SCALABLE ROBUSTNESS FOR CONSENSUS PROTOCOLS WITH HETEROGENEOUS DYNAMICS

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Abstract: It is shown in the paper how robustness can be guaranteed for consensus protocols with heterogeneous dynamics in a scalable and decentralized way i.e. by each agent satisfying a test that does not require knowledge of the entire network. Random graph examples illustrate that the proposed certificates are not conservative for classes of large scale networks, despite the heterogeneity of the dynamics, which is a distinctive feature of this work. The conditions hold for symmetric protocols and more conservative stability conditions are given for general nonsymmetric interconnections. Nonlinear extensions in an IQC framework are finally discussed.

Keywords: Agents, Robust Control, Networks, Complex Systems

1. INTRODUCTION

Distributed control of multi-agent systems is a research topic that has received considerable attention in the recent years within various contexts such as unmanned aerial vehicles(UAV's), automated highway systems, cluster of satellites, flocking and sensor networks. In all these examples, consensus problems (i.e. group agreement for a particular state such as speed or attitude) are of major importance and have been addressed extensively in the literature.

In (Fax and Murray, 2002) graph Laplacians are used to analyze formation stability with SISO linear dynamics and generalizations in MIMO cases are given in (Gattami and Murray, 2004). Various aspects of the consensus problem are studied in (Olfati-Saber, Sep 2004) such as the nature of the steady state, the convergence rate and stability issues for the case of identical communication delays. In (Jadbabai *et al.*, Jun 2003) convergence of a discrete time protocol with switching topology is being considered and the conditions derived are based on the graph connectivity.

A common feature in most of the attempts for stability analysis available so far (including the references above), that simplifies the problem considerably, is the fact that all agents have identical dynamics. In this paper we concentrate mainly in deriving conditions for stability for the case of heterogeneous dynamics. If the network topology and the dynamics of each participating agent are known, then by breaking the loop at the output of each agent, the network can easily be analyzed as a feedback interconnection of an input/output multivariable system using tools from robust control theory. However, what is more challenging in large scale networks is to be able to guarantee a degree of robustness in a decentralized and scalable way i.e. by each agent adjusting its parameters using a rule that does not involve knowing the dynamics and topology of the entire network. By employing techniques that have been used extensively for robustness analysis of internet congestion control protocols, we give in this paper scalable decentralized conditions that guarantee stability of consensus protocols with heterogeneous dynamics. The conditions are only sufficient, but we show with random graph examples that they can be not too conservative for classes of dynamics.

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The paper is structured as follows. A formulation of the problem is given in section 2. Stability conditions are then presented in section 3 for the case of SISO, stable, linear agent dynamics with a symmetric adjacency matrix. Random graphs of large scale networks are also generated to show examples where these conditions are not too conservative. In section 4 we suggest how more conservative conditions can be derived in the case of a general nonsymmetric graph topology. Finally, in section 5 we illustrate how these results can be generalized to networks of nonlinear agents, within a context of IQC's and generalized dissipassivity.

2. PRELIMINARIES

2.1 Notation

$\sigma(M)$ denotes the spectrum of a square matrix M and $\rho(M)$ its spectral radius. $Co(S)$ denotes the convex hull of a set S and $\text{diag}(x_i)$ the matrix with elements $x_1, x_2, \ldots$ on the leading diagonal and zeros elsewhere. The Numerical Range of a matrix $M \in \mathbb{C}^{n\times n}$ is the set

$$N(M) = \{v^* M v : v \in \mathbb{C}^n, v^* v = 1\}$$

(see e.g. (Horn and Johnson, 1991) and (Gustafson and Rao, 1997) for more details on its properties). $\mathbf{H}_\infty$ is the Hardy space of transfer functions of stable linear, time-invariant, continuous time systems.

2.2 Problem formulation

We consider in sections 2 and 3 single input single output linear time invariant dynamical systems. These are linearly interconnected i.e. the input to each system is a linear combination of the outputs from other systems. We consider a directed graph representation of the interconnected system.

Using a notation similar to that in (Olfati-Saber, Sep 2004), $G = (V, E, A)$ is a weighted directed graph , where $V = \{v_1, \ldots, v_n\}$ is the set of nodes, $E \subseteq V \times V$ the set of directed edges and $A = [a_{ij}]$ a weighted adjacency matrix. Directed edges are denoted as $e_{ij} = (v_i, v_j)$, such that e_{ij} is defined to be incident to node v_j. The adjacency matrix $A \in \mathbb{R}^{n\times n}$ satisfies $a_{ii} = 0$, $a_{ij} \neq 0 \Leftrightarrow e_{ij} \in E$. The neighbours of a node v_i are defined as $N_i = \{v_j \in V : (v_j, v_i) \in E\}$ and its in-degree as $|N_i|$. In the digraph representation of the network each dynamical element corresponds to a node of the graph. Furthermore in a network of n dynamic agents, each with scalar input $u_i(t)$, scalar output [2] $y_i(t)$ and transfer function $g_i(s)$, the input and output vectors, $u = [u_1, \ldots, u_n]^T$ and $y = [y_1, \ldots, y_n]^T$ respectively, satisfy the relation $u = -(D - A)y$, where A is the adjacency matrix of the graph, D is a diagonal matrix that corresponds to self feedback for each dynamic and $H = D - A$ is defined as the interconnection matrix of the system.

In consensus protocols the input to each dynamic is a linear combination of the differences of its own output and that of its neighbours. We consider the case

$$u_i = -\frac{1}{d_i} \sum_{j:v_j \in N_i} a_{ij}(y_i - y_j) \qquad (1)$$

$$\text{where} \quad d_i = \sum_{k=1, k\neq i}^{n} a_{ik}$$

The interconnection matrix in this case is the Graph Laplacian defined as

$$L = I - \hat{D}^{-1} A \quad \text{where } \hat{D} = \text{diag}(d_i)$$

Note that the Laplacian is diagonally dominant and hence positive semidefinite. This is a very important property for the subsequent analysis.

3. STABILITY OF SYMMETRIC PROTOCOLS

3.1 Stability conditions

The interconnection in (1) can be represented by the block diagram in figure 1, where $G(s) = \text{diag}(g_i(s))$. Using the multivariable nyquist criterion, the system

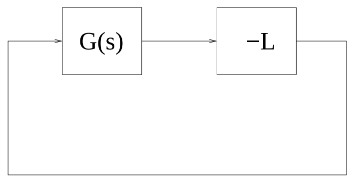

Fig. 1. Block diagram representation of interconnected system.

is stable if and only if the eigenloci of the return ratio $G(s)L$ do not encircle the point -1. In case of identical system dynamics $g_i(s)$ the problem reduces trivially to a test that involves checking the encirclements of -1 by the Nyquist plot of $g_i(s)$ scaled by the eigenvalues of the Laplacian L (see also derivation in (Fax and Murray, 2002)). Note that if the graph is connected L has only a single eigenvalue at the origin with the eigenvector being the vector of ones.

In the case of different agent dynamics analogous sufficient results are given in Proposition 1 for the case the adjacency matrix is symmetric. The integral action by the agents ensures that consensus is achieved at steady state.

Proposition 1. Let $g_i(s) = h_i(s)/s$, $h_i(s) \in \mathbf{H}_\infty$, $h_i(0) \neq 0$ for $i = 1, \ldots, n$. The interconnection of the dynamics described in (1) with $A = A^T$ is stable if

$$-1 \notin \rho(L) Co(0 \cup \{g_i(j\omega) : i = 1, \ldots, n, \omega \in \mathbb{R}^+\})$$

[2] we use u_i, y_i for brevity and denote their Laplace transform as $u_i(s)$ and $y_i(s)$ respectively.

Proof [of Proposition 1] Lemma 3 in the appendix can be used to bound the eigenloci of the return ratio $G(s)L$, since L is symmetric and positive semidefinite. The proposition then follows directly from the multivariable Nyquist criterion (Desoer and Yang, 1980). $\square$

Remark 1. Using Corollary 1 in the appendix, Proposition 1 can be extended to the case $g_i(s)$ are square transfer matrices instead of just scalar, by replacing $g_i(j\omega)$ in the convex hull bound with $N(g_i(j\omega))$. Note that this can be conservative for general unstructured transfer matrices $g_i(j\omega)$, since the numerical range is not always a tight bound for the spectrum of a matrix, particularly when this is highly non symmetric.

Even though a convex hull bound for the union of all agent frequency responses appears as a global condition, it can be given a decentralized interpretation using a hyperplane argument as shown in Proposition 2.

Proposition 2. (Scalable stability). Let $g_i(s) = h_i(s)/s$, $h_i(s) \in \mathbf{H}_\infty$, $h_i(0) \neq 0$ for $i = 1, \ldots, n$. The interconnection of the dynamics described in (1) with $A = A^T$ is stable if given a fixed line l_1 defined on $\mathbb{C}$ such that $-1 \in l_1$, then

$$\left\{l_1 \cap Co(0 \cup \{2g_i(j\omega) : \omega \in \mathbb{R}^+\})\right\} = \emptyset \qquad (2)$$
$$\forall i \in \{1, \ldots, n\}$$

Remark 2. The line l_1 is a global design parameter that is the same for all dynamics and has to be decided beforehand based on the nature of the dynamics that are expected to participate. In the worst case this is a line vertical to the real line and the condition is equivalent to all frequency responses lying to the right of the point -1.

Remark 3. A degree of robustness can be guaranteed with this condition since a bound for the eigenloci of the return ratio is provided. Most importantly this bound is achieved in a scalable and decentralized manner since each dynamic carries out a test that involves only its own dynamic and the global fixed parameter l_1. No other knowledge of the network is required by individual agents.

Proof [of Proposition 2] Since L is diagonally dominant we have a Greshgorin disc type of bound for its spectrum i.e. $\sigma(L) \subseteq \{z : z \in \mathbb{C}, |z-1| \leq 1\}$. Hence we have the bound $\rho(L) \leq 2$. We now apply Proposition 1, but use a hyperplane condition instead of the convex hull of all frequency responses. Let

$$P = Co(0 \cup \{2g_i(j\omega) : i = 1, \ldots, n, \omega \in \mathbb{R}^+\})$$
$$\text{and} \quad P_i = Co(0 \cup \{2g_i(j\omega) : \omega \in \mathbb{R}^+\})$$

Note that $P = Co\left(\bigcup_{i=1}^n P_i\right)$. Since P is a convex set the condition $-1 \notin P$ is equivalent to the existence of a hyperplane through -1 that does not intersect P and this is also equivalent to each of the P_i lying on the same side of the hyperplane. This is what condition (2) states and note also that if all P_i do not intersect the hyperplane they necessarily lie on the same side of the hyperplane since they all include 0. $\square$

In most applications the adjacency matrix A is just a $0-1$ matrix and d_i in (1) is the in-degree.

3.2 Examples

We now try to demonstrate the fact that the stability conditions in Propositions 1 and 2 are in general not too conservative even though they are only sufficient. It is easy to generate examples where they are also necessary. Consider, for example, the simple case of the delayed interconnection of two first order systems

$$n = 2,\ g_1(s) = g_2(s) = k\frac{e^{-sT}}{s},\ L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (3)$$

Noting that $\rho(L) = 2$, the eigenloci of the return ratio $G(s)L$ do not encircle the point -1 if and only if the conditions in propositions 1 or 2 are satisfied.

It is also important to show that the conditions are still not conservative in large scale heterogeneous networks. We therefore consider a graph which has been randomly generated using the Waxman algorithm (Waxman, 1988). The number of nodes is Poisson distributed with the intensity proportional to the area of the domain and their position is uniformly distributed on the plane. Edges [3] are randomly added according to an exponential distribution the intensity of which is inversely proportional to the length of the edge.

The graph in figure 2 consists of 61 nodes and has a mean degree of 4.1 with a standard deviation of 1.9. The hyperplane chosen for design is the line through -1 in figures 3 and 4 and in each case, the gain of each dynamic was adjusted to the maximum that satisfies condition (2) in Proposition 2.

Figures 3 and 4 show the eigenloci of the return ratio for different classes of random dynamics in the frequency range $[0.03, 3]$. In fig. 3 the dynamics are of the form $g_i(s) = k_i\frac{e^{-sT_i}}{s}$ with the delay T_i chosen randomly for each agent from a uniform distribution in $[0, 2]$. In fig. 4 the dynamics are of the form $g_i(s) = k_i\frac{e^{-sT_i}}{s(s+a_i)}$. Parameters T_i and a_i were chosen chosen from uniform distributions in $[0, 5]$ and $[1, 6]$.

[3] An edge in this graph corresponds to two edges of unit weight and opposite orientation between the same nodes in the diagraph representation given in section 2.2.

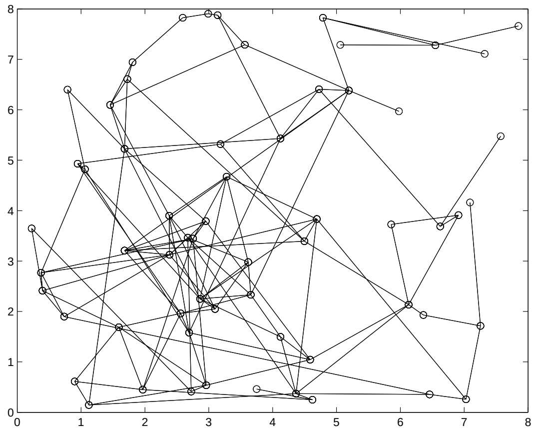

Fig. 2. A random Waxman type graph with 61 nodes.

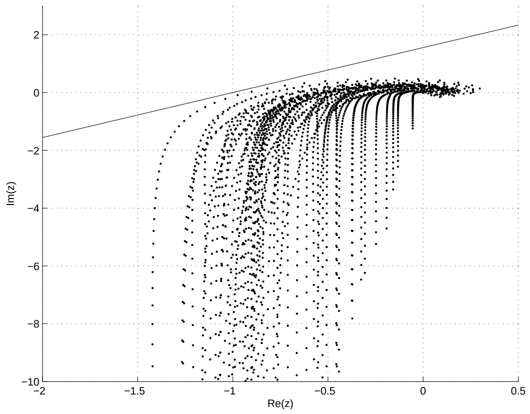

Fig. 3. $\sigma(\mathrm{diag}(g_i(j\omega))A)$, $\omega \in \mathbb{R}^+$, $g_i(s) = k_i \frac{e^{-sT_i}}{s}$

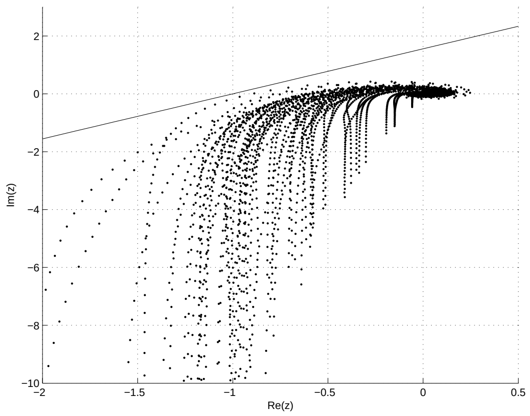

Fig. 4. $\sigma(\mathrm{diag}(g_i(j\omega))A)$, $\omega \in \mathbb{R}^+$, $g_i(s) = k_i \frac{e^{-sT_i}}{s(s+a_i)}$

We observe that the half plane bound we have imposed is not at all conservative for the actual eigenloci of the return ratio. Note also that in this example $\rho(L) = 1.94$ which is close to 2 and hence the corresponding bound for $\rho(L)$ in Proposition 2 is quite tight.

4. NON-SYMMETRIC PROTOCOLS

A major assumption in the previous section is that that A is symmetric which means that communication between agents is bidirectional and equally weighted. We consider now the case of general non-symmetric adjacency matrix with heterogeneous dynamics. This means that Lemma 3 cannot be used, nevertheless, the Laplacian is still diagonally dominant which is what we try to exploit to derive a frequency response stability condition.

Proposition 3. Let $g_i(s) \in \mathbf{H}_\infty$ for $i = 1, \ldots, n$. The interconnection of the dynamics described in (1) is stable if

$$-1 \notin \bigcup_{i=1}^{n} \{g_i(j\omega)S : \omega \in \mathbb{R}^+\}$$
$$\text{where} \quad S = \{z : z \in \mathbb{C}, |z-1| \leq 1\}$$

Proof The spectrum of the frequency response of the return ratio is bounded using Greshgorin discs i.e.

$$\sigma(G(j\omega)L) \subseteq \bigcup_{i=1}^{n} \{z : z \in \mathbb{C}, |z - g_i(j\omega)| \leq |g_i(j\omega)|R_i\}$$
$$\text{where} \quad R_i = \sum_{j=1, i\neq j}^{n} |L_{ij}|$$

and note that $R_i = 1$. Nyquist criterion is then applied. □

This condition can be more conservative than the ones derived in section 3.1. We illustrate this with a simple example of interconnection of two identical dynamics of the form $g(s) = k\frac{e^{-s}}{s+0.5}$.

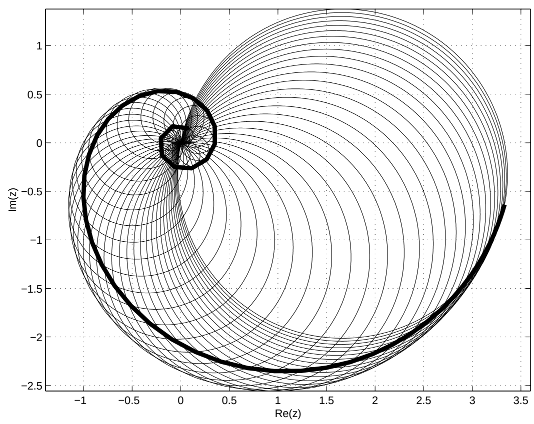

Fig. 5. $\{g(j\omega)S, \omega \in \mathbb{R}^+\}$ where $g(s) = 0.85\frac{e^{-s}}{s+0.5}$. The bold line is $\{2g(j\omega), \omega \in \mathbb{R}^+\}$.

Figure 5 shows that the Greshgorin disc bound for the eigen-loci of the return ratio does not include the point -1 for $k \leq$ 0.85. Nevertheless the necessary and sufficient condition which is predicted by Propositions 1 and 2 is $k \leq 0.95$ (bold line). In fact, for $g(s) = k\frac{e^{-s}}{s}$ the Greshgorin disc bound is arbitrarily close to -1 irrespective of the delay and hence no robustness guarantees can be given.

5. EXTENSIONS TO NONLINEAR AGENTS

The half-plane bound for the agent frequency response in Proposition 2 is likely to have an interpretation within the framework of an interconnection of dissipative systems, bearing in mind the equivalence between passivity and positive realness for linear systems. This is what we show in this section by deriving stability conditions that involve IQC bounds for the agent dynamics which are equivalent to the convex hull bounds for linear systems used in section 3.1.

5.1 Notation

$\mathbf{L}_2^l[0,\infty)$ is the space of functions $f:[0,\infty)\to\mathbb{R}^l$ with finite energy $\|f\|^2=\int_0^\infty |f(t)|^2dt$. This is a subspace of $\mathbf{L}_{2e}^l[0,\infty)$ whose elements need to be integrable on finite intervals. The fourier transform of $f\in\mathbf{L}_2^l[0,\infty)$ is denoted by $\hat{f}(j\omega)=\int_0^\infty e^{-j\omega t}f(t)dt$. The operator $\Delta:\mathbf{L}_{2e}^l[0,\infty)\to\mathbf{L}_{2e}^m[0,\infty)$ is bounded if the gain $\|\Delta\|=sup\{\|\Delta(f)\|/\|f\|:f\in\mathbf{L}_2^l[0,\infty),f\neq 0\}$ exists and is finite. We use definitions of well-posedness and stability as in (Megretski and Rantzer, 1997) and also quote the following IQC definition from the same paper.

Definition 1. (IQC). A bounded operator $\Delta:\mathbf{L}_{2e}^l[0,\infty)\to\mathbf{L}_{2e}^m[0,\infty)$ is said to satisfy the IQC defined by Π if

$$\sigma_\Pi(v,w):=\int_{-\infty}^{\infty}\begin{bmatrix}\hat{v}(j\omega)\\ \hat{w}(j\omega)\end{bmatrix}^*\Pi(j\omega)\begin{bmatrix}\hat{v}(j\omega)\\ \hat{w}(j\omega)\end{bmatrix}d\omega\geq 0 \tag{4}$$

for all $v\in\mathbf{L}_2^l[0,\infty)$ and $w=\Delta(v)$, where $\Pi: j\mathbb{R}\to\mathbb{C}^{(l+m)\times(l+m)}$ can be any measurable Hermitian valued function.

5.2 Stability conditions

We prove first that a hyperplane bound for a frequency response is equivalent to an IQC bound for the corresponding operator.

Lemma 1. The following are equivalent for a transfer function $g(s)\in\mathbf{H}_\infty$ and a parameter $\eta\in\mathbb{R}$

- $\{l_1\cap Co(0\cup\{g(j\omega):\omega\in\mathbb{R}^+\})\}=\emptyset$, where l_1 is the line $\{-1+(\eta+j)t:t\in\mathbb{R}\}$.
- The linear operator with transfer function g(s) satisfies an IQC defined by

$$\Pi(j\omega)=\begin{bmatrix}2 & 1+j\eta\,\mathrm{sign}(\omega)\\ 1-j\eta\,\mathrm{sign}(\omega) & 0\end{bmatrix} \tag{5}$$

where $\mathrm{sign}(\omega)=\begin{cases}1 & \text{if}\quad \omega\geq 0\\ -1 & \text{if}\quad \omega<0\end{cases}$ and with (4) satisfied with a strict inequality.

Proof Note that the first statement is equivalent to

$$1+\Re(g(j\omega))-\eta\Im(g(j\omega))>0,\quad \forall\omega\in\mathbb{R}^+ \tag{6}$$

and let $M(j\omega)=1+\Re(g(j\omega))-\eta\,\mathrm{sign}(\omega)\Im(g(j\omega))$

Also

$$\begin{aligned}&\begin{bmatrix}\hat{v}(j\omega)\\ g(j\omega)\hat{v}(j\omega)\end{bmatrix}^*\Pi(j\omega)\begin{bmatrix}\hat{v}(j\omega)\\ g(j\omega)\hat{v}(j\omega)\end{bmatrix}=\\ &\hat{v}^*(j\omega)(2+g^*(j\omega)[1-j\eta\,\mathrm{sign}(\omega)]+\\ &\qquad g(j\omega)[1+j\eta\,\mathrm{sign}(\omega)])\hat{v}(j\omega)=\\ &\hat{v}^*(j\omega)2\Big(1+\frac{1}{2}[g^*(j\omega)+g(j\omega)]+\\ &\frac{1}{2}[j\eta\,\mathrm{sign}(\omega)g(j\omega)+(j\eta\,\mathrm{sign}(\omega)g(j\omega))^*]\Big)\hat{v}(j\omega)=\\ &\hat{v}^*(j\omega)2(1+\Re(g(j\omega))-\eta\,\mathrm{sign}(\omega)\Im(g(j\omega)))\hat{v}(j\omega)=\\ &\qquad\hat{v}^*(j\omega)2M(j\omega)\hat{v}(j\omega)\end{aligned}$$

Now, since $M(j\omega)=M(-j\omega)$, from Theorem 3.1 in (Megretski and Treil, 1993)

$$\int_{-\infty}^{\infty}\hat{v}^*(j\omega)M(j\omega)\hat{v}(j\omega)d\omega>0\quad\forall v\in\mathbf{L_2}[0,\infty)$$

if and only if $M(j\omega)>0\ \ \forall\omega\in\mathbb{R}^+$ i.e. if and only if (6) is true. □

We now state a stability condition based on the IQC theorem in (Megretski and Rantzer, 1997). As a result of Lemma 1, this reduces to Proposition 2 for linear time invariant systems with transfer functions in $\mathbf{H}_\infty$.

Lemma 2. Let Δ_i $i=1,\ldots,n$ be bounded causal operators, and $u_i, y_i\in\mathbf{L_2}[0,\infty)$ satisfy $y_i=\Delta(u_i)$. Assume linear interconnections between the operators as given in (1) with a symmetric adjacency matrix A and that:

i) $\forall\tau\in[0,1]$ the interconnection between the operators $\tau\Delta_i$ is well posed.
ii) each $2\Delta_i$ satisfies $\sigma_\Pi(u_i,y_i)>0\quad\forall u_i\in\mathbf{L_2}[0,\infty)$ for $\Pi(j\omega)$ as in (5) for some fixed $\eta\in\mathbb{R}$.

Then the interconnected network is stable.

Proof The IQC stability theorem is applied on the interconnection in fig. 1 with $G(s)$ replaced by $\mathrm{diag}(\Delta_i)$. $\mathrm{diag}(2\Delta_i)$ satisfies the IQC defined by $\Pi(j\omega)\otimes I$ where operator $\otimes$ denotes the Kronecker product between the matrices and I is the $n\times n$ identity matrix. We hence need to show that

$$\begin{bmatrix}-L/2\\ I\end{bmatrix}^*(\Pi(j\omega)\otimes I)\begin{bmatrix}-L/2\\ I\end{bmatrix}\leq 0\ \ \forall\omega\in\mathbb{R} \tag{7}$$

Note that the inequality is not strict as we have a strict inequality in assumption (ii) of the Lemma. (7) is true since

$$\begin{aligned}&2\frac{L}{2}^*\frac{L}{2}-\frac{L}{2}^*(1+j\eta\,\mathrm{sign}(\omega))-\frac{L}{2}(1-j\eta\,\mathrm{sign}(\omega))=\\ &\qquad(\text{using } L=L^T)\qquad L^2-2L=\\ &\qquad(\text{using } \rho(L)\leq 2)\qquad L(L-2I)\leq 0\end{aligned}$$

□

Finally it should be noted that the IQC analysis above is closely related to the stability conditions derived in (Moylan and Hill, 1978) and (Vidyasagar, 1981) for the interconnection of dissipative systems. The idea there is to consider quadratic supply rates for the individual dissipative systems and then close the loop with an adjacency matrix such that these supply rates are always negative. In this way the summation of the storage functions will be a common Lyapunov function for the interconnected network. Nevertheless the conditions suggested are more conservative since $\Pi(j\omega)$ is only considered to be real. For linear systems this is equivalent to constraining the frequency responses to be to the right of -1.

6. CONCLUSIONS

Conditions are presented that guarantee robustness for consensus protocols with heterogeneous dynamics in a scalable and decentralized way. These hold for symmetric networks, with more conservative conditions for non symmetric protocols. Finally extensions to the interconnection of nonlinear agents are given within an IQC framework.

APPENDIX

Lemma 3. Let $Q \in \mathbb{C}^{n\times n}$, $Q = Q^* \geq 0$ and $G = \mathrm{diag}(gi), g_i \in \mathbb{C}, \forall i \in \{1,\dots n\}$. Then

$$\sigma(GQ) \subset \rho(Q)Co(0 \cup \{g_i : i \in \{1,\dots,n\}\})$$

Remark 4. Lemma 3 was proved in (Vinnicombe, 2000) where it was used to derive Internet congestion control stability results. We give here a slightly modified proof that is readily extended in Corollary 1 to the case where G is block diagonal instead of just diagonal.

Proof $\{0\} \cup \sigma(GQ) = \{0\} \cup \sigma(Q^{1/2}GQ^{1/2})$. But

$$\begin{aligned}\sigma(Q^{1/2}GQ^{1/2}) &\subset N(Q^{1/2}GQ^{1/2})\\ &= \{v^*Q^{1/2}GQ^{1/2}v : v \in \mathbb{C}^n, \|v\| = 1\}\\ &\subset \rho(Q)\{w^*Gw : w \in \mathbb{C}^n, \|w\| \leq 1\}\\ &= \rho(Q)\{\sum_{i=1}^{n} |w_i|^2 g_i : w_i \in \mathbb{C},\\ &\qquad \sum_{i=1}^{n} |w_i|^2 \leq 1\}\\ &= \rho(Q)Co(0 \cup \{g_i :\ i = 1,\dots,n\}) \quad \square\end{aligned}$$

Corollary 1. Let $Q \in \mathbb{C}^{n\times n}$, $Q = Q^* \geq 0$ and $G = \mathrm{diag}(g_i)$, $g_i \in \mathbb{C}^{n_i\times n_i}$, $\forall i \in \{1,\dots k\}$, $\sum_{i=1}^{k} n_i = n$. Then

$$\sigma(GQ) \subset \rho(Q)Co(0 \cup \{N(g_i) : i \in \{1,\dots,n\}\})$$

Proof Following the proof of Lemma 3

$$\{0\} \cup \sigma(GQ) = \{0\} \cup \sigma(Q^{1/2}GQ^{1/2}) \qquad \text{and}$$

$$\begin{aligned}\sigma(Q^{1/2}GQ^{1/2}) \;&\subset\; \rho(Q)\{w^*Gw : w \in \mathbb{C}^n, \|w\| \leq 1\}\\ &\subset\; \rho(Q)\left\{\sum_{i=1}^{k} \|w_i\|^2 h_i : w_i \in \mathbb{C}^{n_i},\right.\\ \sum_{i=1}^{k}\|w_i\|^2 \leq\; & 1,\quad h_i \in \left.\left\{\frac{w_i^*}{\|w_i\|} g_i \frac{w_i}{\|w_i\|} : w_i \in \mathbb{C}^{n_i}\right\}\right\}\\ &= \rho(Q)\left\{\sum_{i=1}^{k}\|w_i\|^2 N(g_i) : \sum_{i=1}^{k}\|w_i\|^2 \leq 1\right\}\\ &= \rho(Q)\, Co(0 \cup \{N(g_i) : i = 1,\dots,k\}) \quad \square\end{aligned}$$

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