STABILITY RESULTS FOR CONTINUOUS AND DISCRETE TIME LINEAR PARAMETER VARYING SYSTEMS

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Abstract: In this paper, stabilizability problems for discrete and continuous time linear systems with switching or polytopic uncertainties are dealt with. Both state feedback and full-information controllers are considered, possibly with an integral action. Several cases will be analyzed, depending on the control, the uncertainty and the presence of uncertainties in the input matrix. A complete overview of the stabilizability implications among these different cases will be given.

Keywords: Stabilizability, LPV systems, Lyapunov functions, uncertainties, switching.

1. INTRODUCTION

Linear parameter varying systems are an important class of systems from a theoretical and practical point of view. In this paper, the stabilization problem of LPV systems is investigated by focussing on two main factors: the uncertainty characterization and the class of adopted controllers. As far as the time-varying parameter is concerned, two cases are considered: the case of switched systems, in which the time-varying parameter is allowed to take values in a discrete set of points, and the polytopic case, in which the parameter ranges in the convex hull of such points. A further relevant distinction that will come into play is the presence or absence of uncertainty in the input matrices.

As far as the class of controllers is concerned, depending on which are the variables measured for control purposes, different concepts of stabilizability will be defined. We will talk about robust stabilizability when no online information concerning the uncertainty is available to the controller. The case in which such information is available will be referred to as gain-scheduling or full information stabilizability. Also, a certain class of integral controllers, will be analyzed by means of properly expanded systems. Introducing an integrator in a loop has several well-known advantages (such as that of imposing a null steady-state error or handling rate-bounded control problems). Furthermore, the resulting ad-hoc built expanded system has no uncertainties on the control input matrix. This fact has several implications (Barmish (1983)), including the property that gradient-based controllers can be applied.

The class of functions that will be used in this work to establish the several interconnections that hold among the mentioned stabilizability concepts is that of polyhedral Lyapunov functions. Such class is wide enough for our purposes, as it has been established that the existence of polyhedral Lyapunov functions is a necessary and sufficient condition for stability (Molchanov and Pyatnitskiĭ (1986)) and for stabilizability of LPV systems (Blanchini (1995); Blanchini and Miani (2003)). Recently these functions have been considered for the stabilization of switched systems (De Santis et al. (2004); Sun and Ge (2005)). We will review some results already known in the literature

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and we will introduce some new statements to complete the scenario about the subject.

2. PRELIMINARIES AND DEFINITIONS

The systems considered are of the form

$$\dot{\mathbf{x}}(t) = A(\mathbf{w}(t))\mathbf{x}(t) + B(\mathbf{w}(t))\mathbf{u}(t) \tag{1}$$

in the continuous time case and

$$\mathbf{x}(t+1) = A(\mathbf{w}(t))\mathbf{x}(t) + B(\mathbf{w}(t))\mathbf{u}(t) \tag{2}$$

in the discrete time case, where $\mathbf{x}(t) \in \mathbb{R}^{\mathrm{n}}$ is the state variable, $\mathbf{u}(t) \in \mathbb{R}^{\mathrm{q}}$ is the control input, $\mathbf{w}(t) \in W \subset \mathbb{R}^{\mathrm{m}}$ is a time varying parameter. A and B are polytopes of matrices:

$$A(\mathbf{w}) = \sum_{i=0}^{m} w_i A_i, \quad B(\mathbf{w}) = \sum_{i=0}^{m} w_i B_i$$
$$W = \{\mathbf{w} : \sum_{i=1}^{m} w_i \leq 1,\ w_i \geq 0\} \tag{3}$$

We distinguish different kinds of system depending on the function $\mathbf{w}(t)$.

Definition 2.1. The systems is said to be switched if A and B assume values only on the vertices, precisely $A(\mathbf{w}(t)) = A_i$ and $B(\mathbf{w}(t)) = B_i$ for all t (i.e. at each time instant the components of the signal $\mathbf{w}(t)$ take only values $w_i = 1$ and $w_j = 0$ for $i \neq j$, for some i). In the following, $\mathbf{w}_i$ (to be intended as $\mathbf{w}_{i(t)}$) will represent this kind of signals.

By default (i.e. without "switching" specification $\mathbf{w}$), the system is intended as a polytopic LPV, say $\mathbf{w}$ is an arbitrary piecewise-continuous function satisfying (3). As a special case, we will consider systems with no uncertainty on the input matrix B:

$$\dot{\mathbf{x}}(t) = A(\mathbf{w}(t))\mathbf{x}(t) + B\mathbf{u}(t) \tag{4}$$
$$\mathbf{x}(t+1) = A(\mathbf{w}(t))\mathbf{x}(t) + B\mathbf{u}(t) \tag{5}$$

in the continuous and discrete time case respectively.

We investigate the following concepts of stabilization.

Definition 2.2. The state feedback $\mathbf{u} = \Phi_R(\mathbf{x})$ is robustly stabilizing (RS) for system (1) if the closed loop system

$$\dot{\mathbf{x}} = A(\mathbf{w}(t))\mathbf{x}(t) + B(\mathbf{w}(t))\Phi_R(\mathbf{x}) \tag{6}$$

is globally uniformly asymptotically stable (GUAS) with respect to the origin.

Definition 2.3. The full information feedback $\mathbf{u} = \Phi_{GS}(\mathbf{x},\mathbf{w})$ is gain-scheduling stabilizing (GSS) for system (1) if the closed loop system

$$\dot{\mathbf{x}} = A(\mathbf{w}(t))\mathbf{x}(t) + B(\mathbf{w}(t))\Phi_{GS}(\mathbf{x},\mathbf{w}) \tag{7}$$

is GUAS with respect to the origin.

Definition 2.4. The full information feedback $u = \Phi_{SGS}(\mathbf{x},\mathbf{w}_i)$ is switched gain-scheduling stabilizing (SGSS) for system (1) if the closed loop switched system

$$\dot{\mathbf{x}} = A(\mathbf{w}(t))\mathbf{x}(t) + B(\mathbf{w}(t))\Phi_{SGS}(\mathbf{x},\mathbf{w}) \tag{8}$$

is GUAS with respect to the origin.

Similar definitions can be given for discrete time systems. We anticipate that, for robust state feedback controllers, the switched and the "non–switched" case (i.e. $\mathbf{w}$ as in (3)) are indistinguishable, since they are strictly equivalent as later on will be explained.

Let us now introduce the notation and some basic results that will be used in the sequel.

- $\|P\|_1$ represents the one norm of the matrix P ($\|P\|_1 = \max_j \sum_i |p_{ij}|$);
- given a continuous function $\Psi : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$, we denote by $\mathcal{N}(\Psi,k)$ the set $\{\mathbf{x} \in \mathbb{R}^{\mathrm{n}} : \Psi(\mathbf{x}) \leq \mathrm{k}\}$;
- given a matrix X, $conv(X)$ represents the convex hull of the vectors given from the columns of X;
- the time dependency of the variables will be sometimes omitted to simplify the notation ($\mathbf{x}(t+1)$ will be indicated with $\mathbf{x}^+$).

Definition 2.5. The locally Lipschitz positive definite and radially unbounded function $\Psi(\mathbf{x})$ is a Lyapunov function for system (1) with the control $\Phi(\mathbf{x},\mathbf{w})$ if for all $k > 0$ there exist $\beta > 0$ such that

$$D^+\Psi(\mathbf{x},\mathbf{w}) = \limsup_{\tau\to 0^+} \frac{\Psi(\mathbf{x} + \tau[A(\mathbf{w})\mathbf{x} + B(\mathbf{w})\Phi(\mathbf{x},\mathbf{w})]) - \Psi(\mathbf{x})}{\tau} \leq -\beta \tag{9}$$

$\forall \mathbf{w} \in W$ and $\forall \mathbf{x} \notin \mathcal{N}(\Psi,k)$

Definition 2.6. The continuous positive definite and radially unbounded $\Psi(\mathbf{x})$ is a Lyapunov function for system (2) with the control $\Phi(\mathbf{x},\mathbf{w})$ if for all $k > 0$ there exists $\lambda > 0$ such that

$$\Psi(\mathbf{x}) - \Psi\left(A(\mathbf{w}(t))\mathbf{x}(t) + B(\mathbf{w}(t))\Phi(\mathbf{x},\mathbf{w})\right) > \lambda \tag{10}$$

$\forall \mathbf{w} \in W$ and $\forall \mathbf{x}$.

Definition 2.7. A function $\Psi : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$ is polyhedral if it is defined as follows

$$\Psi(\mathbf{x}) = \|F\mathbf{x}\|_\infty \tag{11}$$

where F is a full column rank matrix.

A crucial point for the considered class of systems is the following theorem.

Theorem 2.1. Consider the following system:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t),\mathbf{w}(t)), \tag{12}$$

where $\mathbf{x}(t) \in \mathbb{R}^{\mathrm{n}}$ is the state variable and $\mathbf{w}(t) \in W \subset \mathbb{R}^{\mathrm{m}}$ is the time varying parameter (W is a compact set and $\mathbf{w}(t)$ a piecewise continuous function). f is continuous and locally Lipschitz on $\mathbf{x}$ uniformly in $\mathbf{w}$. The following statements are equivalent:

(1) System (12) is GUAS with respect to the origin.
(2) There exists a smooth Lyapunov function for (12).

Proof 2.1. See Sontag et al. (1996). An earlier version was provided Meılakhs (1979) which holds for exponential stability only.

A tailored version of this theorem is stated below.

Corollary 2.1. The following statements are equivalent:

(1) System (1) is GUAS with respect to the origin with the locally Lipschitz controller $\mathbf{u}(t) = \Phi(\mathbf{x},\mathbf{w})$.
(2) There exists a smooth Lyapunov function for
$$\dot{\mathbf{x}}(t) = A(\mathbf{w}(t))\mathbf{x}(t) + B(\mathbf{w}(t))\Phi(\mathbf{x},\mathbf{w}). \quad (13)$$

Based on this fact, the following can be shown (Liberzon (2003)).

Proposition 1. The closed loop system (6) (or the corresponding discrete–time version) is GUAS if and only if its switched version $\dot{\mathbf{x}} = A(\mathbf{w}_i)\mathbf{x}(t) + B(\mathbf{w}_i)\Phi_R(\mathbf{x})$ is GUAS.

The same property holds for discrete-time systems. Indeed, any trajectory of the LPV system is formed by vectors included in the convex hull of the vertices generated by the switching system. This is why, when we deal with state feedback, we will not distinguish between switching and non-switching w.

Lemma 2.1. Assume there exists a Lyapunov function $\Psi_1(\mathbf{x})$ for system (1) with the continuous control $\Phi_1(\mathbf{x},\mathbf{w})$ locally Lipschitz uniformly with respect to w. Then there exists a polyhedral (and therefore convex) Lyapunov function $\Psi_2(\mathbf{x})$ and a control function $\Phi_2 : (vert\{\mathcal{N}(\Psi_2,1)\},\mathbf{w}) \to \mathrm{I\!R}^{q}$ for system (1) such that any of the following equivalent conditions hold:

- there exists $\beta > 0$ such that
$$D^+\Psi_2(\mathbf{x},\Phi_2(\mathbf{x},\mathbf{w}),\mathbf{w}) \le -\beta \qquad \forall \mathbf{w} \in W,\ \forall \mathbf{x} \in vert\{\mathcal{N}(\Psi,1)\} \quad (14)$$
- there exist $\tau > 0$ and $0 \le \lambda < 1$ such that
$$\Psi_2(\mathbf{x} + \tau[A(\mathbf{w})\mathbf{x} + B(\mathbf{w})\Phi_2(\mathbf{x},\mathbf{w})]) \le \lambda \qquad \forall \mathbf{w} \in W,\ \forall \mathbf{x} \in vert\{\mathcal{N}(\Psi_2,1)\} \quad (15)$$

Proof 2.2. It is basically the same of that provided in Blanchini (1995).

Remark 2.1. The first part of the previous lemma states that there is no restriction in considering polyhedral Lyapunov functions when dealing with stability and the second part shows that such functions can be computed by considering the discrete-time Euler approximating system of (1), defined as
$$x(t+1) = (I + \tau A(w(t)))x + \tau B(w(t))u$$

3. EQUIVALENCES FOR CONTINUOUS TIME SYSTEMS

3.1 Matrix B with uncertainties

It is known that for a continuous time polytopic system, the gain-scheduling stabilizability implies robust stabilizability (and, of course, viceversa).

Definition 3.1. An $r \times r$ matrix H belongs to the set $\mathcal{H}$ if and only if it can be written as
$$H = \tau^{-1}(P - I), \quad for some \quad \tau > 0 \qquad (16)$$
where $\|P\|_1 < 1$.

Theorem 3.1. (Blanchini (2000)) The following statements are equivalent.

(1) There exists a locally Lipschitz stabilizing controller of the form $\Phi_{GS}(\mathbf{x},\mathbf{w})$ for system (1).
(2) There exists a globally Lipschitz stabilizing controller of the form $\Phi_R(\mathbf{x})$ for system (1).
(3) There exists $r \ge n$ and matrices $X \in \mathrm{I\!R}^{n\times r}$ (full rank), $U \in \mathrm{I\!R}^{q\times r}$, and $H_i \in \mathcal{H}$ such that, for $k = 1,\dots,m$
$$A_iX + B_iU = XH_i \qquad (17)$$
where A_i, B_i are the vertices of the polytopic system.

From the previous theorem it follows that the knowledge of the disturbance acting on the system is not necessary to achieve stability is the system state is available for feedback. When switched gain-scheduling stabilizability is considered the scenario is different. Gain-scheduling (an therefore robust) stabilizability implies switched gain-scheduling stabilizability, but the opposite implications is not true in general as shown next.

Example 3.1. Consider the uncertain system
$$\dot{x} = [\alpha + 2(1-\alpha)]x + [\alpha - (1-\alpha)]u$$
For $\alpha = 0$ or $\alpha = 1$ (switched case) $\Phi_{SGS} = -3sgn[\alpha - (1-\alpha)]x$ stabilizes the system. When $0 \le \alpha \le 1$ the stabilizability is lost because for $\alpha = 0.5$ the system is unstable and not reachable.

The next theorem holds for switched systems.

Theorem 3.2. The following statements are equivalent.

(1) There exists a continuous stabilizing controller of the form $\Phi_{SGS}(\mathbf{x},\mathbf{w}_i)$ for (1).

(2) There exists $r \geq n$ and matrices $X \in \mathbb{R}^{n\times r}$ (full rank), $U_i \in \mathbb{R}^{q\times r}$, and $H_i \in \mathscr{H}$ such that, for $k = 1,\ldots,m$

$$A_i X + B_i U_i = X H_i. \qquad (18)$$

Proof 3.1. From section 2 it follows that the first statement is equivalent to the existence of a polyhedral Lyapunov function $\Psi_2(\mathbf{x})$ and a control $\Phi_2(\mathbf{x},\mathbf{w}_i)$. Therefore it just needs to be proved that (18) is equivalent to (15). Equation (15) implies (see remark 2.1) that for every vertex x_j of the unit ball $\Omega = \mathscr{N}(\Psi_2,1)$ there exist $\mathbf{u}_{j_i} = \Phi_2(\mathbf{x}_{j_i},\mathbf{w}_i)$ that assure

$$(I+\tau A_i)x_j + \tau B_i \mathbf{u}_{j_i} \in \lambda\Omega \qquad (19)$$

$\forall i = 1,...,m$. Equation (19) can be rewritten in the following way

$$(I+\tau A_i)x_j + \tau B_i \mathbf{u}_{j_i} = X p_{j_i} \qquad (20)$$

$\forall i = 1,...,m$, where $\left\|p_{j_i}\right\|_1 \leq 1$ and X represents the matrix $X = [\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_r]$. Defining

$$P_i = [p_{1_i}\ p_{2_i}\ \ldots\ p_{r_i}]$$
$$U_i = [\mathbf{u}_{1_i}\ \mathbf{u}_{2_i}\ \ldots\ \mathbf{u}_{r_i}]$$

a compact version of (20) can be achieved:

$$(I+\tau A_i)X_i + \tau B_i U_i = X P_i, \qquad (21)$$

that finally can be rewritten as

$$A_i X_i + B_i U_i = X(P_i - I)/\tau = X H_i \qquad (22)$$

obtaining (18).
This procedure can be reversed to prove the existence of a polyhedral Lyapunov function starting from the existence of the matrices H_i.

3.2 Matrix B without uncertainties

When the matrix B is not affected by uncertainty, the additional property that switched gain-scheduling stabilizability implies gain-scheduling stabilizability holds.

Theorem 3.3. The next statements are equivalent.

(1) There exists a globally Lipschitz stabilizing controller of the form $\Phi_R(\mathbf{x})$ for system (4).
(2) There exists a locally Lipschitz stabilizing controller of the form $\Phi_{GS}(\mathbf{x},\mathbf{w})$ the system (4).
(3) There exists a locally Lipschitz stabilizing controller of the form $\Phi_{SGS}(\mathbf{x},\mathbf{w})$ the system (4).

Proof 3.2. (1⇔2) follows from Theorem 3.1. (2⇒3) is trivial.

(3⇒2) Statement 3 implies the existence of a smooth Lyapunov function $\Psi(\mathbf{x})$ such that $\forall \mathbf{w}_i$

$$\dot{\Psi}(\mathbf{x},\mathbf{w}_i) = \nabla\Psi(A(\mathbf{w}_i)\mathbf{x} + B\Phi_{SGS}(\mathbf{x},\mathbf{w}_i)) < 0$$

Choosing the following gain-scheduling controller (that is also locally Lipschitz)

$$\Phi_{GS}(\mathbf{x},\mathbf{w}) = \sum_{i=1}^{m} \Phi_{SGS}(\mathbf{x},\mathbf{w}_i) w_i$$

the non-positivity of $\dot{\Psi}(\mathbf{x})$ holds:

$$\begin{aligned}\dot{\Psi}(\mathbf{x}) &= \nabla\Psi(A(\mathbf{w})\mathbf{x} + B\Phi_{GS}(\mathbf{x},\mathbf{w}))\\ &= \nabla\Psi(\sum_{i=1}^{m} A(\mathbf{w}_i) w_i \mathbf{x} + B\sum_{i=1}^{m}\Phi_{SGS}(\mathbf{x},\mathbf{w}_i) w_i)\\ &= \sum_{i=1}^{m} w_i \nabla\Psi(A(\mathbf{w}_i)\mathbf{x} + B\Phi_{SGS}(\mathbf{x},\mathbf{w}_i)) < 0\end{aligned}$$

3.3 Expanded systems

The expanded system is a concept already introduce in the quadratic stabilization framework (Barmish (1983)). In this subsection it will be shown that robust stabilizability implies robust stabilizability of the expanded system and viceversa.

Theorem 3.4. The following statements are equivalent.

(1) There exists a globally stabilizing controller of the form $\Phi_R(\mathbf{x})$ for

$$\dot{\mathbf{x}} = A(\mathbf{w})\mathbf{x} + B(\mathbf{w})\mathbf{u} \qquad (23)$$

(2) There exists a globally stabilizing controller of the form $\Phi'_R(\mathbf{x},\mathbf{u})$ for

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{u}}\end{bmatrix} = \begin{bmatrix}A(\mathbf{w}) & B(\mathbf{w})\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} + \begin{bmatrix}0\\ I_q\end{bmatrix}\mathbf{v} \qquad (24)$$

where $\mathbf{v} \in \mathbb{R}^q$ is the input variable of the expanded system.

Proof 3.3. (2⇒1) From theorem (3.1) it follows that

$$\begin{bmatrix}A_i & B_i\\ 0 & 0\end{bmatrix}\begin{bmatrix}X\\ U\end{bmatrix} + \begin{bmatrix}0\\ I_q\end{bmatrix} V = \begin{bmatrix}X\\ U\end{bmatrix} H'_i$$

The equation given from the first row guarantees the stability of (23).

(1⇒2) As a consequence of theorem 3.1, there exists $r \geq n$ and matrices $X \in \mathbb{R}^{n\times r}$ (full rank), $U \in \mathbb{R}^{q\times r}$, and $H_i \in \mathscr{H}$ such that, for $i = 1,\ldots,m$

$$A_i X + B_i U = X H_i \qquad (25)$$

An equivalent gain-scheduling form for (24) is now sought. The above equation can be written as follows:

$$\begin{bmatrix}A_i & B_i\\ 0 & 0\end{bmatrix}\begin{bmatrix}X\\ U\end{bmatrix} + \begin{bmatrix}0\\ I_q\end{bmatrix} V_i = \begin{bmatrix}X\\ U\end{bmatrix} H_i$$

with $V_i = U H_i$. Theorem 3.2 can be used to proof the switched gain-scheduling stability of (24), but this is not guaranteed if $[X^T\ U^T]^T$ is not a full row rank matrix. In this case q zero columns are added to the vector X obtaining the following equation (equivalent to (25))

$$A_i[X\ 0] + B_i[U\ 0] = [X\ 0]\begin{bmatrix}H_i & 0\\ 0 & 0\end{bmatrix}$$

and its expanded version

$$\begin{bmatrix}A_i & B_i\\ 0 & 0\end{bmatrix}\begin{bmatrix}X & 0\\ U & 0\end{bmatrix} + \begin{bmatrix}0\\ I_q\end{bmatrix} V'_i = \begin{bmatrix}X & 0\\ U & 0\end{bmatrix}\begin{bmatrix}H_i & 0\\ 0 & 0\end{bmatrix}$$

where $V_i' = [V_i\, 0]$. A perturbation to the above equation is now introduced

$$\begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ U & \gamma I \end{bmatrix} + \begin{bmatrix} 0 \\ I_q \end{bmatrix} Z_i'' = \begin{bmatrix} X & 0 \\ U & \gamma I \end{bmatrix} \hat{H}_i$$

where $\gamma > 0$, $V_i'' = [V_i\, \overline{V_i}]$ and

$$\hat{H}_i = \begin{bmatrix} H_i & H_i' \\ 0 & H_i'' \end{bmatrix}$$

It is alway possible to find V_i'' and $\hat{H}_i$ such that the perturbed equation holds. The full row rank condition is now satisfied. The last thing to be proved is the existence of γ such that $\hat{H}_i \in \mathscr{H}$. Since $H_i \in \mathscr{H}$, it follows that $H_i = \tau^{-1}(P_i - I)$, where $\|P_i\|_1 < 1$. Using the same value of τ, the following conditions can be checked:

$$\left\| [H_i'^T \quad (H_i'' - \tau I)^T]^T \right\|_1 < \tau$$
$$B_i \gamma I = X H_i'$$

It is always possible to find H_i' to satisfy the last equation since X is full row rank (finding such matrices corresponds to solve $m \times n \times q$ linear equations with $m \times r \times q$ variables). Decreasing the value of γ also $\|H_i'\|_1$ becomes smaller and therefore a suitable H_i'' to satisfy the first inequality can be found. So far only the switched gain-scheduling stabilizability of (24) has been proved. Due to theorem 3.1, the expanded system is also gain-scheduling and robustly stabilizable.

4. EQUIVALENCES FOR DISCRETE TIME SYSTEMS

4.1 Matrix B with uncertainties

In this subsection it will be shown that for discrete time systems not all the implications that hold in the continuous time case are still valid. Anyway, some theorems that resemble the results of the previous section can be stated.

In Blanchini (1995) the following theorem has been proved.

Theorem 4.1. The following statements are equivalent:

(1) There exists a stabilizing controller of the form $\Phi_{RS}(\mathbf{x})$ for system (2).
(2) There exists $r \geq n$ and matrices $X \in \mathrm{IR}^{n\times r}$ (full rank), $P_i \in \mathrm{IR}^{q\times r}$ and $\|P\|_1 < 1$ such that, for $k = 1,\ldots,m$

$$A_i X + B_i U = X P_i. \tag{26}$$

For the gain-scheduling case there is a similar result only when the system is switched.

Theorem 4.2. The following statements are equivalent:

(1) There exists a stabilizing controller of the form $\Phi_{SGS}(\mathbf{x},\mathbf{w})$ for system (2).
(2) There exists $r \geq n$ and matrices $X \in \mathrm{IR}^{n\times r}$ (full rank), $P_i \in \mathrm{IR}^{q\times r}$ and $\|P\|_1 < 1$ such that, for $k = 1,\ldots,m$

$$A_i X + B_i U_i = X P_i. \tag{27}$$

Proof 4.1. The proof is similar to one given for theorem 3.1 in Blanchini and Miani (2003) and it will be omitted for brevity.

For a discrete time system robust stabilizability obviously implies gain-scheduling stabilizability ($RS \Rightarrow GS$). Unfortunately for discrete time systems the equivalence does not hold, as it can be easily shown by means of a counterexample.

Example 4.1. Consider the following system with one state and one input variable

$$x(t+1) = w(t)x(t) + u(t)$$

where $|w(t)| \leq 3$. It is easy to show that the system is gain-scheduling stabilized by $u(t) = -w(t)x(t)$ but it is not robustly stabilizable.

Also for discrete time systems switched gain-scheduling stabilizability does not imply gain-scheduling stabilizability as it can be shown with the following example.

Example 4.2. Consider the uncertain system

$$x(t+1) = 2x(t) + [\alpha - (1-\alpha)]u(t)$$

For $\alpha = 0$ or $\alpha = 1$ (switched case) it is possible to find Φ_{SGS} to stabilize the system. When $0 \leq \alpha \leq 1$ the stabilizability is lost because for $\alpha = 0.5$ the system is unstable and not reachable.

4.2 Matrix B without uncertainties

When there are no uncertainties on the matrix B the following theorem can be stated.

Theorem 4.3. The following statements are equivalent.

(1) There exists a controller of the form $\Phi_{GS}(\mathbf{x},\mathbf{w})$ for (5).
(2) There exists a controller of the form $\Phi_{SGS}(\mathbf{x},\mathbf{w})$ for (5).
(3) There exists $r \geq n$ and matrices $X \in \mathrm{IR}^{n\times r}$ (full rank), $P_i \in \mathrm{IR}^{q\times r}$ and $\|P\|_1 < 1$ such that, for $k = 1,\ldots,m$

$$A_i X + B U_i = X P_i. \tag{28}$$

Proof 4.2. (1)⇔(3) was proved in Blanchini and Miani (2003). (2)⇔(3) is a particular case of Theorem 4.2.

4.3 Expanded systems

Theorem 4.4. The following statements are equivalent.

(1) There exists a globally Lipschitz stabilizing controller of the form $\Phi_R(\mathbf{x})$ for

$$\mathbf{x}^+ = A(\mathbf{w})\mathbf{x} + B(\mathbf{w})\mathbf{u} \tag{29}$$

(2) There exists a globally Lipschitz stabilizing controller of the form $\Phi'_{GS}(\mathbf{x},\mathbf{w})$ for

$$\begin{bmatrix} \mathbf{x}^+ \\ \mathbf{u}^+ \end{bmatrix} = \begin{bmatrix} A(\mathbf{w}) & B(\mathbf{w}) \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} + \begin{bmatrix} 0 \\ I_q \end{bmatrix} \mathbf{v} \tag{30}$$

where $\mathbf{v}(t) \in \mathbb{R}^q$ is the input variable of the expanded system.

Proof 4.3. Almost identical to that of Theorem 3.4.

For the discrete time case, the expanded system achieved from a robustly stabilizable system is only gain-scheduling stabilizable but it may fail to be robustly stabilizable as in the case of the next example.

Example 4.3. Consider the following system:

$$x(t+1) = 2x(t) + (1+w(t))u(t) \tag{31}$$

where $|w(t)| \leq 0.4$. The controller $u(t) = -2x(t)$ stabilizes the system, but there is no stabilizing controller for the expanded system. Note that the expansion of a system correspond to the insertion of a delay. If the expanded robust stabilizability would be maintained, the theorem, recursively applied, would imply robust stabilizability with an arbitrary delay on the control input.

5. SUMMARY OF THE IMPLICATIONS AND DISCUSSION

Let us denote by

RS = robust stabilizability;
GSS = gain-scheduling stabilizability;
SGSS = switched gain-scheduling stabilizability;
ERS = expanded robust stabilizability;
EGSS = expanded gain-scheduling stabilizability;
ESGSS = expanded switched gain-scheduling stabilizability.

In the continuous–time case, the implications among these concepts are reported next.

- **B with uncertainties**

$$RS \iff GSS \implies SGSS$$

- **B without uncertainties**

$$RS \iff GSS \iff SGSS$$

- **Equivalences for the expanded system**

$$\begin{array}{c} RS \\ \Updownarrow \\ ERS \iff EGSS \iff ESGSS \end{array}$$

In the corresponding discrete-time table, reported below, several of the "$\iff$" become "$\implies$".

- **B with uncertainties**

$$RS \implies GSS \implies SGSS$$

- **B without uncertainties**

$$RS \implies GSS \iff SGSS$$

- **Equivalences for the expanded system**

$$\begin{array}{ccccc} & & RS & & \\ & & \Updownarrow & & \\ ERS & \implies & EGSS & \iff & ESGSS \end{array}$$

In conclusion, we have seen how several options in the uncertainty specification and in the controller class can be crucial in the stabilization of LPV systems. We have also seen how continuous and discrete time problems are differently affected by these options.

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