LQ OPTIMAL CONTROL SYNTHESIS FOR A CLASS OF PULSE MODULATED SYSTEMS

H. Fujioka * **C.-Y. Kao** **,2 **S. Almér** ***,3 **U. Jönsson** ***,3

* *Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan*
** *Department of Electrical and Electronic Engineering University of Melbourne,Parkville 3010,Victoria, Australia*
*** *Optimization and Systems Theory, Royal Institute of Technology, 10044 Stockholm, Sweden*

Abstract: We consider linear quadratic optimal control for a class of pulse-width-modulated systems. The problem is motivated from a practical application – digital control of switching power converters. The control synthesis problem is posed based on a sampled data model of the original switching dynamics and a linear quadratic criterion that takes the inter sampling behavior into account.

Keywords: Pulse width modulation, Sampled-data systems, LQ optimal control

1. INTRODUCTION

We consider linear quadratic control of a class of pulse-width modulated systems (PWM). The system consist of two affine vector fields that are periodically switched in a given order. The only control variable is the duty ratio which determines the fraction of the period time in which each of the two dynamics is active. This assumption on the switching is more restrictive than what is normally considered for switched dynamical systems but is nevertheless important from a practical point of view since it appears in digital control of power converters (Kassakian *et al.*, 1991). Feedback controllers for such power converters are often designed based on the averaged dynamics. This is not suitable if the switching frequency is low as is the case in many high power applications.

In this paper we consider a two step design procedure. In the first step we design the stationary duty ratio such that the ripple of the steady state output has as small power as possible. In the second step we design a dynamic output feedback controller based on a sampled data model involving the lifted dynamics and a linear quadratic criterion that takes the inter sampling behavior into account, see (Chen and Francis, 1995) and the references therein. The corresponding optimal control problem has a linear quadratic structure but with the duty ratio appearing as a parameter in the system matrices and the cost function. This makes the design highly nonlinear and we simplify by considering a linearization of the cost function and the dynamics. A region of attraction can easily be estimated due to the special structure of the sampled data model. We apply the proposed method to a step down DC-DC converter.

There have recently been several works that apply new results for switched dynamical systems to the design and analysis of power converters (Geyer *et al.*, 2004; Lincoln and Rantzer, 2002; Rubensson and Lennartsson, 2002). Our model distinguishes itself from the ones considered for design in (Lincoln and Rantzer, 2002; Geyer *et al.*, 2004) because we only consider sampling of the system output at the rate of the switching frequency.

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For complementary details, readers are referred to (Fujioka *et al.*, 2004).

Notation

For a given continuous-time signal f, a discrete-time signal defined by ideal sampling of f will be denoted by $\bar{f}$:

$$\bar{f}_k = f(kT)$$

where $T > 0$ is the sampling period. $\hat{f}$ denotes the lifted signal of f:

$$\hat{f}_k(\theta) = f(kT + \theta), \quad \theta \in [0,\, T).$$

We denote the finite observability Grammian by

$$\Xi\,(A,\, C,\, T) := \int_0^T \mathrm{e}^{A' t} C' C \mathrm{e}^{At}\, \mathrm{d}t.$$

2. PROBLEM FORMULATION

In this section, we formulate the control synthesis problem considered in this paper.

2.1 System Description

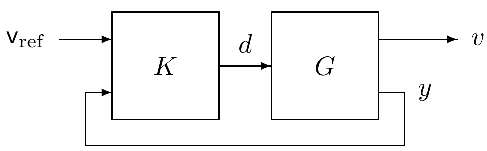

Fig. 1. Feedback Control System

Consider a feedback system depicted in Fig. 1. Here G is a switching system governed by

$$\begin{aligned} \dot{x}(t) &= \begin{cases} A_1 x(t) + B_1, & t \in [kT,\, (k + d_k)T) \\ A_2 x(t) + B_2, & t \in [(k + d_k)T,\, (k + 1)T) \end{cases} \\ v(t) &= C_1 x(t) \\ y_k &= C_2 x(kT) \end{aligned} \tag{1}$$

where $T > 0$ is the sampling period. The duty ratio d is a discrete-time signal satisfying

$$d_k \in [0,\, 1] \tag{2}$$

for any k. v is a continuous-time scalar output signal which will be regulated. y is a discrete-time measurement output signal. Finally, K is a discrete-time controller which determines d from y and the reference $\mathsf{v}_{\mathrm{ref}} \in \mathbb{R}$. We present the controller in Section 5.

The lifting representation of (1) is given by

$$\begin{cases} \bar{x}_{k+1} = \Phi_{d_k} \bar{x}_k + \Upsilon_{d_k} \\ \hat{v}_k(\theta) = \Psi_{d_k}(\theta) \bar{x}_k + \Lambda_{d_k}(\theta) \\ \quad y_k = M \bar{x}_k \end{cases} \tag{3}$$

where $\Phi_{\mathsf{d}} \in \mathbb{R}^{n\times n}$, $\Upsilon_{\mathsf{d}} \in \mathbb{R}^{n\times 1}$, Ψ_{d}: $[0,\, T] \to \mathbb{R}^{1\times n}$ and Λ_{d}: $[0,\, T] \to \mathbb{R}$ are defined with a parameter $\mathsf{d} \in [0,\, 1]$ by

$$[\, \Phi_{\mathsf{d}}\ \ \Upsilon_{\mathsf{d}} \,] := [\, I_n\ \ 0 \,]\, \Omega_{\mathsf{d}}(T),$$

$$[\, \Psi_{\mathsf{d}}(\theta)\ \ \Lambda_{\mathsf{d}}(\theta) \,] := \check{C} \Omega_{\mathsf{d}}(\theta), \quad \check{C} := [\, C\ \ 0 \,]\,.$$

Ω_{d}: $[0,\, T] \to \mathbb{R}^{(n+1)\times(n+1)}$ is defined by

$$\Omega_{\mathsf{d}}(\theta) := \begin{cases} \mathrm{e}^{\check{A}_1 \theta}, & \theta \in [0,\, \mathsf{d}T] \\ \mathrm{e}^{\check{A}_2(\theta - \mathsf{d}T)} \mathrm{e}^{\check{A}_1 \mathsf{d}T}, & \theta \in [\mathsf{d}T,\, T] \end{cases}$$

where $\check{A}_1$, $\check{A}_2 \in \mathbb{R}^{(n+1)\times(n+1)}$ are given by

$$\check{A}_1 := \begin{bmatrix} A_1 & B_1 \\ 0 & 0 \end{bmatrix}, \quad \check{A}_2 := \begin{bmatrix} A_2 & B_2 \\ 0 & 0 \end{bmatrix}.$$

The derivation of (3) is straightforward by noting that one has the following expressions for x:

$$x(kT + \theta) = \begin{cases} \mathrm{e}^{A_1 \theta} x(kT) + \displaystyle\int_0^\theta \mathrm{e}^{A_1(\theta - \tau)} B_1\, \mathrm{d}\tau, \\ \hfill \theta \in [0,\, d_k T) \\ \mathrm{e}^{A_2(\theta - d_k T)} x(kT + d_k T) \\ \quad + \displaystyle\int_0^{\theta - d_k T} \mathrm{e}^{A_2(\theta - d_k T - \tau)} B_2\, \mathrm{d}\tau, \\ \hfill \theta \in [d_k T,\, T) \end{cases}$$

and the standard formula for matrix exponentials:

$$\mathrm{e}^{\check{A}_i \theta} = \begin{bmatrix} \mathrm{e}^{A_i \theta} & \displaystyle\int_0^\theta \mathrm{e}^{A_i(\theta - \tau)} B_i\, \mathrm{d}\tau \\ 0 & 1 \end{bmatrix}.$$

We assume that the plant (1) attains a periodic solution $x^0(t)$ of period T if the duty ratio d is set to d^0 and the system is initialized by $x(0) = \mathsf{x}^0$ where $\mathsf{x}^0 := x^0(0)$. The periodicity of $x^0(t)$ implies that $\mathsf{x}^0 = \Phi_{\mathsf{d}^0} \mathsf{x}^0 + \Upsilon_{\mathsf{d}^0}$, or equivalently

$$(I - \Omega_{\mathsf{d}^0}(T)) \check{\mathsf{x}}^0 = 0, \quad \text{where } \check{\mathsf{x}}^0 := \begin{bmatrix} \mathsf{x}^0 \\ 1 \end{bmatrix}. \tag{4}$$

The continuous-time output $v(t)$ corresponding to $x^0(t)$, denoted by $v^0(t)$, is periodic. The lifted signal of $v^0(t)$ is given by

$$\hat{v}_k^0(\theta) = \Psi_{\mathsf{d}^0}(\theta) \mathsf{x}^0 + \Theta_{\mathsf{d}^0}(\theta). \tag{5}$$

2.2 Problem Formulation

The objective of our control design is to ensure asymptotic convergence of the solution of (1) to a T-periodic solution $x^0(t)$ in such a way that

(1) the power of the stationary error

$$\begin{aligned} J_s &:= \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau |v^0(t) - \mathsf{v}_{\mathrm{ref}}|^2\, \mathrm{d}t \\ &= \frac{1}{T} \int_0^T |\hat{v}_k^0(\theta) - \mathsf{v}_{\mathrm{ref}}|^2\, \mathrm{d}\theta \end{aligned} \tag{6}$$

is minimized.

(2) the rate of convergence which is measured by the error energy integral

$$\begin{aligned} J &:= \int_0^\infty |v(t) - v^0(t)|^2\, \mathrm{d}t \\ &= \sum_{k=1}^\infty \int_0^T |\hat{v}_k(\theta) - \hat{v}_k^0(\theta)|^2\, \mathrm{d}\theta \end{aligned} \tag{7}$$

is minimized.

In the next section we discuss the first problem while the remaining sections are devoted to the second problem.

3. DESIGN OF STATIONARY DUTY-RATIO

The following proposition provides a computational formula for J_s in (6).

Proposition 1. For given x^0 and d^0 satisfying (4), one has

$$J_s(\mathsf{d}^0, \mathsf{x}^0) = \frac{1}{T}\begin{bmatrix} \check{\mathsf{x}}^0 \\ \mathsf{v}_{\text{ref}} \end{bmatrix}' Q_s(\mathsf{d}^0) \begin{bmatrix} \check{\mathsf{x}}^0 \\ \mathsf{v}_{\text{ref}} \end{bmatrix}$$

where

$$Q_s(\mathsf{d}^0) := \Xi\left(A_{s1},\, C_s,\, \mathsf{d}^0 T\right) + \mathrm{e}^{A_{s1}'\mathsf{d}^0 T}\,\Xi\left(A_{s2},\, C_s,\, (1-\mathsf{d}^0)T\right)\mathrm{e}^{A_{s1}\mathsf{d}^0 T}$$

$$A_{s1} := \begin{bmatrix} \check{A}_1 & 0 \\ 0 & 0 \end{bmatrix},\ A_{s2} := \begin{bmatrix} \check{A}_2 & 0 \\ 0 & 0 \end{bmatrix},\ C_s := \begin{bmatrix} \check{C} & -1 \end{bmatrix}.$$

Using this lemma the minimization of the stationary error can be formulated as

$$\min_{\mathsf{d}^0\in[0,\,1]} J_s^*(\mathsf{d}^0) \tag{8}$$

where

$$J_s^*(\mathsf{d}^0) := \min_{\mathsf{x}^0\in\mathbb{R}^n} J_s(\mathsf{d}^0, \mathsf{x}^0) \quad \text{subj. to } \mathsf{x}^0 = \Phi_{\mathsf{d}^0}\mathsf{x}^0 + \Upsilon_{\mathsf{d}^0} \tag{9}$$

The inner optimization problem is trivial when $I - \Phi_{\mathsf{d}^0}$ is nonsingular for all $\mathsf{d}^0 \in [0,1]$. In this case $J_s^*\colon [0,\,1] \to \mathbb{R}$ is the smooth function

$$J_s^*(\mathsf{d}^0) = J_s(\mathsf{d}^0,\, (I-\Phi_{\mathsf{d}^0})^{-1}\Upsilon_{\mathsf{d}^0}) \tag{10}$$

and the optimization in (8) can be done easily.

We note that non-singularity of $I - \Phi_{\mathsf{d}^0}$ corresponds to the case where the homogeneous state equation obtained, by setting $B_1 = B_2 = 0$ in (1), cannot have a T-periodic solution. This is generically true.

4. ERROR SYSTEM

We next derive a sampled-data model for the error dynamics in (7).

Let the error between x and x^0 be $z := x - x^0$ and let $e := v - v^0$. Using (3), (4) and (5), one can derive a lifted system for the error dynamics:

$$\begin{cases} \bar{z}_{k+1} = \Phi_{d_k}\bar{z}_k + \acute{\Gamma}_{d_k} \\ \hat{e}_k(\theta) = \Psi_{d_k}(\theta)\bar{z}_k + \acute{\Theta}_{d_k}(\theta) \end{cases} \tag{11}$$

where $\acute{\Gamma}_{\mathsf{d}} \in \mathbb{R}^n$ and $\acute{\Theta}_{\mathsf{d}}\colon [0,\,T] \to \mathbb{R}$ are defined by

$$\acute{\Gamma}_{\mathsf{d}} := \begin{bmatrix} I_n & 0 \end{bmatrix}(\Omega_{\mathsf{d}}(T) - \Omega_{\mathsf{d}^0}(T))\check{\mathsf{x}}^0,$$
$$\acute{\Theta}_{\mathsf{d}}(\theta) := \check{C}(\Omega_{\mathsf{d}}(\theta) - \Omega_{\mathsf{d}^0}(\theta))\check{\mathsf{x}}^0.$$

Then the criterion (7) can be computed as in the following lemma:

Lemma 1. The criterion (7) satisfies

$$J = \sum_{k=0}^{\infty}\begin{bmatrix} \bar{z}_k \\ \check{\mathsf{x}}^0 \end{bmatrix}' \check{Q}(d_k)\begin{bmatrix} \bar{z}_k \\ \check{\mathsf{x}}^0 \end{bmatrix} \tag{12}$$

where

$$\check{Q}(\mathsf{d}) := S'\begin{bmatrix} \check{Q}_1(\mathsf{d}) & \check{Q}_2(\mathsf{d}) \\ \check{Q}_2'(\mathsf{d}) & \check{Q}_1(\mathsf{d}^0) \end{bmatrix} S,$$

$$S := \left[\begin{array}{c|cc} I & I & 0 \\ 0 & 0 & 1 \\ \hline 0 & -I & 0 \\ 0 & 0 & -1 \end{array}\right]$$

$$\check{Q}_1(\mathsf{d}) := \Xi\left(\check{A}_1,\, \check{C},\, \mathsf{d}T\right) + \mathrm{e}^{\check{A}_1'\mathsf{d}T}\,\Xi\left(\check{A}_2,\, \check{C},\, (1-\mathsf{d})T\right)\mathrm{e}^{\check{A}_1\mathsf{d}T},$$

$$\check{Q}_2(\mathsf{d}) := \begin{cases} \check{Q}_{2a}(\mathsf{d}), & \text{if } \mathsf{d} \le \mathsf{d}^0 \\ \check{Q}_{2b}(\mathsf{d}), & \text{if } \mathsf{d} \ge \mathsf{d}^0 \end{cases}.$$

and $\check{Q}_{2a}$ and $\check{Q}_{2b}$ are respectively defined by

$$\check{Q}_{2a}(\mathsf{d}) := \Xi\left(\check{A}_1,\, \check{C},\, \mathsf{d}T\right) + \mathrm{e}^{\check{A}_1'\mathsf{d}T}\left(\int_0^{(\mathsf{d}^0-\mathsf{d})T}\mathrm{e}^{\check{A}_2'\theta}\check{C}'\check{C}\mathrm{e}^{\check{A}_1\theta}\,\mathrm{d}\theta\right)\mathrm{e}^{\check{A}_1\mathsf{d}T} + \mathrm{e}^{\check{A}_1'\mathsf{d}T}\mathrm{e}^{\check{A}_2'(\mathsf{d}^0-\mathsf{d})T}\,\Xi\left(\check{A}_2,\, \check{C},\, (1-\mathsf{d}^0)T\right)\mathrm{e}^{\check{A}_1\mathsf{d}^0T},$$

$$\check{Q}_{2b}(\mathsf{d}) := \Xi\left(\check{A}_1,\, \check{C},\, \mathsf{d}^0T\right) + \mathrm{e}^{\check{A}_1'\mathsf{d}^0T}\left(\int_0^{(\mathsf{d}-\mathsf{d}^0)T}\mathrm{e}^{\check{A}_1'\theta}\check{C}'\check{C}\mathrm{e}^{\check{A}_2\theta}\,\mathrm{d}\theta\right)\mathrm{e}^{\check{A}_1\mathsf{d}^0T} + \mathrm{e}^{\check{A}_1'\mathsf{d}T}\,\Xi\left(\check{A}_2,\, \check{C},\, (1-\mathsf{d})T\right)\mathrm{e}^{\check{A}_2'(\mathsf{d}-\mathsf{d}^0)T}\mathrm{e}^{\check{A}_1\mathsf{d}^0T}.$$

5. DESIGN OF FEEDBACK CONTROL

In our control problem we have access to the measurement output $\mathcal{Y}_k = \{y_1,\, \ldots,\, y_k\}$, the reference signal v_{ref}, the stationary duty ratio d^0 and the initial condition z_0. If we let $\mathrm{d}_k := d(\mathcal{Y}_k,\, z_0) \in [0,1]$ be the control policy to be optimized then ideally we would like to consider the optimal control problem

$$\begin{aligned} J(z_0) &= \min_{d(\mathcal{Y}_k,\, z_0)}\sum_{k=0}^{\infty}\int_0^T |\hat{e}_k(\theta)|^2 d\theta \\ &\qquad \text{subj. to } \bar{z}_{k+1} = \Phi_{d_k}\bar{z}_k + \acute{\Gamma}_{d_k} \\ &= \min_{d(\mathcal{Y}_k,\, z_0)}\sum_{k=0}^{\infty}\int_0^T |\Psi_{d_k}(\theta)\bar{z}_k + \acute{\Theta}_{d_k}(\theta)|^2 \\ &\qquad \text{subj. to } \bar{z}_{k+1} = \Phi_{d_k}\bar{z}_k + \acute{\Gamma}_{d_k} \\ &= \min_{d(\mathcal{Y}_k,\, z_0)}\sum_{k=0}^{\infty} L(d_k, \bar{z}_k) \\ &\qquad \text{subj. to } \bar{z}_{k+1} = \Phi_{d_k}\bar{z}_k + \acute{\Gamma}_{d_k} \end{aligned}$$

where

$$L(\mathsf{d},\, \mathsf{z}) := \begin{bmatrix} \mathsf{z} \\ \check{\mathsf{x}}^0 \end{bmatrix}' \check{Q}(\mathsf{d})\begin{bmatrix} \mathsf{z} \\ \check{\mathsf{x}}^0 \end{bmatrix} \tag{13}$$

and $\check{Q}$ is defined in Lemma 1. Despite the nice structure of this optimization problem it is highly nonlinear and the cost function is generally only finite on some neighborhood of the origin where the error model is stabilizable.

In general, the preferable control policy $\mathrm{d}(\mathcal{Y}_k,\, z_0)$ is defined as a dynamical controller of the form

$$\begin{aligned} z^K_{k+1} &= F_K(z^K_k,\, z^i_k,\, y_k,\, \mathsf{v}_{\mathrm{ref}}) \\ d_k &= H_K(z^K_k,\, z^i_k,\, y_k,\, \mathsf{v}_{\mathrm{ref}}) \end{aligned}$$

where F_K and H_K generally are nonlinear functions to be designed. The range of H_K must be $[0,\,1]$ to ensure (2). The integrator state is defined as $z^i_{k+1} = z^i_k + \tilde{M}(y_k - C_2\mathsf{x}^0)$, where $\tilde{M}$ must be selected in such a way that $C_1 = \tilde{M}C_2$. For control design purpose we will let the state of the integrator be included in the plant. The full error state can then be decomposed as $(\bar{z}_k, z^i_k)$. In this case we have the following predefined structure for the closed loop error system.

$$\begin{aligned} \begin{bmatrix} \bar{z}_{k+1} \\ z^i_{k+1} \\ z^K_{k+1} \end{bmatrix} &= \begin{bmatrix} \Phi_{d_k}\bar{z}_k + \acute{\Gamma}_{d_k} \\ z^i_k + C_1\bar{z}_k \\ F_K(z^K_k,\, z^i_k,\, C_2(\bar{z}_k + \mathsf{x}^0),\, \mathsf{v}_{\mathrm{ref}}) \end{bmatrix}, \\ &=: \breve{F}(\bar{z}_k,\, z^i_k,\, z^K_k), \\ d_k &= H_K(z^K_k,\, z^i_k,\, C_2(\bar{z}_k + \mathsf{x}^0),\, \mathsf{v}_{\mathrm{ref}}), \\ &=: \breve{H}(\bar{z}_k,\, z^i_k,\, z^K_k), \\ \hat{e}_k(\theta) &= \Psi_{d_k}(\theta)\bar{z}_k + \acute{\Theta}_{d_k}(\theta) \end{aligned}$$

To the LQ cost in (13) we add a penalty on the integrator state in order to ensure convergence. The complete cost function becomes

$$\sum_{k=0}^{\infty} \breve{L}(d_k,\, \bar{z}_k,\, \bar{z}^i_k)$$

where $\breve{L}(\mathsf{d},\, \mathsf{z},\, \mathsf{z}^i) = L(\mathsf{d},\, \mathsf{z}) + (\mathsf{z}^i)'\hat{Q}\mathsf{z}^i$.

A means of obtaining suboptimal controllers is to jointly search for controller parameters and a value function corresponding to a suboptimal cost function. In other words, we consider a positive definite convex function $V\colon \mathcal{W} \to \mathbb{R}^+$ such that

$$\begin{aligned} V(\mathsf{z},\ \mathsf{z}^i, \mathsf{z}^K) \geq L(\breve{H}(\mathsf{z},\, \mathsf{z}^i\, \mathsf{z}^K),\, \mathsf{z}) + (\mathsf{z}^i)'\hat{Q}\mathsf{z}^i \\ + V(\breve{F}(\mathsf{z},\, \mathsf{z}^i\, \mathsf{z}^K)) \end{aligned} \tag{14}$$

for all $(\mathsf{z},\, \mathsf{z}^i\, \mathsf{z}^K) \in \mathcal{W}$, where $\mathcal{W}$ is an open domain for the approximative value function. A natural optimization problem would be

$$\min_{V,\, F_K,\, H_K}\ \min_{\mathsf{z}^i, \mathsf{z}^K} V(z_0,\, \mathsf{z}^i,\, \mathsf{z}^K) \quad \text{subj. to} \quad (14).$$

The nonlinear nature of the error dynamics and the cost function make the problem intractable in general. We consider a linear quadratic approximation of J.

5.1 *Approximating J*

We proceed with computing the dominating term of J in a small neighborhood around the origin.

Suppose that $\bar{z}_k \approx 0$ and $u_k := d_k - \mathsf{d}^0 \approx 0$. Then it can be shown that $L(d_k,\, \bar{z}_k)$ is dominated by $\tilde{L}(u_k,\, \bar{z}_k)$, where $\tilde{L}$ is defined by

$$L(d_k,\, \bar{z}_k) = \tilde{L}(u_k,\, \bar{z}_k) + O((\bar{z}_k,\, u_k)^3). \tag{15}$$

The following lemma gives an explicit form of $\tilde{L}$:

Lemma 2. $\tilde{L}$ is of the form

$$\begin{bmatrix} \bar{z}_k \\ u_k \end{bmatrix}' Q \begin{bmatrix} \bar{z}_k \\ u_k \end{bmatrix} = \begin{bmatrix} \bar{z}_k \\ u_k \end{bmatrix}' \begin{bmatrix} Q_1 & Q_3 \\ Q_3' & Q_2 \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ u_k \end{bmatrix}, \tag{16}$$

where

$$\begin{aligned} Q_1 &:= \Xi\left(A_1,\, C,\, \mathsf{d}^0 T\right) \\ &\qquad + \mathrm{e}^{A_1'\mathsf{d}^0 T}\,\Xi\left(A_2,\, C,\, (1-\mathsf{d}^0)T\right)\mathrm{e}^{A_1\mathsf{d}^0 T}, \\ Q_2 &:= 2T^2\eta'\,\Xi\left(A_2,\, C,\, (1-\mathsf{d}^0)T\right)\eta, \\ Q_3 &:= T\mathrm{e}^{A_1'\mathsf{d}^0 T}\,\Xi\left(A_2,\, C,\, (1-\mathsf{d}^0)T\right)\eta. \end{aligned}$$

and $\eta \in \mathbb{R}^{n\times 1}$ is defined by

$$\eta := [\, I_n\ \ 0\,]\,(\breve{A}_1 - \breve{A}_2)\mathrm{e}^{\breve{A}_1\mathsf{d}^0 T}\breve{\mathsf{x}}^0$$

Hence, when $(\bar{z}_k,\, u_k) \approx 0$,

$$J \approx \tilde{J} := \sum_{k=0}^{\infty} \tilde{L}(u_k,\, \bar{z}_k), \tag{17}$$

which is a quadratic function of $(\bar{z}_k,\, u_k)$. Consider the linearized error dynamics and this cost function, one arrives at a discrete time LQ optimal control problem.

5.2 *LQ controller for linearized error dynamics with the approximate cost function $\tilde{J}$*

Let the control law be

$$\begin{aligned} z^K_{k+1} &= A_K z^K_k + B_{K1}(y_k - M\mathsf{x}^0) + B_{K2}z^i_k \\ u_k &= C_K z^K_k + D_{K1}(y_k - M\mathsf{x}^0) + D_{K2}z^i_k \end{aligned} \tag{18}$$

where $z^i_{k+1} = z^i_k + \tilde{M}C_2\bar{z}_k = z^i_k + C_1\bar{z}_k$. Around $(\mathsf{x}^0,\, \mathsf{d}^0)$, the linearized error dynamics satisfies

$$\bar{z}_{k+1} = \Phi\bar{z}_k + \Gamma u_k$$

$$y_k - C_2\mathsf{x}^0 = C_2\bar{z}_k$$

where $\Phi := \Phi_{\mathsf{d}^0}$ and

$$\begin{aligned} \Gamma &:= \frac{\partial\acute{\Gamma}_{\mathsf{d}}}{\partial\mathsf{d}}(\mathsf{d}^0) \\ &= T\,[I_n\ \ 0]\,\mathrm{e}^{\breve{A}_2(1-\mathsf{d}^0)T}(\breve{A}_1 - \breve{A}_2)\mathrm{e}^{\breve{A}_1\mathsf{d}^0 T}\breve{\mathsf{x}}^0 \end{aligned}$$

by (11). If we include the integrator dynamics in the plant we obtain the augmented system

$$\left[\begin{array}{c} \bar{z}_{k+1} \\ z^i_{k+1} \\ \hline y_k - C_2\mathsf{x}^0 \\ z^i_k \end{array}\right] = \begin{bmatrix} \Phi_a & \Gamma_a \\ C_{2a} & 0 \end{bmatrix} \left[\begin{array}{c} \bar{z}_k \\ z^i_k \\ \hline u_k \end{array}\right]$$

where

$$\begin{bmatrix} \Phi_a & \Gamma_a \\ C_{2a} & 0 \end{bmatrix} := \left[\begin{array}{cc|c} \Phi & 0 & \Gamma \\ C_1 & I & 0 \\ \hline C_2 & 0 & 0 \\ 0 & I & 0 \end{array}\right]$$

Hence, the closed loop dynamics is

$$\begin{bmatrix} \bar{z}_{k+1} \\ z^i_{k+1} \\ z^K_{k+1} \end{bmatrix} = (\mathbf{\Phi} + \mathbf{\Gamma K M}) \begin{bmatrix} \bar{z}_k \\ z^i_k \\ z^K_k \end{bmatrix},$$

where

$$\mathbf{\Phi} := \begin{bmatrix} \Phi_a & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{\Gamma} := \begin{bmatrix} \Gamma_a & 0 \\ 0 & I \end{bmatrix}, \quad \mathbf{M} := \begin{bmatrix} C_{2a} & 0 \\ 0 & I \end{bmatrix},$$

$$\mathbf{K} := \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix}.$$

Now let

$$\hat{V}(\mathsf{z}, \mathsf{z}^i, \mathsf{z}^K) := \left[\begin{array}{c} \mathsf{z} \\ \mathsf{z}^i \\ \hline \mathsf{z}^K \end{array}\right]' P \left[\begin{array}{c} \mathsf{z} \\ \mathsf{z}^i \\ \hline \mathsf{z}^K \end{array}\right]$$

where we will use the partitions

$$P = \begin{bmatrix} P_1 & P_3 \\ P_3' & P_2 \end{bmatrix}, \quad P_1 = \begin{bmatrix} P_{11} & P_{13} \\ P_{13}' & P_{12} \end{bmatrix}.$$

The inequality (14) with L replaced by $\tilde{L}$ can be expressed as a matrix inequality:

$$\begin{aligned} &(\mathbf{\Phi} + \mathbf{\Gamma K M})' P (\mathbf{\Phi} + \mathbf{\Gamma K M}) - P \\ &\quad + (\mathbf{\Psi} + \mathbf{\Theta K M})'(\mathbf{\Psi} + \mathbf{\Theta K M}) < 0 \end{aligned} \tag{19}$$

where

$$\mathbf{\Psi} := \left[\begin{array}{cc|c} \Psi & 0 & 0 \\ 0 & \hat{\Psi} & 0 \end{array}\right], \quad \mathbf{\Theta} := \begin{bmatrix} \Theta & 0 \\ 0 & 0 \end{bmatrix}.$$

Ψ and Θ are given by any factorization of Q defined in Lemma 2:

$$Q = \begin{bmatrix} \Psi' \\ \Theta' \end{bmatrix} [\, \Psi \;\; \Theta \,],$$

and $\hat{\Psi}'\hat{\Psi} = \hat{Q}$.

LMI techniques can be used to solve for P and $\mathbf{K}$:

Theorem 1. The following two statements are equivalent:

(i) There exist $P = P' > 0$ and $\mathbf{K}$ satisfying (19).

(ii) There exist $P_{11} = P_{11}' > 0$ and

$$Y_1 = \begin{bmatrix} Y_{11} & Y_{13} \\ Y_{13}' & Y_{12} \end{bmatrix} = Y_1' > 0$$

satisfying the following three LMIs:

$$(C_2')_\perp (\Phi' P_{11} \Phi - P_{11} + Q_1)(C_2')_\perp' < 0, \tag{20}$$

$$\begin{bmatrix} \begin{bmatrix} \Gamma \\ \Theta \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix} \left(\Pi Y_1 \Pi' - \hat{Y}_1 \right) \begin{bmatrix} \begin{bmatrix} \Gamma \\ \Theta \end{bmatrix}_\perp & 0 \\ 0 & I \end{bmatrix}' < 0, \tag{21}$$

where

$$\Pi = \begin{bmatrix} \Phi & 0 \\ \Psi & 0 \\ C_1 & I \\ 0 & \hat{\Psi} \end{bmatrix}, \quad \hat{Y}_1 = \begin{bmatrix} Y_{11} & 0 & Y_{13} & 0 \\ 0 & I & 0 & 0 \\ Y_{13}' & 0 & Y_{12} & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

and

$$\begin{bmatrix} P_{11} & I & 0 \\ I & Y_{11} & Y_{13} \\ 0 & Y_{13}' & Y_{12} \end{bmatrix} \geq 0. \tag{22}$$

where the subscript $_\perp$ denotes the orthogonal complement of a matrix.

If z_0 is given then a reasonable performance criterion would be

$$\min_{\mathsf{z}^i,\, \mathsf{z}^K} V(z_0, \mathsf{z}^i, \mathsf{z}^K) = z_0'(P_{11} - P_{31} P_{21}^{-1} P_{31}') z_0 \tag{23}$$

where P is partitioned as below ($P_{11} \in \mathbb{R}^{n \times n}$):

$$P = \begin{bmatrix} P_{11} & P_{31} \\ P_{31}' & P_{21} \end{bmatrix}.$$

Proposition 2. The minimization of (23) subject to (19) is feasible if and only if the following semi-definite program is feasible. Then both optimization problems have the same objective value:

$$\text{minimize } \gamma \quad \text{subj. to (20), (21), (22), and}$$

$$Y_1 > 0, \quad P_{11} > 0, \quad \begin{bmatrix} \gamma & z_0' \\ z_0 & Y_{11} \end{bmatrix} \geq 0.$$

Another possibility is to assume z_0 is a random vector with zero expected value and a unit covariance matrix. Then we consider the cost

$$\min_{\mathsf{z}^i,\, \mathsf{z}^K} E\{V(z_0, \mathsf{z}^i, \mathsf{z}^K)\} = \min \operatorname{tr}(P_{11} - P_{31} P_{21}^{-1} P_{31}')$$

5.3 Controller Construction

Suppose that the LMIs in Theorem 1 can be solved. Then there exist a solution (P_{11}, Y_1) which satisfies (22) strictly because of the strictness of (20) and (21). One can then construct an n-dimensional controller. This means that the full controller including the integral part is of dimension $n + 1$. We omit the details of the controller reconstruction.

5.4 Integrator Windup Compensation

The controller needs to be equipped with an integrator anti-windup scheme to perform well in presence of large load disturbances and set-point changes. In the simulation examples we considered the following simple scheme

$$\begin{aligned} \begin{bmatrix} z^K_{k+1} \\ z^i_{k+1} \end{bmatrix} &= \begin{bmatrix} A_K & B_{K2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z^K_{k+1} \\ z^i_{k+1} \end{bmatrix} + \begin{bmatrix} B_{K1} \\ 0 \end{bmatrix} \delta y_k \\ &\quad + \begin{bmatrix} 0 \\ \tilde{M} \end{bmatrix} (y_k - C_2 \mathsf{x}^0 + T_w(d_k - u_k)) \\ u_k &= [\, C_K \;\; D_{K2} \,] \begin{bmatrix} z^K_k \\ z^i_k \end{bmatrix} + D_{K1}(y_k - C_2 \mathsf{x}^0) \\ d_k &= [u_k + \mathsf{d}^0]_0^1 \end{aligned}$$

where $\delta y_k = (y_k - C_2 \mathsf{x}^0)$ and T_w is chosen to obtain a large region of stability.

6. EXAMPLE

We consider a boost converter with topology as in Figure 2. The example is adopted from (Jeltsema and Scherpen, 2004). When the switch is in posi-

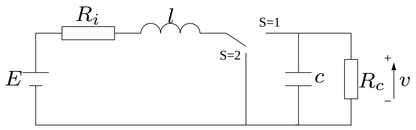

Fig. 2. A boost converter.

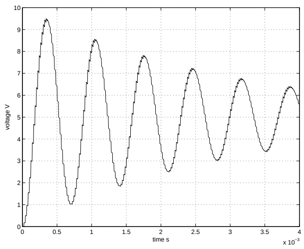

Fig. 3. Open loop response of the output voltage.

tion $s = 1$ and $s = 2$, respectively, we have the system matrices (for the case $R_i = 0$)

$$A_1 = \begin{bmatrix} 0 & -1/l \\ 1/c & -1/(Rc) \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1/(Rc) \end{bmatrix},$$

$$B_1 = \begin{bmatrix} E/l \\ 0 \end{bmatrix}, \; B_2 = B_1, \; C_1 = C_2 = [0 \; 1] .$$

with parameter values $E = 1$ V, $R_c = 30\ \Omega$, $R_i = 0\ \Omega$, $l = 10\ \mu H$, $c = 50\ \mu F$, period time $T = (50 \cdot 10^3)^{-1}$ and reference output voltage $\mathrm{v}_{\mathrm{ref}} = 5$ V. If we design a controller based on the algorithms developed in this paper with with integrator cost coefficient $Q_e = 0.01$, and an additional cost $Q_d = 0.01$ added to Q_2 then we obtain the controller

$$A_K = \begin{bmatrix} -0.2560 & 0.0361 \\ -0.2255 & 0.0191 \end{bmatrix}, B_K = \begin{bmatrix} 2.2604 & 0.3991 \\ 1.0747 & -0.4429 \end{bmatrix}$$

$$C_K = [-0.2123 \; 0.0260] \, , D_K = [0.3528 \; 0.0803]$$

The anti windup feedback gain is $T_w = 14$.

Fig. 3 shows the output voltage when starting with all initial states (plant and controller) at zero and the duty ratio fixed at $\mathsf{d}^0 = 1/5$. Fig. 4 shows the controlled output voltage when the nominal duty ratio d^0 is ramped up to the final value $\mathsf{d}^0 = 1/5$. At time $t = 2ms$ the load resistance drops from 30 to 15 Ω. Finally, Fig. 5 shows the controlled output voltage when the nominal duty ratio d^0 is ramped up to the final value $\mathsf{d}^0 = 1/5$ and the source voltage increases by 50 percent.

Note that we use a different modulation compared to (Jeltsema and Scherpen, 2004) so any fair comparison of the results cannot be done. Our controller provides good damping of the load disturbance. The initial part of the step response and the large input voltage change brings the system state far away from the equilibrium and into the nonlinear range where saturation and integrator windup affects the response very much.

7. CONCLUSIONS

We have considered control synthesis for a class of pulse-width-modulated systems based on a linearized sampled data model. The proposed design methodology is applied to design a controller for a boost converter. As demonstrated in the example, the nonlinear closed loop system has a good performance as well as a large region of stability.

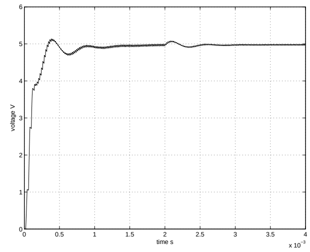

Fig. 4. Output voltage response when the load resistance R_c drops 50 percent.

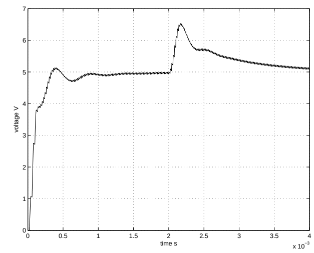

Fig. 5. Output voltage response when the source voltage E increases 50 percent

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