THE POSSIBILISTIC FILTER: AN ALTERNATIVE APPROACH FOR STATE ESTIMATION

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Abstract: A new method to implement fuzzy Kalman filters is introduced in this paper. This has special application in fields where inaccurate models or sensors are involved, such as in mobile robotics. The innovation consists in using possibility distributions, instead of gaussian distributions. The main advantage of this approach is that uncertainty is not needed to be symmetric, while a region of possible solutions is allowed. The contribution of this work also includes a method to propagate uncertainty through both the process and the observation models. This one is based on quantifying uncertainty as trapezoidal possibility distributions. Finally, an example of a mobile robot during a localization process using landmarks is shown. *Copyright ©2005 IFAC*

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1. INTRODUCTION

Mobile robots localization is traditionally carried out using probabilistic techniques. The well known Extended Kalman Filter (EKF) provides an accurate solution to mobile robots localization. Apparently, the only condition is to initiate appropriately uncertainty matrices of the initial state estimation $\mathbf{P}(0|0)$, the process model $\mathbf{Q}(k+1)$ and the measurement model $\mathbf{R}(k+1)$.

Nevertheless, localization is done by combining incoming measures with an accurate map of the environment. This implies three main inherent problems to the EKF. First, initial maps of the environment are usually fuzzy. Second, measure uncertainty is not gaussian and, indeed, is not symmetric. Third, uncertainty propagation through non-linear equations produce accumulative errors, which have demonstrated to become important when the robot moves hundreds of meters.

At least the two first problems suggest the idea of representing uncertainty by using fuzzy logic. A novel Fuzzy Kalman filter (FKF) is presented in this paper.

Many authors propose to combine fuzzy logic and the Kalman estimation process. (L. Jetto and Venturini, 1999) and (K. Kobayashi and Watanabe, 1995) present a probabilistic EKF for mobile robots, where fuzzy rules are used to adapt uncertainty matrices $\mathbf{Q}(k)$ and $\mathbf{R}(k+1)$. (D. Longo and Sacco, 2002) applies fuzzy rules combined with a Kalman filter for mobile robots localization, but fuzzy logic is only used for sensor fusion outside the state estimation process. (Layne and Passino, 2001) goes further and uses fuzzy relations to represent both the observation model and the system model. Nevertheless, noise is white. (Trajanoski and Wach, 1996) presents a fuzzy filter for a glucoregulatory system that has better per-

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formance that a conventional EKF, but still includes probability. Finally, (Wan and van der Merwe, 2000) presents the unscented Kalman filter as a method to minimize the inconsistency problem of the EKF. This method does not use fuzzy logic, and proposes the use of higher order probabilistic measures.

While existent works on FKF focus on using fuzzy rules and fuzzy relations, we propose to include fuzzy logic in variables modelling (Oussalah and Schutter, 2000). This implies that gaussian probability distributions are replaced by possibility distributions. (Zadeh, 1978) introduces what a possibility distribution is and states the probability / possibility consistency principle. (Dubois and Prade, 1988) formalizes management of possibility distributions using sup-min rules. These operators are replaced in this work to facilitate the implementation of the Kalman filter. Additionally, the expected value of a possibility distribution is defined in (C. Carlsson and Majlender, 2003), but we redefine their concept of variance matrix.

In our work, possibility distributions are considered at several α-cuts (in fact only $\alpha = \{0, 1\}$ are taken into account). This means that many operations can be done using interval arithmetic (L. Jaulin and Walter, 2001). Along this article we show how the three problems stated above disappear or are reduced with this new approach, maintaining at least the same accuracy of the localization process.

2. UNCERTAINTY REPRESENTATION

2.1 Fuzzy variables

In many applications, information about variables behaviour is possibilistic rather than probabilistic. Furthermore, a person is able to reason using qualitative criteria instead of highly precise information. Humans are able to make estimations in the presence of uncertainty, and so an algorithm should be.

In order to use aqualitative estimation algorithm, the possibilistic representation of the uncertainty is recalled. A fuzzy variable defined over the universe of discourse X is said to be modelled by a possibility distribution such as

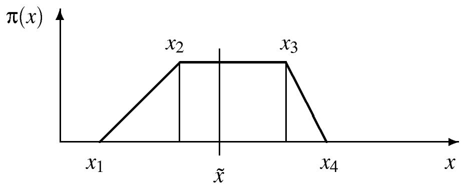

Fig. 1. Possibility distribution

$$\pi(x) = \begin{cases} 1, \; \forall x \in [x_2, x_3] \\ 0, \; \forall x \notin [x_1, x_4] \end{cases} \tag{1}$$

where $[x_2, x_3]$ is the possible region and $[x_1, x_4]$ is the not impossible region, being $x_1 \le x_2 \le x_3 \le x_4$. We represent the possibility distribution by the expectaion of x

$$E[x] \sim \Pi(x_1, x_2, x_3, x_4) \tag{2}$$

The function may be, in general, non-symmetric, which is the usual case of a sensor uncertainty, for example.

2.2 Uncertainty measures

If we define the area of the distribution as

$$\alpha_x = \int_X \pi(x)dx \tag{3}$$

the central value

$$C[f(x)] = \frac{\int_X f(x)\; \pi(x)dx}{\alpha_x} \tag{4}$$

and the center of gravity as

$$\tilde{x} = C[x] = \frac{\int_X x\; \pi(x)dx}{\alpha_x} \tag{5}$$

which can be considered an average between the most possible value and the less impossible value, then we define the uncertainty of x as

$$\begin{aligned} Unc[x] = C[(x-\tilde{x})^2] &= \frac{\int_X (x-\tilde{x})^2 \pi(x)dx}{\alpha_x} \\ &= C[x^2] - \tilde{x}^2 \end{aligned} \tag{6}$$

with

$$\begin{aligned} C[x-\tilde{x}] &= \frac{\int_X (x-\tilde{x})\pi(x)dx}{\alpha_x} \\ &= \frac{\int_X x\pi(x)dx}{\alpha_x} - \tilde{x} = 0 \end{aligned} \tag{7}$$

2.3 Multivariable systems

In the case of a multivariable system, with x and y defined over the universes of discourse X and Y, respectively, a joint possibility distribution $\pi(x,y)$ is used with

$$sup\; \pi(x,y) = 1, \quad \forall x,y \tag{8}$$

which satisfies that its volume is

$$\alpha_{x,y} = \int_X \int_Y \pi(x,y)dxdy \tag{9}$$

The marginal distributions verify that

$$\frac{\pi(x)}{\alpha_x} = \frac{\int_Y \pi(x,y)dy}{\alpha_{x,y}} \tag{10}$$

$$\frac{\pi(y)}{\alpha_y} = \frac{\int_X \pi(x,y)dx}{\alpha_{x,y}} \tag{11}$$

being $E[x] \sim \Pi(x_1,x_2,x_3,x_4)$ and $E[y] \sim \Pi(y_1,y_2,y_3,y_4)$ respectively.

The dependency between both fuzzy variables is given by

$$\begin{aligned} Dep[x,y] &= C[(x-\tilde{x})(y-\tilde{y})] \\ &= \frac{\int_X \int_Y (x-\tilde{x})(y-\tilde{y})\pi(x,y)dxdy}{\alpha_{x,y}} \end{aligned} \tag{12}$$

and the correlation index is

$$\eta(x,y) = \frac{Dep[x,y]}{\sqrt{Unc[x]Unc[y]}} \tag{13}$$

When x and y are independent,

$$\pi(x,y) = \pi(x)\pi(y) \tag{14}$$

this means

$$\alpha_{x,y} = \int_X \pi(x)dx \int_Y \pi(y)dy = \alpha_x \alpha_y \tag{15}$$

$$Dep[x,y] = 0 \tag{16}$$

and

$$\eta(x,y) = 0 \tag{17}$$

If x and y are fully correlated, i.e., $y = ax + b$

$$\alpha_{x,y} = 0 \tag{18}$$

$$Dep[x,y] = \frac{a}{|a|}\sqrt{Unc[x]Unc[y]} \tag{19}$$

and

$$\eta(x,y) = \pm 1 \tag{20}$$

Then, the following array notation may be used:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{21}$$

with $E[\mathbf{x}] \sim \Pi(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)$ and uncertainty matrix

$$\begin{aligned} Unc[\mathbf{x}] &= C[(\mathbf{x}-\tilde{\mathbf{x}})(\mathbf{x}-\tilde{\mathbf{x}})^T] \\ &= \begin{bmatrix} Unc[x] & Dep[x,y] \\ Dep[x,y] & Unc[x] \end{bmatrix} \end{aligned} \tag{22}$$

3. UNCERTAINTY PROPAGATION

Uncertainty is propagated as follows. Suppose the affine relation given by $z = ax + b$. Then,

$$\pi(z) = \pi(x), \ \forall x,y,z \mid z = ax + b \tag{23}$$

This implies that

$$\forall l \in \{1,\dots,4\}, \quad z_l = \begin{cases} a\,x_l + b & if a > 0 \\ a\,x_{5-l} + b & if a < 0 \end{cases} \tag{24}$$

$$Unc[z] = a^2\, Unc[x] \tag{25}$$

Note that this problem (the sign of a) does not appear in the probabilistic case, because gaussian functions are symmetric. Anyway, the shape is maintained, instead of the area. As we can see, this is the main difference with respect to the probabilistic case.

Table 1. Distribution functions.

	Probability	**Possibility**
Area	always 1	changes
Shape	symmetric	asymmetric

On the other hand, if $z = x + y$,

$$z_l = x_l + y_l \tag{26}$$

and

$$\begin{aligned} Unc[z] &= Unc[x] + Unc[y] \\ &+ 2\eta_{x,y}\sqrt{Unc[x]Unc[y]} \end{aligned} \tag{27}$$

although the trapezoidal shape is not longer maintained. And finally, when $\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{b}$,

$$\mathbf{z}_l = \mathbf{A}\mathbf{x}_l + \mathbf{b} \tag{28}$$

and

$$Unc[z] = \mathbf{A}\; Unc[x]\; \mathbf{A}^T \tag{29}$$

4. DYNAMIC ESTIMATION

4.1 The Kalman filter

Let there be now the process and observation models

$$\begin{aligned} \mathbf{x}(k+1) &= \phi(k)\mathbf{x}(k) \\ &+ \mathbf{G}(k+1)\mathbf{u}(k+1) + \mathbf{v}(k+1) \end{aligned} \tag{30}$$

$$\mathbf{z}(k+1) = \mathbf{H}(k+1)\mathbf{x}(k+1) + \mathbf{w}(k+1) \tag{31}$$

with possibility distributions

$$E[\mathbf{w}(k+1)] \sim \Pi(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_4) \text{ (measure noise)} \tag{32}$$

$$E[\mathbf{v}(k+1)] \sim \Pi(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4) \quad \text{(process noise)} \tag{33}$$

$$E[\tilde{\mathbf{x}}(k|k)] \sim \Pi(\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2,\tilde{\mathbf{x}}_3,\tilde{\mathbf{x}}_4) \quad \text{(state estimation)} \tag{34}$$

with $\tilde{\mathbf{w}}(k+1) = \mathbf{0}$, $\tilde{\mathbf{v}}(k+1) = \mathbf{0}$ center of gravities, $\mathbf{x}(k) \in [\tilde{\mathbf{x}}_2, \tilde{\mathbf{x}}_3]$ and $\mathbf{R}(k+1) = Unc[\mathbf{w}(k+1)]$, $\mathbf{Q}(k+1) = Unc[\mathbf{v}(k+1)]$, $\mathbf{P}(k|k) = Unc[\tilde{\mathbf{x}}(k|k)]$ uncertainty matrices. $\mathbf{u}(k+1)$ is a non fuzzy variable.

The Kalman filter steps follow:

a) Prediction:

$$\tilde{\mathbf{x}}_l(k+1|k) = \phi(k)\tilde{\mathbf{x}}_l(k|k) + \mathbf{G}(k+1)\mathbf{u}(k+1) + \mathbf{v}_l(k+1) \quad (35)$$

$$\mathbf{P}(k+1|k) = \phi(k)\mathbf{P}(k|k)\phi^T(k) + \mathbf{Q}(k+1) \quad (36)$$

$$\tilde{\mathbf{z}}_l(k+1) = \mathbf{H}(k+1)\tilde{\mathbf{x}}_l(k+1|k) + \mathbf{w}_l(k+1) \quad (37)$$

$$\mathbf{S}(k+1) = \mathbf{H}(k+1)\mathbf{P}(k+1|k)\mathbf{H}^T(k+1) + \mathbf{R}(k+1) \quad (38)$$

b) Observation:

$$\mathbf{z}(k+1) \quad (39)$$

with

$$\mathbf{z}_l(k+1) = \mathbf{z}(k+1), \quad \forall i \in \{1,\ldots,4\} \quad (40)$$

c) Matching:

A possibilistic criteriun to accept or reject the observation is

$$\pi(\mathbf{z}(k+1)) \geq \alpha \quad (41)$$

with α a confidence value.

d) Correction:

It can be proved that the fuzzy Kalman gain $\mathbf{W}(k+1)$ that minimizes the uncertainty $\mathbf{P}(k+1|k+1)$ is the same than in the probabilistic case:

$$\mathbf{W}(k+1) = \mathbf{P}(k+1|k)\mathbf{H}^T(k+1)\mathbf{S}^{-1}(k+1) \quad (42)$$

$$\tilde{\mathbf{x}}_l(k+1|k+1) = [I - \mathbf{W}(k+1)\mathbf{H}(k+1)]\tilde{\mathbf{x}}_l(k+1|k) + + \mathbf{W}(k+1)\mathbf{z}_l(k+1) \quad (43)$$

$$\mathbf{P}(k+1|k+1) = \mathbf{P}(k+1|k) - \mathbf{W}(k+1)\mathbf{S}(k+1)\mathbf{W}^T(k+1) \quad (44)$$

4.2 *Example*

Let there a mobile robot located at

$$E[\tilde{x}(4|4)] \sim \Pi(0.173, 0.373, 0.673, 0.773) \quad (45)$$

$$E[\tilde{y}(4|4)] \sim \Pi(0.283, 0.483, 0.783, 0.883) \quad (46)$$

$$\mathbf{P}(4|4) = \begin{bmatrix} 0.019 & 0 \\ 0 & 0.019 \end{bmatrix} \quad (47)$$

If we suppose again that the robot is completely stopped, i.e., $\mathbf{u}(5) = 0$, $\mathbf{Q}(5) = \mathbf{0}$ and $\phi(4) = \mathbf{I}$,

$$E[\mathbf{v}(k)] \sim \Pi(0,0,0,0) \quad (48)$$

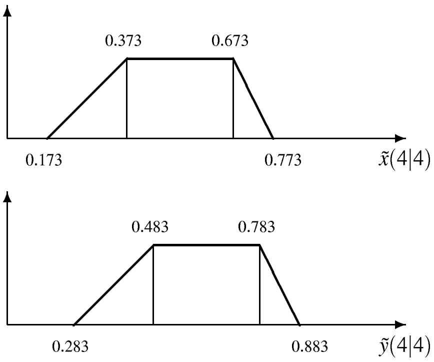

Fig. 2. Possibility distributions at (4|4)

and that the sensor model is given by

$$\mathbf{H}(5) = [0 \;\; 1] \quad (49)$$

$$E[w(5)] \sim \Pi(-0.233, -0.033, -0.033, 0.267) \quad (50)$$

$$\mathbf{R}(5) = 0.011 \quad (51)$$

and that the observation at $k = 5$ is 0.498, then the estimation at this instant follows:

a) Prediction:

$$E[\tilde{x}(5|4)] \sim \Pi(0.173, 0.373, 0.673, 0.773) \quad (52)$$

$$E[\tilde{y}(5|4)] \sim \Pi(0.283, 0.483, 0.783, 0.883) \quad (53)$$

$$\mathbf{P}(5|4) = \begin{bmatrix} 0.019 & 0 \\ 0 & 0.019 \end{bmatrix} \quad (54)$$

$$\tilde{z}_l(5) = H(5)\tilde{x}_l(5|4) + w_l(5) \quad (55)$$

$$E[\tilde{z}(5)] \sim \Pi(0.05, 0.45, 0.75, 1.15) \quad (56)$$

$$\mathbf{S}(5) = \mathbf{H}(5)\mathbf{P}(5|4)\mathbf{H}^T(5) + \mathbf{R}(5) = 0.030 \quad (57)$$

b) Observation:

$$z(5) = 0.498 \quad (58)$$

$$E[z(5)] \sim \Pi(0.498, 0.498, 0.498, 0.498) \quad (59)$$

c) Matching:

$$\pi(0.498) = 1 \quad (60)$$

d) Correction:

$$\mathbf{W}(5) = \mathbf{P}(5|4)\mathbf{H}^T(5)\mathbf{S}^{-1}(5) = \begin{bmatrix} 0 \\ 0.633 \end{bmatrix} \quad (61)$$

$$\tilde{\mathbf{x}}_l(5|5) = [I - \mathbf{W}(5)\mathbf{H}(5)]\tilde{\mathbf{x}}_l(5|4) + \mathbf{W}(5)\mathbf{z}_l(5) \quad (62)$$

$$E[\tilde{x}(5|5)] \sim \Pi(0.173, 0.373, 0.673, 0.773) \quad (63)$$

$$E[\tilde{y}(5|5)] \sim \Pi(0.419, 0.492, 0.603, 0.639) \quad (64)$$

$$\mathbf{P}(5|5) = \mathbf{P}(5|4) - \mathbf{W}(5)\mathbf{S}(5)\mathbf{W}^T(5) = \begin{bmatrix} 0.019 & 0 \\ 0 & 0.007 \end{bmatrix} \quad (65)$$

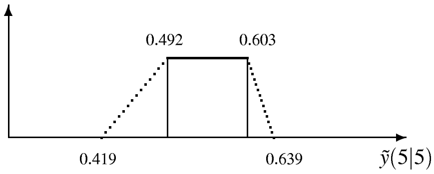

Fig. 3. Possibility distribution at (5|5)

It can be seen that the uncertainty of $\tilde{y}$ also decreases, although the trapezoidal shape is not maintained. The example shows that, although the accuracy is not necessary to navigate, it can be also achieved using fuzzy logic.

5. NON-LINEAR ESTIMATION

5.1 The extended possibilistic filter

The probabilistic estimation has two error sources, originated by the linearization. The first one affects the state expectation, which is an approximation to force to obtain a symmetric (gaussian) projection. The second one affects the uncertainty matrices, which are calculated through the jacobian matrices of the non-linear functions, and which directly affects the Kalman gain calculation,

$$\mathbf{W}(k+1) = \mathbf{P}(k+1|k)\mathbf{H}^T(k+1)\mathbf{S}^{-1}(k+1) \quad (66)$$

which uses those uncertainty matrices and is used to correct the state prediction. These errors can lead to inconsistencies in the filter when $k \to \infty$, a phenomena that appears, for example, when a mobile robot navigates for hundreds of meters.

In the case of the proposed FKF, the first problem, i.e., the error due to projection of the center of gravity is minimised by projecting the four points of the trapezoidal funcion:

$$\tilde{\mathbf{x}}_l(k+1|k) = \mathbf{f}(k+1, \tilde{\mathbf{x}}_l(k|k), \mathbf{u}(k+1)) + \mathbf{v}_l(k+1) \quad (67)$$

$$\tilde{\mathbf{z}}_l(k+1) = \mathbf{h}(k+1, \tilde{\mathbf{x}}_l(k+1|k)) + \mathbf{w}_l(k+1) \quad (68)$$

$$\nu_l(k+1) = \mathbf{z}(k+1) - \tilde{\mathbf{z}}_l(k+1) \quad (69)$$

$$\tilde{\mathbf{x}}_l(k+1|k+1) = \tilde{\mathbf{x}}_l(k+1|k) + \mathbf{W}(k+1)\nu_l(k+1) \quad (70)$$

so

$$\tilde{z} \neq h(\tilde{x}) \quad (71)$$

It must be taken into account that this phylosophy of uncertainty propagation can not be applied to the probabilistic Kalman filter, because its distributions are gaussian and so, they must always keep their symmetric shape.

The second problem, i.e. the error due to the uncertainty matrices, also appears as we are suppossing trapezoidal shape,

$$\alpha_z \neq h'(\tilde{x})\, \alpha_x \quad (72)$$

$$Unc[z] \neq [h'(\tilde{x})]^2\, Unc[x] \quad (73)$$

which has also a small effect on the Kalman gain calculation.

5.2 Application to the localization of a mobile robot

The following experiments reflects a mobile robot following a straight line, and localizing itself by using one landmark. Figure 4 represents the evolution of the possibility distributions of the estimation components $\tilde{x}$, $\tilde{y}$, $\tilde{\theta}$. Each graph shows the corners of the trapezoidal distribution. The probabilistic estimation should always be inside the possible region.

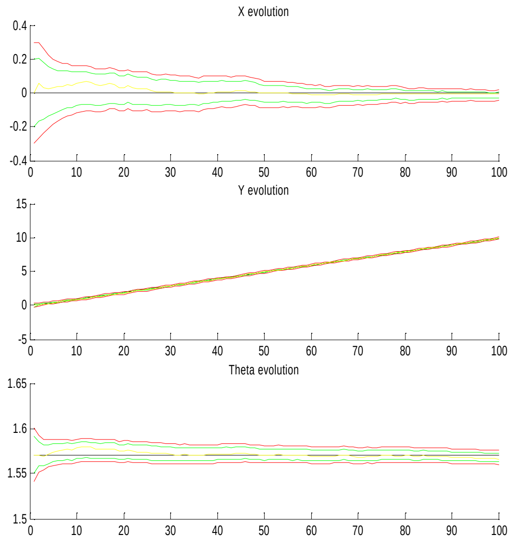

Fig. 4. Possibility distributions evolution

Figure 5 shows the evolution of the uncertainty of the estimation components, $Unc[\tilde{x}]$, $Unc[\tilde{y}]$ and $Unc[\tilde{\theta}]$.

Finally, 6 reflects a more complex experiment in which a b21r robot moves inside a room, using three landmarks and three walls represented as segments.

The figure shows both the odometry and the estimated paths. Uncertainty in (x, y) is shown as a set of rectangles which apply for the possible and not impossible regions, respectively.

6. CONCLUSION

A new FKF has been introduced which is a classic EKF able to manage possibilistic uncertainty. Its main characteristic is that fuzzy logic is included in the uncertainty model of the state estimation, instead of using fuzzy rules for process and observation models, as other authors do. Uncertainty propagation through those models has been explained, and proved to be reasonable even in the non linear case.

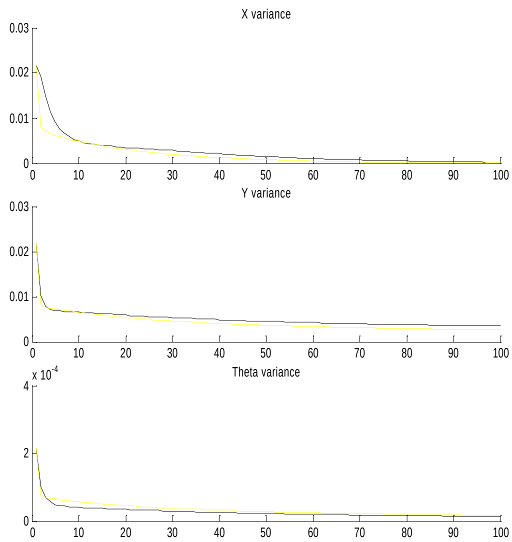

Fig. 5. Estimation uncertainty evolution

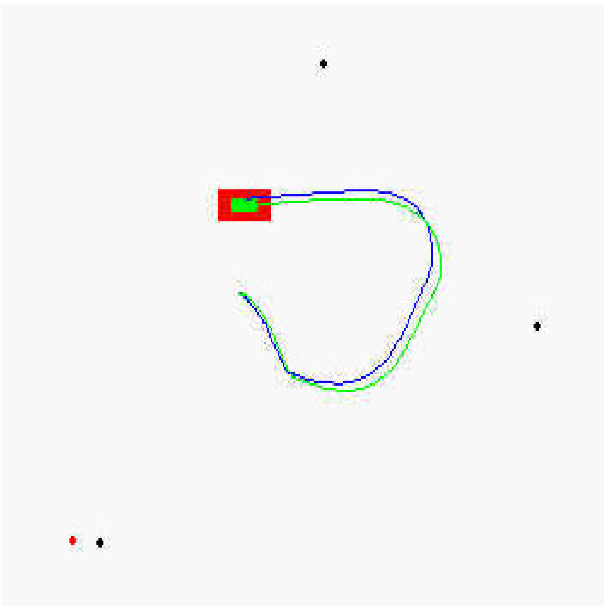

Fig. 6. Navigation with landmarks and segments

The main differences between the probabilistic and the possibilistic approaches are illustrated in table 2.

This means that the FKF has more sense in applications where uncertainty is managed in a qualitative manner, as in navigation issues in mobile robotics. Furthermore, the selection of one method does not excludes the other. In fact, the higher advantage of the presented ideas is that both types of models may be applied in parallel.

Table 2. Comparison between approaches.

Gaussian	**Fuzzy**
Observation noise is estimated by experimentation	Observation noise is estimated by approximation
An accurate model is needed, or a lot of measures will be rejected	Allows higher model uncertainty that will be corrected later
Strict: rejects smaller errors	It allows bigger errors
It only accepts probable data	It only rejects impossible data
It must start with an accurate estimation	By default it works with uncertain estimations

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