REMARKS ON INPUT-TO-OUTPUT STABILITY FOR DISCRETE TIME SYSTEMS

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Abstract: In this work we discuss several notions on input-to-output stability for discrete time systems. We first discuss how the notions differ from each other, then we provide Lyapunov characterizations for these input-to-output stability properties.

Keywords: input-to-output stability, partial stability, nonlinear discrete time systems, Lyapunov methods.

1. INTRODUCTION

Due to increasing popularity of digital computers in control engineering applications, the analysis and synthesis of discrete-time control systems have received attention of many researchers in the past two decades, see e.g., (Agarwal, 1992; Ogata, 1987; Nešić *et al.*, 1999*a*; Nešić *et al.*, 1999*b*; Nešić and Laila, 2002; Nešić and Teel, to appear; Nešić and Angeli, 2002; Kellett and Teel, 2002), and (Lan and Huang, 2005).

In this work we consider several discrete-time output stability notions, inspired by the recent work of (Sontag and Wang, 1999) and (Sontag and Wang, 2001) on the stability of continuous time systems. We begin with introducing the discrete-time variants of the output stability notions from (Sontag and Wang, 1999). Then, we turn our attention to their Lyapunov characterizations in the context of discrete-time systems. Most of the results can be considered as discrete analogues to the results in (Sontag and Wang, 1999) and (Sontag and Wang, 2001). However, many results cannot be obtained by merely carrying out some obvious "discretization" of the arguments used in the continuous time case. For instance, in the discrete Lyapunov formulations, it is not enough to only require the usual "proper, positive definite" and the usual decay conditions on V. One has to impose an extra "one-step bounded-input-bounded-output" condition (see (15)) to guarantee that the existence of a Lyapunov function indeed implies a corresponding stability property (see Section 3 for details). On the other hand, in many cases, it is much simpler technically to deal with the discrete results than with the continuous case. For instance, the existence and uniqueness of solutions (in the forward case) is not an issue in the discrete time case. This enables one to present simpler proofs for some results than the proofs of their continuous counterparts, and indeed, this helps us to gain further insights into the ideas of the proofs.

This work may be considered as a continuation of the ISS work developed in (Jiang *et al.*, 1999), (Jiang and Wang, 2001) and (Jiang and Wang, 2002). As one will see, similar to the continuous-time case, the input-to-output stability properties are more general than the ISS property. The corresponding Lyapunov characterizations are therefore more complicated, and indeed demand more

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conditions on Lyapunov functions than in the ISS case. More precisely, in addition to the usual conditions on the Lyapunov functions, one requires additionally the one-step-bounded-input-bounded-state property as mentioned earlier.

In this work, we will consider systems whose state spaces are metric spaces in contrast to the continuous time case where the state spaces are usually Euclidean or differentiable manifolds. The consideration of metric spaces as state spaces has several motivations in addition to just being "more general". One motivation is to consider stability of systems with parameters in a subset Λ (whether open or closed or neither) of a metric space. The parameters may be some designing parameters, or the step length in the discretization of a continuous time system, see (Nešić *et al.*, 1999*b*). In the later case, the set Λ is in general not compact. For a family of systems with parameters λ taking values in a subset Λ and the state space X,

$$x(k+1) = f(x(k), u(k), \lambda),$$

the stability properties such as robust stability or input-to-state stability, whether uniform in λ or not, can be treated as some output stability properties discussed in this work for the augmented systems

$$\lambda(k+1) = \lambda(k),\ x(k+1) = f(x(k), u(k), \lambda(k))$$

with the output map defined by $y = h(\lambda, x) := x$, and the state space defined by $\Lambda \times X$. Even if X is an Euclidean space, $\Lambda \times X$ rarely is. When the set Λ is not compact, the stability property of the parameterized systems is in general not uniform. The output notions discussed in this work should provide convenient tools in handling such cases.

Another motivation to study the case when the state space is a metric space occurs in dealing with time varying systems. This will be discussed in more details in Section 2.3.

Finally, it also does not take too much extra work when one enlarges the type of state spaces from Euclidean to metric spaces. This may be considered as another helpful feature of the discrete time case: in contrast to the continuous time case, the state space of a discrete time system does not have to be a differentiable manifold.

This paper is organized as follows. In Section 2, we introduce several notions on input-to-output stability, and discuss the relations among them. We also discuss in this section how different stability properties for time varying systems can be treated as stability properties regarding the output variables of some augmented time invariant systems. In Section 3, we formulate the Lyapunov descriptions of the notions on input-to-output stability. In Section 4, we discuss several notions of robust output stability and their Lyapunov formulations. The main result in this section underlies the proofs of the Lyapunov theorems in Section 3.

Due to the length limit, we have omitted most proofs. The detailed proofs will be provided in the forthcoming paper (Jiang *et al.*, to be submitted).

2. NOTIONS ON OUTPUT STABILITY

Consider a system as in the following:

$$x(k+1) = f(x(k), u(k)), \quad y = h(x(k)), \qquad (1)$$

where the states $x(\cdot)$ take values in a metric space $\mathcal{X}$, the inputs $u(\cdot)$ and outputs $y(\cdot)$ take values in $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively, and where the maps $f : \mathcal{X} \times \mathbb{R}^m \to \mathcal{X}$ and $h : \mathcal{X} \to \mathbb{R}^p$ are continuous. We assume, for the system (1) being considered, *controls* or *inputs* are functions $u : \mathbb{Z}_+ \to \mathbb{R}^m$.

For each $\xi \in \mathcal{X}$ and each input u, we denote by $x(\cdot, \xi, u)$ (and $y(\cdot, \xi, u)$) the trajectory (and the output, respectively) of system (1) with initial state $x(0) = \xi$ and the input u. We also assume that for some $p_0 \in \mathcal{X}$, $h(p_0) = 0$.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, positive definite, and strictly increasing; and is of class $\mathcal{K}_\infty$ if it is also unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each fixed $s \geq 0$, $\beta(s, t)$ decreases to 0 as $t \to \infty$.

2.1 Basic Definitions

Definition 2.1. Let ω_o be a continuous function from $\mathcal{X}$ to $\mathbb{R}_{\geq 0}$. Assume that, for some $\alpha_0 \in \mathcal{K}$, $|h(\xi)| \leq \alpha_0(\omega_o(\xi))$ for all ξ. We say that system (1) is

• *input-to-output stable* (IOS) *with respect to* ω_o if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that

$$|y(k, \xi, u)| \leq \beta(\omega_o(\xi), k) + \gamma(\|u\|) \ \ \forall\, k \geq 0; \quad (2)$$

• *output-Lagrange input-to-output stable* (OLIOS) *with respect to* ω_o if it is IOS with respect to ω_o and for some $\mathcal{K}$-functions σ_1 and σ_2,

$$|y(t, \xi, u)| \leq \max\{\sigma_1(|h(\xi)|),\ \sigma_2(\|u\|)\} \ \ \forall\, k \geq 0; (3)$$

• *state-independent input-to-output stable* (SIIOS) if (2) can be strengthened to

$$|y(k, \xi, u)| \leq \beta(|h(\xi)|, k) + \gamma(\|u\|) \ \ \forall\, k \geq 0. \quad (4)$$

In each case, we interpret the estimates as holding for all states $\xi \in \mathcal{X}$ and all inputs u. □

Note that in the above definition, we do not assume that $\omega_o(p_0) = 0$. However, at any point p such that $\omega_o(p) = 0$, (2) implies that

$$|y(k, p, u)| \leq \gamma(\|u\|) \qquad \forall\, k \geq 0,$$

and particularly, $y(k, p, \mathbf{0}) = 0$ for all $k \geq 0$, where $\mathbf{0}$ denotes the zero input function.

As discussed in (Sontag and Wang, 1999), for any $\beta \in \mathcal{KL}$ and any $\sigma \in \mathcal{K}$, there exist some $\hat{\beta} \in \mathcal{KL}$ and some $\kappa \in \mathcal{K}$ such that

$$\min\{\sigma(s), \beta(r,t)\} \le \hat{\beta}\left(s,\ \frac{t}{1+\kappa(r)}\right).$$

Hence, a system is OLIOS if and only if for some $\mathcal{KL}$-function β and some $\mathcal{K}$-functions κ and γ,

$$|y(k,\xi,u)| \le \beta\left(|h(\xi)|\,,\ \frac{k}{1+\kappa(\omega_o(\xi))}\right) + \gamma(\|u\|)$$

for all $k \ge 0, \xi \in \mathcal{X}$ and all u. The significance of such an estimate is that it nicely encapsulates both the IOS and the output-Lagrange aspects of the OLIOS property.

To a given system as in (1) and a continuous function $\lambda : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, we associate the following system with inputs $d(\cdot)$:

$$\begin{aligned} x(k+1) &= g(x(k), d(k)) \\ &:= f(x(k), d(k)\lambda(|h(x(k))|)), \\ y &= h(x), \end{aligned} \tag{5}$$

where the inputs $d(\cdot)$ are functions from $\mathbb{R}_{\ge 0}$ to the closed unit ball in $\mathbb{R}^m$. We let $x_\lambda(\cdot,\xi,d)$ (and $y_\lambda(\cdot,\xi,d)$) denote the trajectory (and the output function, respectively) of (5) corresponding to each initial state ξ and each input function d. Note that for each ξ and each d, $x_\lambda(k,\xi,d) = x(k,\xi,u)$, where $u(k) = d(k)\lambda(|y_\lambda(k,\xi,d)|)$.

Definition 2.2. A system as in (1) is *robustly output stable* (ROS) with respect to ω_o if there exists some $\mathcal{K}_\infty$-function λ such that for the corresponding system (5), there exists some $\beta \in \mathcal{KL}$ such that

$$|y_\lambda(k,\xi,d)| \le \beta(\omega_o(\xi), k) \tag{6}$$

for all $k \ge 0$, all $\xi \in \mathcal{X}$, and all d. □

The function λ in (5) is called a robust output gain margin. It specifies the magnitude of output feedback that can be tolerated without destroying output stability.

2.2 Relations among the Output Stability Notions

As in the continuous time case, we have the following result whose proof follows exactly the same idea as in (Sontag and Wang, 1999).

Lemma 2.3. If a system is IOS with respect to ω_o, then it is ROS with respect to ω_o. □

Hence, we have the following implications with respect to a given ω_o:

$$\text{SIIOS} \Rightarrow \text{OLIOS} \Rightarrow \text{IOS} \Rightarrow \text{ROS}.$$

In the special case when $\mathcal{X} = \mathbb{R}^n, h(\xi) = \xi$ and $\omega_o(\xi) = |\xi|$, all the four notions coincide with the standard ISS notion (c.f. (Jiang and Wang, 2001) and (Jiang *et al.*, 1999)).

When $\mathcal{X} = \mathbb{R}^n$ and $\omega_o(\xi) = |\xi|$, the IOS, OLIOS, SIIOS, and the ROS properties with respect to ω_o become the IOS, OLIOS, SIIOS and ROS properties respectively as studied in (Sontag and Wang, 1999) and (Sontag and Wang, 2001) for the continuous case.

Similarly to the case of continuous time, we say that a system as in (1) is OLIOS with respect to some ω_o *under output redefinition* if there exist some continuous map $h_0 : \mathcal{X} \to \mathbb{R}_{\ge 0}$ with $h_0(p_0) = 0$, some $\chi_1 \in \mathcal{K}_\infty$, and $\chi_2 \in \mathcal{K}$ such that

$$\chi_1(|h(\xi)|) \le h_0(\xi) \le \chi_2(\omega_o(\xi)) \tag{7}$$

for all $\xi \in \mathcal{X}$, and that the system

$$x(k+1) = f(x(k), u(k)), \quad y = h_0(\xi) \tag{8}$$

is OLIOS with respect to ω_o. As in the continuous time case, the following holds:

Proposition 2.4. The following are equivalent for a system as in (1) with respect to any given ω_o:

(1) The system is IOS.
(2) The system is OLIOS under output redefinition. □

The implication (2) ⇒ (1) is obvious. The implication (1) ⇒ (2) can be proved by following the same ideas as in the proof of Theorem 6 of (Sontag and Wang, 1999). To be more specific, if the system (1) is IOS, then without loss of generality, one can assume that the following holds for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$:

$$|y(t)| \le \max\{\beta(\omega_o(\xi),\ k),\ \gamma(\|u\|)\} \qquad \forall k \ge 0.$$

The h_0 to be used in the output redefinition is then defined by

$$h_0(\xi) = \sup_{k\ge 0, u} \big\{\max\{|y(k,\xi,u)| - \gamma(\|u\|),\ 0\}\big\}. \tag{9}$$

It then can be shown as in (Sontag and Wang, 1999) that h_0 is continuous,

$$|h(\xi)| \le h_0(\xi) \le \beta_0(\omega_o(\xi)) \qquad \forall\, \xi \in \mathcal{X},$$

where $\beta_0(s) = \beta(s,0)$, and the system (8) is OLIOS.

2.3 Remarks on Time-varying Stability Notions

An immediate advantage for considering the general case when the state space $\mathcal{X}$ is a metric space (instead of an Euclidean space) is that one can easily treat the stability properties for time varying systems as output stability properties of time invariant systems. For a time varying system

$$x(k+1) = f(k, x(k), u(k)), \tag{10}$$

where for each $k \geq 0$, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and the map $f : \mathbb{Z}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous (where the topology on $\mathbb{Z}_+$ is the discrete topology), we say that the system is *uniformly input-to-state stable* (UISS) if for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, it holds that

$$|x(k, k_0, \xi, u)| \leq \beta(|\xi|, k - k_0) + \gamma(\|u\|) \tag{11}$$

for all $k \geq k_0 \geq 0$, all $\xi \in \mathbb{R}^n$ and all u, where we have used $x(\cdot, k_0, \xi, u)$ to denote the trajectory corresponding to the initial value $x(k_0) = \xi$ and the input u.

We say that the system is *input-to-state stable* (ISS) if

• the system is uniformly bounded-input-bounded state, that is, for some $\sigma_1, \sigma_2 \in \mathcal{K}$,

$$|x(k, k_0, \xi, u)| \leq \max\{\sigma_1(|\xi|),\ \sigma_2(\|u\|)\} \tag{12}$$

for all $k \geq k_0 \geq 0$, all $\xi \in \mathbb{R}^n$ and all u; and
• the system has the asymptotic gain property, that is, for some $\gamma \in \mathcal{K}$,

$$\limsup_{k\to\infty} |x(k, k_0, \xi, u)| \leq \gamma(\|u\|) \tag{13}$$

for all $\xi \in \mathbb{R}^n$ and all u. Note that for systems without the inputs, the UISS and ISS properties reduce to the usual UGAS and GAS properties respectively.

It should be clear to see that for a time varying system, UISS$\Rightarrow$ ISS, but the converse is not true, as the converse fails even for the input free case.

For a time varying system as in (10), consider the augmented system:

$$\begin{aligned} x^o(k+1) &= x^o(k) + 1, \\ x(k+1) &= f(x^o(k), x(k), u(k)). \end{aligned} \tag{14}$$

It is easy to show that the time varying system (10) is UISS if and only if the corresponding augmented system (14) is SIIOS with the output map $h(\xi^o, \xi) = |\xi|$. It takes somewhat more work to show that (10) is ISS if and only if the corresponding system (14) is OLIOS with respect to $\omega_o(\xi^o, \xi) := |(\xi^o, \xi)|$. Note that the state space of (14) is given by $\mathbb{Z}_+ \times \mathbb{R}^n$ which is not an Euclidean space.

More generally, one can treat IOS and other related output stability properties of a time varying system as some output stability properties of an augmented time invariant system in a similar manner.

3. LYAPUNOV FORMULATIONS OF THE OUTPUT STABILITY NOTIONS

In this section, we introduce the associate Lyapunov concepts.

Definition 3.1. For system (1) and a given function ω_o, a continuous function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is:

• an IOS-*Lyapunov function* with respect to ω_o if

(a) there exists some $\sigma \in \mathcal{K}$ such that for all ξ and all μ,

$$V(f(\xi, \mu)) \leq \max\{\sigma(V(\xi)),\ \sigma(|\mu|)\}, \tag{15}$$

(b) there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(\omega_o(\xi)) \quad \forall \xi \in \mathcal{X}, \tag{16}$$

(c) there exist $\chi \in \mathcal{K}$ and $\alpha_3 \in \mathcal{KL}$ such that

$$\begin{gathered} V(\xi) \geq \chi(|\mu|) \\ \Downarrow \\ V(f(\xi, \mu)) - V(\xi) \leq -\alpha_3(V(\xi), \omega_o(\xi)); \end{gathered} \tag{17}$$

• an OLIOS-*Lyapunov function* with respect to ω_o if V is an IOS-Lyapunov function with respect to ω_o, and if (16) can be strengthened to

$$\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|h(\xi)|) \qquad \forall \xi \in \mathcal{X}, \tag{18}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$;
• an SIIOS-*Lyapunov function* if V is an OLIOS-Lyapunov function (with any function ω_o such that $\omega_o(\xi) \geq |h(\xi)|$) and (17) can be strengthened to

$$\begin{gathered} V(\xi) \geq \chi(|\mu|) \\ \Downarrow \\ V(f(\xi, \mu)) - V(\xi) \leq -\alpha_3(V(\xi)) \end{gathered} \tag{19}$$

for some $\alpha_3 \in \mathcal{K}_\infty$ and some $\chi \in \mathcal{K}$;
• an ROS-*Lyapunov function* with respect to ω_o if there exist $\chi \in \mathcal{K}$ and $\alpha_3 \in \mathcal{KL}$ such that

$$\begin{gathered} |h(\xi)| \geq \chi(|\mu|) \\ \Downarrow \\ V(f(\xi, \mu)) - V(\xi) \leq -\alpha_3(V(\xi), \omega_o(\xi)) \end{gathered} \tag{20}$$

and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that (16) hold. □

Remark 3.2. Note that condition (15) amounts to the requirement that V should not have a big increment in one step along the trajectories. One may notice that when comparing with the continuous analogues of the Lyapunov formulations, this is an extra requirement of V. We remark that without this condition, conditions (16) and (17) (or even the combination of (18) and (19)) alone are too weak to guarantee any of the correpronding output stability property. A counterexample will be given later. However, we remark that in the following two cases, this requirement is redundant.

Case 1. When the system is input free, that is, when $f(\xi, \mu) = f_0(\xi)$ for all $\mu \in \mathbb{R}^m$ and all $\xi \in \mathcal{X}$, (17) becomes

$$V(f_0(\xi)) - V(\xi) \leq -\alpha_3(V(\xi), \omega_o(\xi)) \tag{21}$$

for some $\alpha_3 \in \mathcal{KL}$. In this case, one automatically gets property (15).

Case 2. When $\mathcal{X} = \mathbb{R}^n$, $h(\xi) = \xi$ and $\omega_o(\xi) = |\xi|$ (that is, the case of ISS), the condition (17) again implies (15). That is, in the ISS case, (15) is again redundant.

Also observe that the condition (15) was not a requirement in the ROS case. □

Remark 3.3. Among all the variations of the IOS properties, SIIOS is the strongest one. In the Lyapunov formulation, the SIIOS-Lyapunov functions also assemble the most features of ISS-Lyapunov functions. First of all, as in the ISS case, it results in an equivalent definition if the function α_3 in (19) is merely required to be a continuous, positive definite function, see (Jiang and Wang, 2001, Remark 3.3) and (Jiang and Wang, 2002, Lemma 2.8). Secondly, observe that for an SIIOS-Lyapunov function, the conjunction of (15) and (19) is equivalent to the existence of some $\rho \in \mathcal{K}$ such that

$$V(f(\xi,\mu)) - V(\xi) \le -\alpha_3(V(\xi)) + \rho(|\mu|) \qquad (22)$$

It is clear that (22) implies both (15) and (19). Now suppose (15) and (19) hold for some $\sigma \in \mathcal{K}$, $\alpha_3 \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$. Then for any μ and ξ such that $V(\xi) \le \chi(|\mu|)$, one has

$$V(f(\xi,\mu)) \le \max\{\sigma(\chi(|\mu|)), \sigma(|\mu|)\}.$$

With ρ defined by $\rho(s) := \max\{\sigma(\chi(s)), \sigma(s)\} - \chi(s) + \alpha_3(\chi(s))$, it holds that

$$\begin{gathered} V(\xi) \le \chi(|\mu|) \\ \Downarrow \\ V(f(\xi,\mu)) - V(\xi) \le -\alpha_3(V(\xi)) + \rho(|\mu|). \end{gathered}$$

Combining this with (19) and modify ρ to a $\mathcal{K}$-function if necessary, (22) follows readily. □

Below we summerize our main results on the Lyapunov characterizations of the output stability properties. We say that system (1) is ω_o-*uniformly bounded input bounded state* stable (ω_o-UBIBS) if there exists some function $\sigma \in \mathcal{K}$ such that

$$\omega_o(x(k,\xi,u)) \le \max\{\sigma(\omega_o(\xi)),\ \sigma(\|u\|)\}$$

for all $k \ge 0$, all ξ and all u.

Theorem 1. An ω_o-UBIBS system is:

(1) IOS if and only if it admits an IOS-Lyapunov function;
(2) OLIOS if and only if it admits an OLIOS-Lyapunov function;
(3) SIIOS if and only if it admits an SIIOS-Lyapunov function; and
(4) ROS if and only if it admits an ROS-Lyapunov function. ■

In estimates (17) and (20), the decay rate of $V(x(k))$ depends on both the value of $\omega_o(x(k))$ and the value of $V(x(k))$. The main feature of α_3 is that it allows for slower decay rate if $V(x(k))$ is very small or if $\omega_o(x(k))$ is very large.

Also note that it follows from Lemma A.2 in (Sontag and Wang, 2001) that (17) is equivalent to

$$\begin{gathered} V(\xi) \ge \chi(|\mu|) \\ \Downarrow \\ V(f(\xi,\mu)) - V(\xi) \le -\frac{\alpha_4(V(\xi))}{1+\kappa(\omega_o(\xi))} \qquad \forall \xi,\ \forall \mu, \end{gathered}$$

for some $\alpha_4 \in \mathcal{K}, \kappa \in \mathcal{K}$. The similar remark also applies to (20).

We now present a counterexample to illustrate that condition (15) is crucial to guarantee output stabilities.

Example 3.4. Consider the two-dimensional system whose output map is $y = x$:

$$\begin{aligned} z(k+1) &= z(k), \\ x(k+1) &= a(|x_k| - |u_k|)z_k, \end{aligned}$$

where $a(\cdot)$ is the function defined by $a(s) = 0$ if $s \ge 0$, and $a(s) = |s|$ if $s < 0$. This system is UBIBS since $z(k) = z(0)$ for all $k \ge 0$, and

$$|x(k)| \le |u_{k-1}|\,|z_{k-1}| \le \|u\|^2 + |z(0)|^2$$

for all $k \ge 0$. Let $V(z,x) = |x|$. It is clear that property (18) holds, and $V(f(z,x)) - V(z,x) = -V(z,x)$ whenever $V(z,x) \ge |u|$. So, V satisfies both properties (18) and (19). However, the system is not IOS (and thus, not SIIOS). This can be seen by the following argument. Suppose, for sake of contradiction, that the system is IOS, i.e., the following estimate holds for some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$:

$$|x(k)| \le \beta(|(z_0,x_0)|\,,k) + \gamma(\|u\|) \qquad \forall\, k \ge 0.$$

Let $r_0 = \gamma(1)$. Let $K > 0$ be such that $\beta(2r_0,k) < r_0/2$ for all $k \ge K$. Hence, for any initial state $|(z_0,x_0)| \le 2r_0$ and any u with $\|u\| \le 1$, it holds that

$$|x(k)| \le \frac{3}{2}r_0 \qquad \forall\, k \ge K. \qquad (23)$$

Now consider the trajectory of the system with the initial state $(z(0),x(0)) = (2r_0,0)$ and the input function u defined by $u(k) = 0$ for all $k \le K-1$, and $u(k) = 1$ for all $k \ge K$. For this trajectory, it holds that $x(k) = 0$ for all $k \le K$ and $x(K+1) = |u(K)|\,z(0) = 2r_0$, contradicting (23). □

4. UNIFORM STABILITY NOTIONS

In this section we establish a technical lemma that underlies the proofs of the necessity of the statements in Theorem 1 (i.e., the converse Lyapunov theorems). For this purpose, we introduce the following notions.

Consider a system as in (1) with inputs taking values in a subset Ω of $\mathbb{R}^m$. We use $\mathcal{M}_\Omega$ to denote the set of all input functions from $\mathbb{Z}_+$ to Ω.

Throughout this section, let Ω be compact, and let ω_o be a continuous function from $\mathcal{X}$ to $\mathbb{R}_{\geq 0}$.

Definition 4.1. A system (1) is *output stable* with respect to ω_o uniformly in $u \in \mathcal{M}_\Omega$, if there exists a $\mathcal{KL}$-function β such that

$$|y(k,\xi,u)| \leq \beta(\omega_o(\xi),k) \qquad \forall k \geq 0 \tag{24}$$

holds for all $u \in \mathcal{M}_\Omega$ and all $\xi \in \mathcal{X}$.

If, in addition, there exists $\sigma \in \mathcal{K}$ such that

$$|y(k,\xi,u)| \leq \sigma(|h(\xi)|) \quad \forall\, k \geq 0 \tag{25}$$

holds for all trajectories of the system with $u \in \mathcal{M}_\Omega$, then the system is *output-Lagrange output stable* with respect to ω_o uniformly in $u \in \mathcal{M}_\Omega$.

Finally, if (24) is strengthened to

$$|y(k,\xi,u)| \leq \beta(|h(\xi)|\,,k) \quad \forall\, k \geq 0 \tag{26}$$

for all trajectories of the system with $u \in \mathcal{M}_\Omega$, then the system is *state-independent output stable* uniformly in $u \in \mathcal{M}_\Omega$. □

We remark that in the special case when $h(\xi) = \xi$ and when $\omega_o(\xi) = |\xi|$, all the three uniform output stability properties given in Definition 4.1 reduce to the robust global asymptotic stability property, see (Jiang and Wang, 2002) and in (Kellett and Teel, 2002). Converse Lyapunov theorems were provided in (Jiang and Wang, 2002) and in (Kellett and Teel, 2002) for the global asymptotic stability property. Below we present a converse Lyapunov theorem for the more general notions of output stability.

Theorem 2. Suppose that a system (1) is output stable with respect to ω_o uniformly in $u \in \mathcal{M}_\Omega$. Then the system admits a continuous Lyapunov function V satisfying the following properties:

- there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

  $$\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(\omega_o(\xi)) \quad \forall\, \xi \in \mathcal{X}, \tag{27}$$

- there exists $\alpha_3 \in \mathcal{KL}$ such that

  $$V(f(\xi,\mu)) - V(\xi) \leq -\alpha_3(V(\xi),\omega_o(\xi)) \tag{28}$$

  for all $\xi \in \mathcal{X}$ and all $\mu \in \Omega$.

Moreover, if the system is output-Lagrange uniformly output stable with respect to inputs in $\mathcal{M}_\Omega$, then (27) can be strengthened to

$$\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|h(\xi)|) \quad \forall\, \xi, \tag{29}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Finally, if the system is state-independent uniformly output stable with respect to inputs in $\mathcal{M}_\Omega$, then (27) can be strengthened to (29) and (28) can be strengthened to:

$$V(f(\xi,\mu)) - V(\xi) \leq -\alpha_4(V(\xi)) \quad \forall\, \xi,\ \forall\, \mu \tag{30}$$

for some $\alpha_4 \in \mathcal{K}$. ∎

REFERENCES

Agarwal, R. P. (1992). *Difference Equations and Inequalities: Theory, Methods and Applications.* Marcel Dekker, Inc.. NY.

Jiang, Zhong-Ping and Yuan Wang (2001). Input-to-state stability for discrete-time nonlinear systems. *Automatica* **37**, 857–869.

Jiang, Zhong-Ping and Yuan Wang (2002). A converse Lyapunov theorem for discrete time systems with disturbances. *Systems & Control Letters* **45**(1), 49–58.

Jiang, Zhong-Ping, Eduardo D. Sontag and Yuan Wang (1999). Input-to-state stability for discrete-time nonlinear systems. In: *Proc. 14th IFAC World Congress, Beijing.* Vol. E. pp. 277–282.

Jiang, Zhong-Ping, Yuandan Lin and Yuan Wang (to be submitted). On input-to-output stability and Lyapunov functions for discrete time systems.

Kellett, Christopher M. and Andrew R. Teel (2002). On robustness of stability and Lyapunov functions for discontinuous difference equations. In: *Proc. 41st IEEE Conf. Decision and Control (CDC'02).* Las Vegas. pp. 4282–4287.

Lan, W. and J. Huang (2005). Robust output regulation for discrete-time nonlinear systems. *International Journal off Robust and Nonlinear Control* **15**, 63–81.

Nešić, Dragan and Andrew Teel (to appear). A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Transactions on Automatic Control.*

Nešić, Dragan and D. Angeli (2002). Integral versions of ISS for sampled-data nonlinear systems via their approximate discrete-time models. *IEEE Transactions on Automatic Control* **47**, 2033–2037.

Nešić, Dragan and D.S. Laila (2002). A note on input-to-state stabilization for nonlinear sampled-data systems. *IEEE Transactions on Automatic Control* **47**, 1153–1158.

Nešić, Dragan, Andrew Teel and Eduardo D. Sontag (1999*a*). Formulas relating $\mathcal{KL}$ stability estimates of discrete-time and sampled-data nonlinear systems. *Systems & Control Letters* **38**, 49–60.

Nešić, Dragan, Andrew Teel and Petar V. Kokotović (1999*b*). Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations. *Systems & Control Letters* **38**, 259–270.

Ogata, K. (1987). *Discrete-Time Control Systems.* Prentice-Hall. Englewood Cliffs.

Sontag, Eduardo D. and Yuan Wang (1999). Notions of input to output stability. *Systems & Control Letters* **38**, 351–359.

Sontag, Eduardo D. and Yuan Wang (2001). Lyapunov characterizations of input to output stability. *SIAM Journal on Control and Optimization* **39**, 226–249.