ON NONUNIFORM AND SEMI-UNIFORM INPUT-TO-STATE STABILITY FOR TIME VARYING SYSTEMS

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Abstract: In this work we discuss nonuniform and semi-uniform input-to-state stability (ISS) properties for time-varying systems. Treating the ISS properties for a time varying system as properties related to output stability for a time invariant auxiliary system, we provide several characterizations in terms of Lyapunov functions and the asymptotic gains; as well as a small gain theorem, for the time varying notions on input-to-state stability.

Keywords: input-to-state stability, time-varying nonlinear systems, Lyapunov functions, asymptotic gains, small gain theorems.

1. INTRODUCTION

Since formulated by E.D.Sontag in the late 1980's, the notion of input-to-state stability (ISS) has found wide applications in system analysis and design. For instance, various small gain theorems (c.f., for instance, (Jiang *et al.*, 1994), (Coron *et al.*, 1995), (Ingalls and Sontag, 2002)) in the ISS framework provide convenient tools for stability of interconnected systems.

In practice, it is very often the case that the systems under consideration are time varying. Such a situation often arises from, e.g., trajectory tracking problems. Another often seen situation is that a time invariant system may fail to be stabilized by a C^1 time invariant feedback, yet a time varying feedback may result in a stable closed-loop system. For instance, it was shown in (Karafyllis and Tsinias, 2003) that if a system can be stabilized by a C^0 time invariant feedback, then it is stabilizable by a *smooth* time varying feedback. It is thus natural to understand the ISS property for time varying systems.

The authors in (Lin, 1996), (Edwards *et al.*, 2000) and (Malisoff and Mazenc, to appear) studied the uniform ISS property for time varying systems with equivalent Lyapunov formulations. In a series of recent work (see e.g., (Karafyllis and Tsinias, 2004) and (Karafyllis and Tsinias, 2003)), notions of nonuniform in time stability and input-to-state stability were introduced. The nonuniform in time ISS property roughly means that, after some transition phases (whose lengths may depend on t_0), the trajectories are bounded by a generalized L_∞ norm of u with a weighting function. The authors of (Karafyllis and Tsinias, 2004) and (Karafyllis and Tsinias, 2003) have obtained Lyapunov results, small gain theorems together with some interesting applications on the nonuniform in time ISS property.

In this work, we will study two notions on ISS, nonuniform ISS and semi-uniform ISS. The nonuniform ISS property in our work is different from the nonuniform ISS property proposed and studied in (Karafyllis and Tsinias, 2004) and (Karafyllis and Tsinias, 2003), mainly in that in our formulation, we do not allow the weighting function in the estimates of the gain functions (see Section 2.3 for more details).

This paper is organized as follows. In Section 2, we introduce the time varying notions on ISS, and compare our notions with other related notions in the literature. In Section 3, we establish Lyapunov characterizations of the time varying notions on ISS. In Section 4, we show that the semi-uniform ISS property is equivalent to the conjunction of the unfiorm (Lagrange) stability and the asymptotic gain property. In Section 5, we present a small gain theorem for the semi-uniform ISS property.

Because of the length restriction, we have omitted most of the proofs. The detailed proofs will be provided in the forth coming paper (Lin *et al.*, n.d.).

2. BASIC DEFINITIONS

Consider the time varying system

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{1}$$

where, for each t, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz. Inputs, denoted by u, are measurable, locally essentially bounded functions from $\mathbb{R}$ to $\mathbb{R}^m$. We use $x(t, \xi, t_0, u)$ to denote the trajectory of the system corresponding to the initial condition $x(t_0) = \xi$ and the input function u. This solution is uniquely defined on some maximum interval $[t_0, T_{t_0,\xi,u})$ with $T_{t_0,\xi,u} \le \infty$. If $T_{t_0,\xi,u} = \infty$ for all t_0, ξ and all u, the system is said to be *forward complete*.

Throughout this work, we use $|\xi|$ to denote the Euclidean norm for $\xi \in \mathbb{R}^n$, and, for $-\infty < a < b \le \infty$, we use $\|u\|_{(a,b)}$ to denote the L^m_∞ norm of u as a function defined on the interval (a, b). We use $\|u\|$ to denote the L^m_∞ norm of u on $[0, \infty)$.

A function $\alpha : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is *of class* $\mathcal{N}$ if it is continuous and nondecreasing; is *of class* $\mathcal{K}$ if it is continuous, positive definite, and strictly increasing; and is *of class* $\mathcal{K}_\infty$ if it is also unbounded. A function $\beta : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is said to be of class $\mathcal{KL}$ if for each fixed $t \ge 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each fixed $s \ge 0$, $\beta(s, t)$ decreases to 0 as $t \to \infty$.

2.1 Notions of Input-to-State Stability

Definition 2.1. A system as in (1) is *input-to-state stable* (ISS) if there exist $\beta \in \mathcal{KL}$, $\sigma_0 \in \mathcal{N}$ and $\gamma \in \mathcal{K}$ such that, for each t_0, each ξ and each u, the following holds for all $t \ge t_0$:

$$\begin{aligned} |x(t, \xi, t_0, u)| \; &\le \; \beta(\sigma_0(t_0)\,|\xi|\,,\; t - t_0) \\ &+ \; \gamma\left(\|u\|_{(t_0,\infty)}\right). \end{aligned} \tag{2}$$

A system as in (1) is *semi-uniformly input-to-state stable* if it is ISS, and if in addition, there exists some $\sigma \in \mathcal{K}$ such that

$$|x(t, \xi, t_0, u)| \le \max\{\sigma(|\xi|),\ \sigma(\|u\|)\} \quad \forall\, t \ge t_0 \tag{3}$$

holds for all $\xi \in \mathbb{R}^n$, all u and all t_0. □

Similar to the discussions as in Section 2.2 of (Sontag and Wang, 2001), one can show that system (1) is semi-uniformly ISS if and only if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $\rho \in \mathcal{K}$ such that

$$|x(t, \xi, t_0, u)| \le \beta\left(|\xi|\,, \frac{t - t_0}{1 + \rho(|t_0|)}\right) + \gamma(\|u\|) \tag{4}$$

for all $t \ge t_0$.

It is not hard to see that it will result in the same definition if one requires ρ in (4) to be an $\mathcal{N}$ function.

Note that it is natural to define a system as in (1) to be *uniformly* ISS (UISS) by requiring the trajectories to satisfy an estimate as in

$$|x(t, \xi, t_0, u)| \le \beta(|\xi|\,, t - t_0) + \gamma(\|u\|_{[t_0,\infty)}) \tag{5}$$

for all $t \ge t_0$ (where $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$), that is, when the function ρ in (4) can be chosen as a constant function. Observe that the ISS notion studied in (Lin, 1996) and (Edwards *et al.*, 2000) is in fact the *uniform* ISS property.

An estimate as in (4) nicely encapsulates both the ISS and the uniform boundedness property (3) aspects of the semi-uniform ISS property. It shows intrinsically how the semi-uniform ISS property sits between the ISS and the uniform ISS properties: the semi-uniform ISS property requires the overshoots of the trajectories be bounded by $|\xi|$ (which is uniform in t_0), yet the rates at which the trajectories decay to the ball of radius $\gamma(\|u\|)$ is not necessarily uniform in t_0. The UISS property requires both the overshoot estimates and the decay rates be uniform in $|t_0|$.

To see the difference beween the semi-uniform ISS and the uniform ISS properties, consider the system

$$\dot{x} = -\frac{x - u}{1 + |t|}. \tag{6}$$

To make the discussions simpler, we will restrict to the case $t_0 \ge 0$. It is not hard to see that the trajectories of the system are given by

$$x(t) = x(0)\frac{1 + t_0}{1 + t} + \frac{1}{1 + t}\int_{t_0}^{t} u(s)\, ds$$

($t \ge t_0 \ge 0$), and thus, we get the following semi-uniform ISS estimate:

$$|x(t, x_0, t_0, u)| \le \beta\left(|x_0|\,, \frac{t - t_0}{1 + t_0}\right) + \gamma(\|u\|),$$

where $\beta(r, s) = \frac{r}{1+s}$, $\gamma(r) = r$. On the other hand, it should not be hard to see that the system is not UISS.

To see the difference between the semi-uniform ISS and the ISS properties, one may consider the system

$$\dot{x}_1 = -2x_1, \quad \dot{x}_2 = x_1 e^t - x_2 + u. \tag{7}$$

which is ISS, but not semi-uniformly ISS.

2.2 Notions of Asymptotic Stability

Observe that in the special case when there is no input signal acting on a system, the ISS property reduces to asymptotical stability properties for systems as in the following:

$$\dot{x}(t) = f(t, x(t)), \tag{8}$$

where f is a locally Lipschitz map. We use $x(t, \xi, t_0)$ to denote the trajectory of (8) with the initial condition $x(t_0) = \xi$. Corresponding to the ISS notions, we have the following:

Definition 2.2. Consider a system as in (8):

- it is *globally asymptotically stable* (GAS) if for some $\beta \in \mathcal{KL}$ and $\sigma_0 \in \mathcal{N}$, the following holds for all ξ and all t_0:

$$|x(t, \xi, t_0)| \le \beta(\sigma_0(t_0)\,|\xi|\,, t - t_0)\ \forall\, t \ge t_0; \tag{9}$$

- it is *semi-uniformly* GAS if
  - it is GAS, and
  - it is uniformly stable, that is, for some $\sigma \in \mathcal{K}$, it holds for all t_0 and all ξ that $|x(t, \xi, t_0)| \le \sigma(|\xi|)$ for all $t \ge t_0$;
- it is uniformly GAS (UGAS) if for some $\beta \in \mathcal{KL}$, the following holds for all ξ and all t_0:

$$|x(t, \xi, t_0)| \le \beta(|\xi|\,, t - t_0) \quad \forall\, t \ge t_0. \tag{10}$$

We remark that the GAS and the UGAS properties defined here are the same as the asymptotic stability in the whole and the uniform asymptotical stability in the whole defined in (Hahn, 1967).

It can be proved that the semi-uniform GAS property is equivalent to the existence of a $\mathcal{KL}$-function β and an $\mathcal{N}$-function ρ such that

$$|x(t, \xi, t_0)| \le \beta\left(|\xi|\,,\ \frac{t - t_0}{1 + \rho(|t_0|)}\right) \ \ \forall\, t \ge t_0. \tag{11}$$

By the definitions, one sees that if system (1) is ISS (semi-uniformly ISS, uniformly ISS, respectively), then its zero-input system $\dot{x}(t) = f(t, x(t), \mathbf{0})$ is GAS (semi-uniformly GAS, UGAS respectively), where $\mathbf{0}$ denotes the zero input function.

2.3 Comparison with Other Notions

We remark that the (nonuniform) ISS property given in Definition 2.1 is not the same as the nonuniform in time ISS property proposed and studied in (Karafyllis and Tsinias, 2004) and (Karafyllis and Tsinias, 2003). The later notion is equivalent to requiring the trajectories of the system satisfy the estimate

$$\begin{aligned} |x(t, \xi, t_0, u)| \ \le\ & \beta(\sigma_0(t_0)\,|\xi|\,,\ t - t_0) \\ & +\ \gamma\left(\|\phi u\|_{(t_0,\infty)}\right), \end{aligned} \tag{12}$$

for some "weighting" function ϕ, where $\beta \in \mathcal{KL}$, $\sigma_0 \in \mathcal{N}$ and $\gamma \in \mathcal{K}$ (Karafyllis and Tsinias, 2003).

It can be seen that the ISS notion defined by (2) is stronger than the one defined by (12). On the other hand, the weighting function ϕ in (12) may allow the trajectories not to converge to a ball with radius "proportional" to $\|u\|$ (and even to diverge from such a ball), especially when $\phi(t) \to \infty$ as $t \to \infty$. For instance, the system

$$\dot{x} = -x + xu$$

is not ISS as defined in Definition 2.1, because the bounded input $u \equiv 2$ results in unbounded trajectories. But since the system is forward complete, and the corresponding 0-input system is GAS, the system does satisfy the ISS property defined by (12) (Karafyllis and Tsinias, 2004, Corollary 3.7).

For systems without input signals as in (8), both of the ISS notions defined by (2) and (12) lead to the same notion of GAS defined by (9) and studied in (Karafyllis and Tsinias, 2003).

The semi-uniform ISS and semi-uniform stability properties describe the special cases of ISS and GAS when the overshoots in the transient phases do not depend on the values of the initial time t_0. Though not found in the past literature, we believe that these notions have interesting applications in practice. While the ISS notion defined by (12) does not coincide with the usual ISS property in the special case for time invariant systems, the UISS, the semi-uniform ISS, and the ISS notions defined by (5), (4) and (2) all reduce to the usual ISS property in the special case for time invariant systems.

We also point out that our blanket assumption that $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz map is stronger than the assumption used in the work along the line (Karafyllis and Tsinias, 2003), where f is not assumed to be locally Lipschtiz with respect to the t variable. Indeed, the Lipschitz condition on the t variable should not be essential. However, because our approach of treating the time varying notions as output stability notions for some time invariant auxiliary systems (see Section 2.4), we need to put the Lipschitz condition on the t variable.

2.4 Time Varying Stability Properties as Output Stability Properties of Time Invariant Systems

We associate with the system (1) the following auxiliary time invariant system

$$\dot{\tau} = 1,\ \dot{x} = f(\tau, x, u) \tag{13}$$

with the output map $h(\tau(t), x(t)) = x(t)$. It turns out that the stability properties of (1) are related to the output stability properties of (13) (see (Sontag and Wang, 1999) for detailed definitions of the output stability properties).

Proposition 2.3. The following holds for (1):

- it is ISS if and only if (13) is input-to-output stable;
- it is semi-uniform ISS if and only if (13) is output-Lagrange input-to-output stable;
- it is UISS if and only if (13) is state-independent input-to-output stable. ∎

Based on Proposition 2.3, many results on the time varying notions of stability can be derived from the results on the output stability properties for time invariant systems.

3. LYAPUNOV FUNCTIONS

In this section we study the Lyapunov concepts associated with the time varying stability properties. The following proposition was shown in the previous work (e.g., (Lin, 1996) and (Edwards *et al.*, 2000)), where for a smooth (C^∞) function $\varphi : (t,\xi) \mapsto \varphi(t,\xi)$, we use $D_t\varphi$ to denote the partial derivative of φ in the variable t, and $D_\xi\varphi(t,\xi)$ for the gradient of φ in the variable $x \in \mathbb{R}^n$.

Proposition 3.1. The system (1) is uniformly ISS if and only if there exists a smooth function $V : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

- for some $\alpha_i \in \mathcal{K}_\infty$ $(i = 1,2)$:

$$\alpha_1(|\xi|) \leq V(t,\xi) \leq \alpha_2(|\xi|) \quad \forall \xi,\ \forall t; \tag{14}$$

- for some $\chi \in \mathcal{K}$ and some $\alpha_3 \in \mathcal{K}_\infty$,

$$\begin{gathered} V(t,\xi) > \chi(|\mu|) \\ \Downarrow \\ D_tV(t,\xi) + D_\xi V(t,\xi) f(t,\xi,\mu) \leq -\alpha_3(\xi)\,. \end{gathered} \tag{15}$$

We call a function V satisfying the conditions in Proposition 3.1 a UISS-*Lyapunov function* for (1).

The proof of the sufficiency implication in Proposition 3.1 is very similar to the proof in the time invariant case. First of all, (15) implies that for some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{KL}$, it holds that

$$V(t, x(t,\xi,t_0,u)) \leq \beta(V(t_0,\xi), t-t_0) + \gamma(\|u\|). \tag{16}$$

Together with (14), one can get a uniform iss estimate on $x(t,\xi,t_0,u)$.

It should not be hard to prove that if (14) is relaxed to

$$\alpha_1(|\xi|) \leq V(t,\xi) \leq \alpha_2(\alpha_0(t)\,|\xi|) \quad \forall \xi,\ \forall t, \tag{17}$$

where $\alpha_0 \in \mathcal{N}$, then together with (15), and hence (16), one can get an ISS estimate for the trajectories of the system. However, the conditions (17) and (15) on the Lyapunov function are too strong in the sense that the converse of the result fails. To establish a Lyapunov formulation that is equivalent to the ISS property, we consider the following:

Definition 3.2. With respect to the system (1), a smooth function $V : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a function $\lambda : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are called an ISS-*Lyapunov function* and an *auxiliary modulus* if

(1) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that (17) holds;
(2) there exist $\chi \in \mathcal{K}$ and $\alpha_3 \in \mathcal{K}_\infty$ such that

$$\begin{gathered} V(t,\xi) > \chi(|\mu|) \\ \Downarrow \\ D_tV(t,\xi) + D_\xi V(t,\xi) f(t,\xi,\mu) \leq -\frac{\alpha_3(V(\xi))}{1+\lambda(t,\xi)}; \end{gathered} \tag{18}$$

(3) the following holds for the auxiliary modulus function λ:
   - for some $\kappa \in \mathcal{K}$, $0 \leq \lambda(t,\xi) \leq \kappa(|(t,\xi)|)$ for all t and all ξ; and
   - λ is locally Lipschitz on $\mathbb{R}\times(\mathbb{R}^n \setminus \{0\})$ and satisfies

$$\begin{gathered} V(t,\xi) > \chi(|\mu|) \\ \Downarrow \\ D_t\lambda(t,\xi) + D_\xi\lambda(t,\xi) f(t,\xi,\mu) \leq 0 \end{gathered} \tag{19}$$

   almost everywhere on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$.

We say that an ISS-Lyapunov function is a *semi-uniform* ISS-*Lyapunov function* for (1) if (17) can be strengthened to (14) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. □

Theorem 1. For a system as in (1):

(1) it is ISS if and only if it admits an ISS-Lyapunov function;
(2) it is semi-uniformly ISS if and only it admits a semi-uniform ISS-Lyapunov function. ■

3.1 Remarks on Lyapunov Functions

In contrast to the uniform ISS-Lyapunov functions (or with ISS-Lyapunov functions in the time invariant case), one has to consider the auxiliary modulus function $\lambda(\cdot,\cdot)$ in the non-uniform case. This is mainly because in the non-uniform case, the decay rate of the Lyapunov function along trajectories are affected by t_0. The following result provides a Lyapunov sufficient condition. The conditions on the decay rate of the Lyapunov function along trajectories are ready to be checked. However, it is still not clear to us if the existence of such Lyapunov functions is necessary for the ISS property.

Proposition 3.3. The system (1) is ISS if there exists a C^1 function $V : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying the following:

(a) there exist some $\alpha_1 \in \mathcal{K}$ and some $\alpha_2 \in \mathcal{K}$ such that (17) holds;
(b) there exist some $\chi \in \mathcal{K}$, $\alpha_3 \in \mathcal{K}_\infty$ and some $\kappa \in \mathcal{N}$ such that

$$\begin{gathered} V(t,\xi) > \chi(|\mu|) \\ \Downarrow \\ D_tV(t,\xi) + D_\xi V(t,\xi) f(t,\xi,\mu) \leq -\frac{\alpha_3(V(t,\xi))}{1+\kappa(t)}; \end{gathered} \tag{20}$$

(c) $\displaystyle\int_{t_0}^{\infty} \frac{1}{1+\kappa(s)}\,ds = \infty$ for some t_0.

Moreover, the system is semi-uniform ISS if there exists a function V as in the above satisfying conditions (a)-(c) with (a) strengthened to the existence of $\alpha_1 \in \mathcal{K}_\infty$ and $\alpha_2 \in \mathcal{K}_\infty$ such that (14) holds.

A related result is an equivalent Lyapunov characterization for UISS given in the recent work

(Malisoff and Mazenc, to appear). By Theorem 6 in (Malisoff and Mazenc, to appear), if the condition (c) in Proposition 3.3 is strengthened to the following:

$$\exists\, h>0,\ \varepsilon>0\ \ni: \int_{t-h}^{t} \frac{1}{1+\kappa(s)}\,ds \ge \varepsilon \quad \forall\, t\ge 0$$

then the system is UISS.

Remark 3.4. Though conditions (b) and (c) in Proposition 3.3 are perhaps too strong for the existence of such a Lyapunov function to be necessary for the ISS property, they are more convenient than the conditions in Definition 3.2 to be checked. For instance, for the system

$$\dot{x}(t) = -\frac{x^3(t)+x(t)u(t)}{1+|t|}, \tag{21}$$

the function $V(\xi) = \xi^2/2$ is such a Lyapunov function, since

$$V(\xi)\ge 2\,|\mu| \;\Rightarrow\; DV(\xi)f(t,\xi,\mu) \le -\frac{\xi^4}{2(1+|t|)};$$

and $\int_0^\infty \frac{1}{1+s}\,ds = \infty$. Consequently, one concludes that the system is semi-uniform ISS. □

Applying Proposition 3.3 to systems without input signals, we get the following sufficient result for time varying stability properties.

Corollary 3.5. Consider a system as (8). The system is GAS if there exists a C^1 function $V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}_{\ge 0}$ satisfying the following:

(a) there exist some $\alpha_1\in\mathcal{K}$ and some $\alpha_2\in\mathcal{K}$ such that (17) holds;
(b) there exist some $\alpha_3\in\mathcal{K}$ and some $\kappa\in\mathcal{N}$ such that

$$D_tV(t,\xi)+D_\xi V(t,\xi)f(t,\xi) \le -\frac{\alpha_3(V(t,\xi))}{1+\kappa(t)};$$

(c) $\displaystyle\int_{t_0}^{\infty} \frac{1}{1+\kappa(s)}\,ds = \infty$ for some t_0.

Moreover, the system is semi-uniform GAS if there exists a function V as in the above satisfying conditions (a)-(c) with (a) strengthened to the existence of $\alpha_1\in\mathcal{K}_\infty$ and $\alpha_2\in\mathcal{K}_\infty$ such that (14) holds. □

4. ASYMPTOTIC CHARACTERIZATIONS OF THE SEMI-UNIFORM ISS PROPERTY

As in the time invariant case, the ISS property for a time varying system implies that the system satisfies the *local stability* (LS) and the *asymptotic gain* (AG) properties, that is,

• there exist $\sigma\in\mathcal{K}$, $\rho\in\mathcal{N}$ and $\delta>0$ such that for all $|\xi|\le\delta$ and all $\|u\|\le\delta$,

$$\text{(LS)}\quad |x(t)| \le \max\{\sigma(\rho(t_0)\,|\xi|),\sigma(\|u\|)\}\ \forall\, t\ge t_0,$$

where $x(t)=x(t,\xi,t_0,u)$; and

• for some $\gamma\in\mathcal{K}$,

$$\text{(AG)}\qquad \overline{\lim_{t\to\infty}}\,|x(t,\xi,t_0,u)| \le \gamma(\|u\|).$$

It can also be seen that if a system is semi-uniform ISS, then it satisfies the *uniform local stability* (ULS) property, that is,

• there exist $\sigma\in\mathcal{K}$ and $\delta>0$ such that for all $|\xi|\le\delta$ and all $\|u\|\le\delta$,

$$\text{(ULS)}\ \ |x(t)| \le \max\{\sigma(|\xi|),\sigma(\|u\|)\}\ \forall\, t\ge t_0,$$

where $x(t) = x(t,\xi,t_0,u)$. Observe that in the time invariant case, the LS and the ULS properties coincide.

A separation principle was established in (Sontag and Wang, 1996) which states that the time invariant ISS property is equivalent to the conjunction of the LS and the AG properties. This result was later generalized to the time invariant output Lagrange input-to-output stability in (Ingalls *et al.*, 2001). Based on (Ingalls *et al.*, 2001, Theorem 1) and Proposition 2.3 of the current paper, we get the following separation principle for the semi-uniform ISS property:

Theorem 2. The system (1) is semi-uniform ISS if and only if it is ULS and AG. ∎

It is natural to ask if the separation principle in the time invariant case can be generalized to the time varying case, that is, if the time varying notion of ISS is equivalent to the conjunction of LS and AG. The answer is still not clear to the authors at this stage.

Applying Theorem 2 to the system (8), we get the following. Though the result appears to be natural, we have not been able to find it in the past literature (note here that the semi-uniform GAS property was defined in terms of (11) for some $\beta\in\mathcal{KL}$ and $\rho\in\mathcal{K}$).

Corollary 4.1. The system (8) is semi-uniform GAS if and only if the followings hold:

(1) (local uniform stability) for any $\varepsilon>0$, there exists $\delta>0$ such that $|x(t,\xi,t_0)|<\varepsilon$ for all $t\ge t_0$ whenever $|\xi|<\delta$;
(2) (attractivity) $\lim_{t\to\infty} x(t,\xi,t_0)=0$ for all ξ and all t_0. ∎

5. A SMALL GAIN THEOREM

An important application of the time invariant notion of ISS is the small gain theorems for the stability analysis for inter-connected systems. The first small gain theorem in the ISS-framework was established in (Jiang *et al.*, 1994). In conjunction with the results in (Sontag and Wang, 1996), a much simplified proof of the small gain theorem was provided in (Coron *et al.*, 1995). In (Jiang *et al.*, 1996), a small gain theorem was provided in terms of ISS-Lyapunov functions. In (Ingalls and Sontag, 2002), the authors presented an ISS-type small gain theorem in terms of operators that recovers the case of state space form. In (Jiang and Wang, 2003), a small gain theorem was generalized to deal with the output-Lagrange input-to-output stability. In the recent work (Karafyllis and Tsinias, 2004), a small gain theorem was proved for the weighted-ISS property for time varying systems followed by an interesting application of smooth output time-varying feedback stabilization for time invariant systems. In this section, we will obtain a small gain theorem in terms of semi-uniform ISS for inter-connected time varying systems, which follows readily from the small gain theorem in terms of time invariant output-Lagrange input-to-output stability obtained in (Jiang and Wang, 2003).

Consider an inter-connected time varying system

$$\begin{aligned} \dot{x}_1(t) &= f_1(t, x_1(t), v_1(t), u_1(t)), \\ \dot{x}_2(t) &= f_2(t, x_2(t), v_2(t), u_2(t)), \end{aligned} \tag{22}$$

subject to the inter-connection constraints

$$v_1(t) = x_2(t), \quad v_2(t) = x_1(t), \tag{23}$$

where for $i = 1, 2$, $x_i(t) \in \mathbb{R}^{n_i}, u_i(t) \in \mathbb{R}^{m_i}$, and where f_i is C^1.

Theorem 3. Suppose both of the subsystems in (22) are semi-uniform ISS, and thus, there exist some $\beta_i \in \mathcal{KL}$, and some $\mathcal{K}$-functions κ_i γ_i^x, γ_i^u, $(i = 1, 2)$, such that the following holds for all $t \geq t_0$:

$$\begin{aligned} |x_1(t)| &\leq \max\left\{\beta_1\left(|x_1(t_0)|,\ \frac{t-t_0}{1+\kappa_1(|t_0|)}\right),\right. \\ &\qquad\qquad \left.\gamma_1^x(\|v_1\|),\ \gamma_1^u(\|u_1\|)\right\}, \\ |x_2(t)| &\leq \max\left\{\beta_2\left(|x_2(t_0)|,\ \frac{t-t_0}{1+\kappa_2(|t_0|)}\right),\right. \\ &\qquad\qquad \left.\gamma_2^x(\|v_2\|),\ \gamma_2^u(\|u_2\|)\right\}; \end{aligned} \tag{24}$$

where we have used $x_i(t)$ to denote $x_i(t, \xi_i, v_i, u_i)$. If the small gain condition $\gamma_1^x \circ \gamma_2^x(s) < s$ hold for all $s > 0$, then the inter-connected system (22)-(23) is semi-uniformly ISS with (u_1, u_2) as the inputs. ∎

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