x^{-1} & (\mathbb{M}_x)_{33} \end{bmatrix} < 0
$$

with

$$
\begin{aligned}
(\mathbb{M}_e)_{11} &= \mathsf{Sym}\{P_eA - R_eC\} + \Psi_{e_1} \\
(\mathbb{M}_e)_{12} &= P_eA_d - \Psi_{x_3} \\
(\mathbb{M}_x)_{11} &= \mathsf{Sym}\left\{P_x^{-1}A^\top + R_x^\top B^\top\right\} + S_1^{-1} + \bar{\Psi}_{x_1} \\
(\mathbb{M}_x)_{33} &= -h^{-1}\left(W_x + W_xS_2W_x\right)^{-1} \\
(\mathbb{M}_x)_{31} &= \left(AP_x^{-1} + BR_x\right) \\
(\mathbb{M}_x)_{12} &= A_dP_x^{-1} - \bar{\Psi}_{x_3}
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{e_1} &= Q_e + hZ_e + Y_e^\top + Y_e \\
&\quad + K^\top B^\top \left(S_1 + hW_x + hS_2^{-1}\right) BK \\
\Psi_{e_2} &= Q_e, \qquad \Psi_{e_3} = Y_e^\top, \\
\Psi_{x_1} &= Q_x + hZ_x + Y_x^\top + Y_x \\
\Psi_{x_2} &= Q_x, \qquad \Psi_{x_3} = Y_x^\top
\end{aligned}
$$

Then system (2-3-4) is asymptotically stable.

To check the result stated by Theorem 3.1 we use the following iterative algorithm

Algorithm

1. Solve the LMI

$$\mathbb{M}_x < 0$$

for the decision variables P_x^{-1}, $R_x = KP_x^{-1}$, Q_x, Z_x, Y_x, S_1^{-1} and $\mathbb{W}_x^{-1} = (W_x + W_xS_2W_x)^{-1}$,

2. Choose $W_x > 0$ in such a way that

$$
\begin{aligned}
&\mathbb{T}_x \geq 0 \\
&S_2 = W_x^{-1}\left(\mathbb{W}_x - W_x\right)W_x^{-1} > 0
\end{aligned}
$$

hold

3. Solve the LMI problem

$$
\begin{aligned}
\mathbb{M}_e &< 0 \\
\mathbb{T}_e &\geq 0
\end{aligned}
$$

for the decision variables P_e, $R_e = P_eL$, Q_e, Z_e, Y_e.

Proof of Theorem 3.1

First, let the observer error be given as

$$e(t) = x(t) - z(t)$$

which leads the observer error dynamics as

$$E\dot{e}(t) = A_0e(t) + A_de(t-h) \tag{7}$$

with

$$A_0 = (A - LC)$$

Note that the system dynamics can be rewritten as

$$E\dot{x}(t) = Ax(t) + A_dx(t-h) + BKx(t) - BKe(t)$$

The system and the observer are given by

$$
\begin{aligned}
E\dot{x}(t) &= A_cx(t) + A_dx(t-h) - BKe(t) \\
E\dot{e}(t) &= A_0e(t) + A_de(t-h)
\end{aligned} \tag{8}
$$

with $\qquad A_c = A + BK$

We recall that regularity and absence of impulses of the pair (E, A) implies that there exist two invertible matrices G and $H \in \mathbb{R}^{n\times n}$ such that (Dai, 1989)

$$
\begin{aligned}
\bar{E} &= GEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \\
\bar{A}_{c,0} &= GA_{c,0}H = \begin{bmatrix} A_{x,e} & 0 \\ 0 & I_{n-r} \end{bmatrix} \\
\bar{A}_d &= GA_dH = \begin{bmatrix} A_{d_{11}} & A_{d_{12}} \\ A_{d_{21}} & A_{d_{22}} \end{bmatrix} \\
\bar{B} &= GB \qquad \bar{C} = CH
\end{aligned} \tag{9}
$$

where $I_r \in \mathbb{R}^{r\times r}$ and $I_{n-r} \in \mathbb{R}^{n-r\times n-r}$ are identity matrices.

This transformation is applied to equations of theorem 3.1 with

$$
\begin{aligned}
\bar{P}_{x,e} &= G^{-\top}P_{x,e}H \qquad \bar{W}_{x,e} = G^{-\top}W_{x,e}G^{-1} \\
\bar{Q}_{x,e} &= H^\top Q_{x,e}H \qquad \bar{Z}_{x,e} = H^\top Z_{x,e}H \\
\bar{R}_x &= R_xG^{-1} \qquad \bar{Y}_{x,e} = H^\top Y_{x,e}H
\end{aligned} \tag{10}
$$

Noting the expression of $\bar{E}$ in (9) and using (5), we can deduce that $\bar{P}_{x_{11},e_{11}} = \bar{P}_{x_{11},e_{11}}^\top \geq 0$ and $\bar{P}_{x_{12},e_{12}} = 0$, therefore $\bar{P}_{x,e}$ reduces to

$$
\bar{P}_{x,e} = \begin{bmatrix} \bar{P}_{x_{11},e_{11}} & 0 \\ \bar{P}_{x_{21},e_{21}} & \bar{P}_{x_{22},e_{22}} \end{bmatrix} \tag{11}
$$

Now, let

$$\zeta_x(t) = H^{-1}x(t) \qquad \zeta_e(t) = H^{-1}e(t) \tag{12}$$

where $\zeta_{x_1,e_1} \in \mathbb{R}^r$, $\zeta_{x_2,e_2} \in \mathbb{R}^{n-r}$. Using (9), the singular delay system (8) can be written as

$$
\begin{aligned}
\dot{\zeta}_{x_1}(t) &= \bar{A}_x\zeta_{x_1}(t) + \bar{A}_{d_{11}}\zeta_{x_1}(t-h) + \bar{A}_{d_{12}}\zeta_{x_2}(t-h) \\
&\quad - \bar{B}_1\bar{K}_1\zeta_{e_1}(t) - \bar{B}_1\bar{K}_2\zeta_{e_2}(t) \\
0 &= \zeta_{x_2}(t) + \bar{A}_{d_{21}}\zeta_{x_1}(t-h) + \bar{A}_{d_{22}}\zeta_{x_2}(t-h) \\
&\quad - \bar{B}_2\bar{K}_1\zeta_{e_1}(t) - \bar{B}_2\bar{K}_2\zeta_{e_2}(t) \\
\dot{\zeta}_{e_1}(t) &= \bar{A}_e\zeta_{e_1}(t) + \bar{A}_{d_{11}}\zeta_{e_1}(t-h) + \bar{A}_{d_{12}}\zeta_{e_2}(t-h) \\
0 &= \zeta_{e_2}(t) + \bar{A}_{d_{21}}\zeta_{e_1}(t-h) + \bar{A}_{d_{22}}\zeta_{e_2}(t-h)
\end{aligned} \tag{13}
$$

In order to investigate the stability of the closed loop system (8), let us consider the Lyapunov functional candidate :

$$V(\zeta_{\mathbf{x}_t}, \zeta_{\mathbf{e}_t}) = V_1(\zeta_{\mathbf{x}_t}, \zeta_{\mathbf{e}_t}) + V_2(\zeta_{\mathbf{x}_t}, \zeta_{\mathbf{e}_t}) + V_3(\zeta_{\mathbf{x}_t}, \zeta_{\mathbf{e}_t})$$

$$+V_4(\zeta_{\mathbf{x}_t},\zeta_{\mathbf{e}_t})$$

with

$$V_1(\zeta_{\mathbf{x}_t},\zeta_{\mathbf{e}_t}) = \zeta_e(t)^\top \bar{P}_e^\top \bar{E}\zeta_e(t) + \zeta_x(t)^\top \bar{P}_x^\top \bar{E}\zeta_x(t)$$

$$V_2(\zeta_{\mathbf{x}_t},\zeta_{\mathbf{e}_t}) = \int_{t-h}^{t}\int_{s}^{t} \dot{\zeta}_e(\tau)^\top \bar{E}^\top \bar{W}_e \bar{E}\dot{\zeta}_e(\tau)d\tau ds + \int_{t-h}^{t}\int_{s}^{t} \dot{\zeta}_x(\tau)^\top \bar{E}^\top \bar{W}_x \bar{E}\dot{\zeta}_x(\tau)d\tau ds$$

$$V_3(\zeta_{\mathbf{x}_t},\zeta_{\mathbf{e}_t}) = \int_{t-h}^{t} \zeta_e(s)^\top \bar{Q}_e\zeta_e(s)ds + \int_{t-h}^{t} \zeta_x(s)^\top \bar{Q}_x\zeta_x(s)ds$$

$$V_4(\zeta_{\mathbf{x}_t},\zeta_{\mathbf{e}_t}) = \int_{0}^{t}\int_{\tau-h}^{\tau} \begin{bmatrix} \zeta_e(\tau)^\top & \dot{\zeta}_e(s)^\top \end{bmatrix} \begin{bmatrix} \bar{Z}_e & \bar{Y}_e \\ \bar{Y}_e^\top & \bar{E}^\top \bar{W}_e \bar{E} \end{bmatrix} \begin{bmatrix} \zeta_e(\tau) \\ \dot{\zeta}_e(s) \end{bmatrix} ds d\tau + \int_{0}^{t}\int_{\tau-h}^{\tau} \begin{bmatrix} \zeta_x(\tau)^\top & \dot{\zeta}_x(s)^\top \end{bmatrix} \begin{bmatrix} \bar{Z}_x & \bar{Y}_x \\ \bar{Y}_x^\top & \bar{E}^\top \bar{W}_x E \end{bmatrix} \begin{bmatrix} \zeta_x(\tau) \\ \dot{\zeta}_x(s) \end{bmatrix} ds d\tau$$

therefore, the first derivative of the Lyapunov functional candidate is :

$$\begin{aligned}
\dot{V}(\zeta_{\mathbf{x}_t},\zeta_{\mathbf{e}_t}) \leq\ & \zeta_e(t)^\top \left(\bar{P}_e^\top \bar{A}_0 + \bar{A}_0^\top \bar{P}_e + \bar{K}^\top \bar{B}^\top S_1 \bar{B}\bar{K}\right)\zeta_e(t) \\
& +2\zeta_e(t)^\top \bar{P}_e^\top \bar{A}_d \zeta_e(t-h) \\
& +\zeta_e(t)^\top \bar{Q}_e\zeta_e(t) - \zeta_e(t-h)^\top \bar{Q}_e \zeta_e(t-h) \\
& +h\left(\bar{A}_0\zeta_e(t) + \bar{A}_d\zeta_e(t-h)\right)^\top \bar{W}_e \\
& \quad \times \left(\bar{A}_0\zeta_e(t) + \bar{A}_d\zeta_e(t-h)\right) \\
& +h\zeta_e(t)^\top \bar{Z}_e\zeta_e(t) + 2\zeta_e(t)^\top \bar{Y}_e\left(\zeta_e(t) - \zeta_e(t-h)\right) \\
& +\zeta_x(t)^\top \left(\bar{P}_x^\top \bar{A}_c + \bar{A}_c^\top \bar{P}_x + \bar{P}_x^\top S_1^{-1}\bar{P}_x\right)x(t) \\
& +2\zeta_x(t)^\top \bar{P}_x^\top \bar{A}_d\zeta_x(t-h) \\
& +\zeta_x(t)^\top \bar{Q}_x\zeta_x(t) - \zeta_x(t-h)^\top \bar{Q}_x\zeta_x(t-h) \\
& +h\underline{\left(\bar{A}_c\zeta_x(t) + \bar{A}_d\zeta_x(t-h) - \bar{B}\bar{K}\zeta_e(t)\right)^\top \bar{W}_x \times} \\
& \quad \underline{\left(\bar{A}_c\zeta_x(t) + \bar{A}_d\zeta_x(t-h) - \bar{B}\bar{K}\zeta_e(t)\right)} \\
& +h\zeta_x(t)^\top \bar{Z}_x\zeta_x(t) + 2\zeta_x(t)^\top \bar{Y}_x\left(\zeta_x(t) - \zeta_x(t-h)\right)
\end{aligned}$$

Note that the underlined expression above can be bounded as follows

$$\begin{aligned}
& \left(\bar{A}_c\zeta_x(t) + \bar{A}_d\zeta_x(t-h) - \bar{B}\bar{K}\zeta_e(t)\right)^\top \bar{W}_x \times \\
& \left(\bar{A}_c\zeta_x(t) + \bar{A}_d\zeta_x(t-h) - \bar{B}\bar{K}\zeta_e(t)\right) \\
= & \left(\bar{A}_c\zeta_x(t) + \bar{A}_d\zeta_x(t-h)\right)^\top \bar{W}_x \times \\
& \left(\bar{A}_c\zeta_x(t) + \bar{A}_d\zeta_x(t-h)\right) \\
& -2\left(\bar{A}_c\zeta_x(t) + \bar{A}_d\zeta_x(t-h)\right)^\top \bar{W}_x\left(\bar{B}\bar{K}\zeta_e(t)\right) \\
& +\left(\bar{B}\bar{K}\zeta_e(t)\right)^\top \bar{W}_x\left(\bar{B}\bar{K}\zeta_e(t)\right)
\end{aligned}$$

and using lemma 2.1, we deduce easily what follows : $\exists S_2 > 0$ such as :

$$\begin{aligned}
\leq & \left(\bar{A}_c\zeta_x(t) + \bar{A}_d\zeta_x(t-h)\right)^\top \bar{W}_x\left(\bar{A}_c\zeta_x(t) + \bar{A}_d\zeta_x(t-h)\right) \\
+ & \left(\bar{A}_c\zeta_x(t) + \bar{A}_d\zeta_x(t-h)\right)^\top \bar{W}_x S_2 \bar{W}_x \\
& \times\left(\bar{A}_c\zeta_x(t) + \bar{A}_d\zeta_x(t-h)\right) \\
+ & \left(\bar{B}\bar{K}\zeta_e(t)\right)S_2^{-1}\left(\bar{B}\bar{K}\zeta_e(t)\right) + \left(\bar{B}\bar{K}\zeta_e(t)\right)^\top \bar{W}_x\left(\bar{B}\bar{K}\zeta_e(t)\right)
\end{aligned}$$

Finally, the first derivative of the Lyapunov functional candidate is

$$\begin{aligned}
\dot{V}(\zeta_{\mathbf{x}_t},\zeta_{\mathbf{e}_t}) \leq\ & 2\zeta_e(t)^\top \left(\bar{P}_e^\top \bar{A}_d - \bar{\Psi}_{e_3} + h\bar{A}_0^\top \bar{W}_e\bar{A}_d\right)\zeta_e(t-h) \\
& +\zeta_e(t)^\top \left(\bar{P}_e^\top \bar{A}_0 + \bar{A}_0^\top \bar{P}_e + \bar{\Psi}_{e_1} + h\bar{A}_0^\top \bar{W}_e\bar{A}_0\right. \\
& \left.+\bar{K}^\top \bar{B}^\top\left(S_1 + h\bar{W}_x + hS_2^{-1}\right)\bar{B}\bar{K}\right)\zeta_e(t) \\
& +\zeta_e(t-h)^\top\left(-\bar{\Psi}_{e_2} + h\bar{A}_d^\top \bar{W}_e\bar{A}_d\right)\zeta_e(t-h) \\
& +2\zeta_x(t)^\top\left(\bar{P}_x^\top \bar{A}_d - \bar{\Psi}_{x_3}\right. \\
& \left.+h\bar{A}_c^\top\left(\bar{W}_x + \bar{W}_x S_2\bar{W}_x\right)\bar{A}_d\right)\zeta_x(t-h) \\
& +\zeta_x(t)^\top\left(\bar{P}_x^\top \bar{A}_c + \bar{A}_c^\top \bar{P}_x + \bar{\Psi}_{x_1}\right. \\
& + h\bar{A}_c^\top\left(\bar{W}_x + \bar{W}_x S_2\bar{W}_x\right)\bar{A}_c \\
& \left.+\bar{P}_x^\top S_1^{-1}\bar{P}_x\right)\zeta_x(t) \\
& +\zeta_x(t-h)^\top\left(-\bar{\Psi}_{x_2}\right. \\
& \left.+ h\bar{A}_d^\top\left(\bar{W}_x + \bar{W}_x S_2\bar{W}_x\right)\bar{A}_d\right)\zeta_x(t-h)
\end{aligned}$$

and using the lemma 2.2, and assuming that

$$\begin{cases} h\bar{A}_d^\top\left(\bar{W}_x + \bar{W}_x S_2\bar{W}_x\right)\bar{A}_d - \bar{\Psi}_{x_2} < 0 \\ h\bar{A}_d^\top \bar{W}_e\bar{A}_d - \bar{\Psi}_{e_2} < 0 \end{cases} \tag{14}$$

we deduce the following expression

$$\begin{aligned}
\dot{V}(\zeta_{\mathbf{x}_t},\zeta_{\mathbf{e}_t}) \leq\ & \zeta_e(t)^\top\Big\{\left(\bar{P}_e^\top \bar{A}_d - \bar{\Psi}_{e_3} + h\bar{A}_0^\top \bar{W}_e\bar{A}_d\right)\times \\
& \left(\bar{\Psi}_{e_2} - h\bar{A}_d^\top \bar{W}_e\bar{A}_d\right)^{-1}\times \\
& \left(\bar{P}_e^\top \bar{A}_d - \bar{\Psi}_{e_3} + h\bar{A}_0^\top \bar{W}_e\bar{A}_d\right)^\top \\
& +\left(\bar{P}_e^\top \bar{A}_0 + \bar{A}_0^\top \bar{P}_e + \bar{\Psi}_{e_1} + h\bar{A}_0^\top \bar{W}_e\bar{A}_0\right. \\
& \left.+\bar{K}^\top \bar{B}^\top\left(S_1 + h\bar{W}_x + hS_2^{-1}\right)\bar{B}\bar{K}\right)\Big\}\zeta_e(t) \\
& \zeta_x(t)^\top\Big\{\left[\bar{P}_x^\top \bar{A}_d - \bar{\Psi}_{x_3}\right. \\
& \left.+h\bar{A}_c^\top\left(\bar{W}_x + \bar{W}_x S_2\bar{W}_x\right)\bar{A}_d\right]\times \\
& \left(\bar{\Psi}_{x_2} - h\bar{A}_d^\top\left(\bar{W}_x + \bar{W}_x S_2\bar{W}_x\right)\bar{A}_d\right)^{-1}\times \\
& \times\left[\bar{P}_x^\top \bar{A}_d - \bar{\Psi}_{x_3} + h\bar{A}_c^\top\left(\bar{W}_x + \bar{W}_x S_2\bar{W}_x\right)\bar{A}_d\right]^\top \\
& +\left(\bar{P}_x^\top \bar{A}_c + \bar{A}_c^\top \bar{P}_x + \bar{\Psi}_{x_1}\right. \\
& + h\bar{A}_c^\top\left(\bar{W}_x + \bar{W}_x S_2\bar{W}_x\right)\bar{A}_c \\
& \left.+\bar{P}_x^\top S_1^{-1}\bar{P}_x\right)\Big\}\zeta_x(t)
\end{aligned}$$

The expression above can be written in a compact form as follows

$$\begin{aligned}
\dot{V}(\zeta_t) \leq\ & \zeta_x^\top(t)(M_{x_{11}} - M_{x_{12}}M_{x_{22}}^{-1}M_{x_{12}}^\top)\zeta_x(t) \\
& +\zeta_e^\top(t)(M_{e_{11}} - M_{e_{12}}M_{e_{22}}^{-1}M_{e_{12}}^\top)\zeta_e(t)
\end{aligned} \tag{15}$$

with

$$M_{x_{11}} = \bar{A}_c^\top \bar{P}_x + \bar{P}_x^\top \bar{A}_c + h A_c^\top \left(\bar{W}_x + \bar{W}_x S_2 \bar{W}_x\right) \bar{A}_c + \bar{P}_x^\top S_1^{-1} \bar{P}_x + \bar{\Psi}_{x_1}$$
$$M_{x_{12}} = \bar{P}_x^\top \bar{A}_d - \bar{\Psi}_{x_3} + h\bar{A}_c^\top \left(\bar{W}_x + \bar{W}_x S_2 \bar{W}_x\right) A_d$$
$$M_{x_{22}} = h\bar{A}_d^\top \left(\bar{W}_x + \bar{W}_x S_2 \bar{W}_x\right) \bar{A}_d - \bar{\Psi}_{x_2}$$
$$M_{e_{11}} = \bar{A}_0^\top \bar{P}_e + \bar{P}_e^\top \bar{A}_0 + h\bar{A}_0^\top \bar{W}_e \bar{A}_0 + \bar{\Psi}_{e_1}$$
$$M_{e_{12}} = \bar{P}_e^\top \bar{A}_d - \bar{\Psi}_{e_3} + h\bar{A}_0^\top \bar{W}_e \bar{A}_d$$
$$M_{e_{22}} = h\bar{A}_d^\top \bar{W}_e \bar{A}_d - \bar{\Psi}_{e_2}$$

It comes then that $\dot{V}(\zeta_t)$ is definite negative if $\left(M_{x_{11},e_{11}} - M_{x_{12},e_{12}} M_{x_{22},e_{22}}^{-1} M_{x_{12},e_{12}}^\top\right) < 0$ which associated with (14) can be expressed as

$$M_x = \begin{bmatrix} M_{x_{11}} & M_{x_{12}} \\ M_{x_{12}}^\top & M_{x_{22}} \end{bmatrix}, \qquad M_e = \begin{bmatrix} M_{e_{11}} & M_{e_{12}} \\ M_{e_{12}}^\top & M_{e_{22}} \end{bmatrix}$$

Notice that matrix M_e and M_x can be expressed as follows:

$$M_e = \begin{bmatrix} \bar{A}_0^\top \bar{P}_e + \bar{P}_e^\top \bar{A}_0 + \bar{\Psi}_{e_1} & \bar{P}_e^\top \bar{A}_d - \bar{\Psi}_{e_3} \\ (\bar{P}_e^\top \bar{A}_d - \bar{\Psi}_{e_3})^\top & -\bar{\Psi}_{e_2} \end{bmatrix} + \begin{bmatrix} h\bar{A}_0^\top \bar{W}_e \\ h\bar{A}_d^\top \bar{W}_e \end{bmatrix} (h\bar{W}_e)^{-1} \left[h\bar{W}_e \bar{A}_0 \quad h\bar{W}_e \bar{A}_d \right] < 0$$
$$M_x = \begin{bmatrix} \bar{A}_c^\top \bar{P}_x + \bar{P}_x^\top \bar{A}_c + \bar{P}_x^\top S_1^{-1} \bar{P}_x + \bar{\Psi}_{x_1} & \bar{P}_x^\top \bar{A}_d - \bar{\Psi}_{x_3} \\ (\bar{P}_x^\top \bar{A}_d - \bar{\Psi}_{x_3})^\top & -\bar{\Psi}_{x_2} \end{bmatrix} + \begin{bmatrix} \bar{A}_c^\top \\ \bar{A}_d^\top \end{bmatrix} \left(h\left(\bar{W}_x + \bar{W}_x S_2 \bar{W}_x\right)\right) \left[\bar{A}_c \quad \bar{A}_d \right] < 0$$

We taking into account of $W_e = \alpha P_e$ for have a LMI on the matrix $\mathbb{M}_e$.
Using the Schur complement, M_e and M_x are negative definite if and only if we have

$$\mathbb{M}_e = \begin{bmatrix} \bar{A}_0^\top \bar{P}_e + \bar{P}_e^\top \bar{A}_0 + \bar{\Psi}_{e_1} & \bar{P}_e^\top \bar{A}_d - \bar{\Psi}_{e_3} & h\bar{A}_0^\top \alpha \bar{P}_e \\ (\bar{P}_e^\top \bar{A}_d - \bar{\Psi}_{e_3})^\top & -\bar{\Psi}_{e_2} & h\alpha A_d^\top \bar{P}_e \\ h\alpha \bar{P}_e \bar{A}_0 & h\alpha \bar{P}_e \bar{A}_d & -h\alpha \bar{P}_e \end{bmatrix} < 0$$
$$\hat{\mathbb{M}}_x = \begin{bmatrix} \bar{A}_c^\top \bar{P}_x + \bar{P}_x^\top \bar{A}_c + \bar{P}_x^\top S_1^{-1} \bar{P}_x + \bar{\Psi}_{x_1} & \bar{P}_x^\top \bar{A}_d - \bar{\Psi}_{x_3} & \bar{A}_c^\top \\ (\bar{P}_x^\top \bar{A}_d - \bar{\Psi}_{x_3})^\top & -\bar{\Psi}_{x_2} & \bar{A}_d^\top \\ \bar{A}_c & \bar{A}_d & -\left(h\bar{W}_x + h\bar{W}_x S_2 \bar{W}_x\right)^{-1} \end{bmatrix} < 0$$

Pre- and post-multiply $\hat{\mathbb{M}}_x$ by $\mathbb{J}_x = \mathrm{diag}\left[P_x^{-1} \quad P_x^{-1} \quad I \right]$ it comes that $\hat{\mathbb{M}}_x$ is negative definite if and only if we have

$$\mathbb{M}_x = \begin{bmatrix} \bar{P}_x^{-\top} \bar{A}_c^\top + \bar{A}_c \bar{P}_x^{-1} + S_1^{-1} + \tilde{\Psi}_{x_1} & \bar{A}_d \bar{P}_x^{-1} - \tilde{\Psi}_{x_3} & \bar{P}_x^{-\top} \bar{A}_c^\top \\ (\bar{A}_d \bar{P}_x^{-1} - \tilde{\Psi}_{x_3})^\top & -\tilde{\Psi}_{x_2} & \bar{P}_x^{-\top} \bar{A}_d^\top \\ \bar{A}_c P_x^{-1} & \bar{A}_d P_x^{-1} & -h^{-1}\left(\bar{W}_x + \bar{W}_x S_2 \bar{W}_x\right)^{-1} \end{bmatrix} < 0$$

with

$$\tilde{\Psi}_{x_{1,2,3}} = \bar{P}_x^{-\top} \bar{\Psi}_{x_{1,2,3}} \bar{P}_x^{-1}$$

It follows from inequality (15) that $\dot{V}(\zeta_t) < 0$ and

$$\begin{aligned} &\lambda_1 \parallel \zeta_{x_1,e_1}(t) \parallel^2 - V(\zeta_0) \qquad (16) \\ &\leq \zeta_{x,e}(t)^\top diag\left(\bar{P}_x, \bar{P}_e\right) diag\left(\bar{E}, \bar{E}\right) \zeta_{x,e}(t) \\ &+ \int_{t-h}^{t} \int_{s}^{t} \dot{\zeta}_{x,e}(\tau)^\top diag\left(\bar{E}, \bar{E}\right) \times \\ &diag\left(\bar{W}_x, \bar{W}_e\right) \dot{\zeta}_{x,e}(\tau) d\tau ds \\ &+ \int_{t-h}^{t} \zeta_{x,e}(s)^\top diag\left(\bar{Q}_x, \bar{Q}_e\right) \zeta_{x,e}(s) ds \\ &+ h\zeta_{x,e}(t)^\top diag\left(\bar{Z}_x, \bar{Z}_e\right) \zeta_{x,e}(t) \\ &+ 2\zeta_{x,e}(t)^\top diag\left(\bar{Y}_x^\top, \bar{Y}_e^\top\right) \int_{t-h}^{t} \dot{\zeta}_{x,e}(s) ds \\ &+ \int_{t-h}^{t} \dot{\zeta}_{x,e}(\tau)^\top diag\left(\bar{E}, \bar{E}\right) \times \\ &\left(\bar{W}_x, \bar{W}_e\right) \dot{\zeta}_{x,e}(\tau) d\tau - V(\zeta_{x,e}(0)) \\ &= \dot{V}(\zeta_{\mathbf{x},\mathbf{e}}(\mathbf{t})) \leq -\lambda_2 \int_{0}^{t} \parallel \zeta_{x,e}(s) \parallel^2 ds \\ &\leq -\lambda_2 \int_{0}^{t} \parallel \zeta_{x_1,e_1}(s) \parallel^2 ds \qquad (17) \end{aligned}$$

with $\zeta_{x,e}(t) = \left[\zeta_x^\top \quad \zeta_e^\top \right]^\top$ $\lambda_1 = \lambda_{min}\{P_{x_{11}}, P_{e_{11}}\} > 0$

$$\lambda_2 = -\lambda_{max}\left(\begin{bmatrix} M_x & \mathbb{O} \\ \mathbb{O} & M_e \end{bmatrix} \right) > 0$$

Taking (16) into account, we deduce that

$$\lambda_1 \parallel \zeta_{x_1,e_1}(t) \parallel^2 + \lambda_2 \int_0^t \parallel \zeta_{x_1,e_1}(s) \parallel^2 ds \leq V(\zeta_0)$$

such that

$$\parallel \zeta_{x_1,e_1}(t) \parallel^2 \leq c_1 \quad \text{and} \quad \int_0^t \parallel \zeta_{x_1,e_1}(s) \parallel^2 \leq c_2 \qquad (18)$$

where

$$c_1 = \frac{1}{\lambda_1} V(\zeta_0) \qquad c_2 = \frac{1}{\lambda_2} V(\zeta_0) \qquad (19)$$

Thus, $\parallel \zeta_{x_1}(t) \parallel$ and $\parallel \zeta_{e_1}(t) \parallel$ are bounded and from (13) it comes out that $\frac{d}{dt} \parallel \zeta_{x_1,e_1}(t) \parallel^2$ is bounded too. By Lemma 2.3, we have that $\parallel \zeta_{x_1,e_1}(t) \parallel^2$ is uniformly continuous. Therefore, taking account of (18) and using Lemma 2.3 we obtain that

$$\lim_{t\to\infty} \parallel \zeta_{x_1}(t) \parallel = 0 \quad \lim_{t\to\infty} \parallel \zeta_{e_1}(t) \parallel = 0 \qquad (20)$$

Now we have to state the same behaviour for $\zeta_{x_2}(t)$ and $\zeta_{e_2}(t)$ respectively. To do so, note that for any $t > 0$, there exists a positive integer k such that $k\bar{h} - \bar{h} \leq t < k\bar{h}$, we have

$$\zeta_{x_2}(t) = B_2 K_1 \zeta_{e_1}(t) - \sum_{i=1}^{k} (-A_{d_{22}})^{i-1} \left[A_{d_{21}} \zeta_{x_1}(t - ih) \right.$$

$$+(B_2K_1A_{d_{22}}+iB_2K_2A_{d_{21}})\zeta_{e_1}(t-ih)\Big]$$
$$-(-A_{d_{22}})^k\,[B_2K_1\zeta_{e_1}(t-kh)$$
$$-kB_2K_2\zeta_{e_2}(t-kh)+\zeta_{x_2}(t-kh)] \quad (21)$$

$$\zeta_{e_2}(t)=-\sum_{i=1}^{k}(-A_{d_{22}}^{i-1}A_{d_{21}}\zeta_{e_1}(t-ih)) +(-A_{d_{22}})^k\zeta_{e_2}(t-kh)$$

Since $\| \zeta_{x_1}(t) \|$ and $\| \zeta_{e_1}(t) \|$ are bounded and if

$$\rho(A_{d_{22}})<1 \quad (22)$$

then it comes out that

$$\lim_{t\to\infty} \| \zeta_{x_2}(t) \|=0 \quad \lim_{t\to\infty} \| \zeta_{e_2}(t) \|=0 \quad (23)$$

Thus, the closed-loop system (1) is stable.

Remark 3.1. The results of Theorem (3.1) are only sufficient and therefore if these conditions are not verified we can't claim that the system under study is not stable.

Example 3.1. In this example, we consider that the singular system under study has one time-delay. Let us assume that the dynamics are described by the following matrices:

$$A_0=\begin{bmatrix}-1 & 0.0\\ -0.01 & 1\end{bmatrix} \quad A_1=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix} \quad \alpha=0.001$$
$$E=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} \quad B=\begin{bmatrix}0\\ 1\end{bmatrix} \quad C=[0 \quad 1] \quad (24)$$

Applying Theorem 3.1 for the overall system leads us to state that this system remains stable for any delay $h\leq\bar{h}=1.15s$ with

$$K=[37.0734 \quad 28.0279] \qquad L=\begin{bmatrix}-0.2206\\ 6.7031\end{bmatrix}$$

4. CONCLUSION

In this paper, we have discussed the design of observer-based feedback controller for singular time-delay systems. Delay-dependent sufficient conditions have been developed to check whether a system of this class of is stable or unstable, a state feedback controller with consequent parameters has been used to stabilize the system. The LMI technique is used in all the development. We provided an iterative algorithm to solve the feasibility of the obtained LMI's. Finally, a numerical example is given to illustrate the validity of the design method.

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