ON THE OBSERVER DESIGN OF MULTI OUTPUT SYSTEMS IN DISCRETE–TIME

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Abstract: The paper deals with the equivalence under coordinates change, of a discrete–time nonlinear multi output system to observer canonical forms. Necessary and sufficient conditions for local equivalence to these forms are given.

Keywords: Discrete–time nonlinear systems, state equivalence, nonlinear observers, differential geometry.

1. INTRODUCTION

The problem of the equivalence to observer canonical forms, first set in a continuous–time context in [Krener et al., 1983], [Bestle et al., 1983] and further developed in [Krener et al., 1985], [Xia et al., 1989], was addressed by several authors both in continuous and discrete time: see for instance [Chung et al., 1990], [Lee et al., 1991], [La Scala et al., 1995], [Song et al., 1995], [Moraal et al., 1995], [Lin et al., 1995], [Besançon et al., 1998], [Lilge, 1998], [Barbot et al., 1999], [Besançon et al., 1999], [Huijeberts, 1999], [Kazantzis et al., 2000]. In [Hou et al., 1999] necessary and sufficient conditions were given for the equivalence under coordinates change to nonlinear multi output continuous time observers with linear error dynamics. However these conditions require the solution of a set of partial differential equations.

Recently in [Califano et al., 2003] the problem was addressed for nonautonomous single output discrete time systems in the geometric framework introduced in [Monaco et al., 1997a] by considering both state and output transformations. Necessary and sufficient conditions for local equivalence were given. The proof of the result is constructive w.r.t. the coordinates change and the required output transformation.

In the present paper a first solution to the problem of the equivalence under coordinates change to observer canonical forms for multi output discrete time systems is proposed. The approach used gives the methodology to cope with nonlinear output transformations as shown in [Califano et al., 2003] for the single output case, thus leading to more general results than those proposed in [Hou et al., 1999] for continuous time systems.

The relevance of the addressed problem stands in the possibility of designing nonlinear discrete observers based on the extended Kalman filter as shown in [Reif et al., 1999].

We will consider a nonlinear discrete-time system

$$\begin{aligned} x(k+1) &= F(x(k), u(k)) \\ y(k) &= h(x(k)) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $F : \mathbb{R}^n \times R^m \to \mathbb{R}^n$ and h are analytic functions, $(0,0)$ is an equilibrium pair, i.e. $F(0,0) = 0$, and $h(0) = 0$. We will assume that the Jacobian matrix $\dfrac{\partial F_0}{\partial x}$

has full rank at $x_0 = 0$. As a consequence, the drift term F_0 is locally invertible, and the complete dynamics too is locally invertible in a neighborhood of $(0,0)$. In the sequel $\mathcal{X}_0$ and $\mathcal{U}_0$ will denote such suitable meighborhoods of $x_0 = 0$ and $u = 0$ respectively.

Definitions *The problem of the equivalence to the generalized observer form has a solution if there exists a locally defined coordinates change $z = \phi(x)$ such that, in the new variables, (1) reads*

$$\begin{aligned} z(k+1) &= A(u(k))z(k) + \Psi(y(k), u(k)) \\ y(k) &= Cz(k) \end{aligned} \tag{2}$$

with $(A(0), C)$ an observable pair in the canonical Brunovskij form, i.e.

$$A(0) = diag(A_1(0) \cdots A_p(0)),\ C = diag(C_1 \cdots C_p),$$

$$A_i(0) = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}_{k_i \times k_i},$$

$$C_i = (0 \ \cdots \ 0 \ \ 1)_{1 \times k_i}$$

where k_i is the observability index associated with the i–th output.

The problem will be locally or globally solvable depending on the properties of the transformations. Dropping the term "generalized" will mean that $A(u) = A(0) = A$.

The paper is organized as follows. Some technical arguments concerning the geometric framework are given in Section 2. In Section 2.1 the single output case is recalled. In Section 3, the problem of local equivalence to the generalized output injection observer form is addressed. The global version of the result is presented in Section 3.1.

2. RECALLS AND NOTATIONS

The following notations are issued from [Monaco et al., 1986], [Sontag, 1986], [Jakubczyk et al., 1990]. Given two vector fields $\tau_1(x), \tau_2(x)$, a real valued function $\lambda(x)$ and a diffeomorphism $\phi(x)$, defined on $\mathbb{R}^n$, $L_{\tau_1}\lambda(x) := \dfrac{\partial \lambda(x)}{\partial x}\tau_1(x)$, is the standard Lie derivative, $ad_{\tau_1}\tau_2(x) := [\tau_1, \tau_2](x) = L_{\tau_1} \circ L_{\tau_2}(Id)|_x - L_{\tau_2} \circ L_{\tau_1}(Id)|_x$ is the Lie bracket of vector fields and $Ad_\phi \tau_1$ is the transport of τ_1 along $\phi(x)$, i.e. $Ad_\phi \tau_1 := \left(\dfrac{\partial \phi}{\partial x}\tau_1\right)\Big|_{\phi^{-1}}$. δ_{rs} denotes the kronecker index which is 0 for $r \neq s$ and 1 for $r = s$, Id denotes the identity function, I the identity operator and $J\phi(x) := \dfrac{\partial \phi(x)}{\partial x}$.

Consider now ([Monaco et al., 1997a], [Monaco et al., 1997b]) the parameterized family of vector fields associated with (1)

$${}_{i_1}G^0(x,u) \doteq \left(\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} F(\cdot, u_1, \cdots, u_{i_1} + \varepsilon, \cdots, u_m)\right)\bigg|_{F^{-1}(x,u)}$$

$i_1 = 1, \cdots, m$, locally well defined around $u = 0$ and set for $i_j \in [1, m]$, and $k > 1$

$${}_{i_1,i_2\cdots i_k}G^0_k(x) \doteq \frac{\partial^{k-1}}{\partial u_{i_2} \cdots \partial u_{i_k}}\bigg|_{u=0} {}_{i_1}G^0(x,u)$$

with ${}_{i_1}G^0_1(x) \doteq {}_{i_1}G^0(x, 0)$. Accordingly (1) admits the following exponential representation

$$\begin{aligned} F(x,u) &= F_0(x) + \sum_{i=1}^m u^i L_{{}_iG^0_1(\cdot)}(Id)|_{F_0(x)} + \sum_{i=1}^m \\ &\sum_{j=1}^m \frac{u^i u^j}{2}\Big(L_{{}_{ij}G^0_2(\cdot)} + L_{{}_iG^0_1(\cdot)} \circ L_{{}_jG^0_1(\cdot)}\Big)(Id)|_{F_0(x)} + 0(u^3) \\ &= e^{\underline{u}\mathcal{G}^0(\cdot,\underline{u})}(Id)\Big|_{F_0(x)} = \left(I + \sum_{p>0}\frac{1}{p!}L^p_{\underline{u}\mathcal{G}^0(\cdot,\underline{u})}\right)(Id)\bigg|_{F_0(x)}. \end{aligned}$$

$\underline{u}\mathcal{G}^0(.,\underline{u}) := R^n \to R^n$, is defined by its series expansion with respect to u. It is a smooth vector field parameterized by $(u_1, \cdots, u_m)$ and a Lie element in the $({}_{i_1\ldots i_k}G^0_k)$'s [Monaco et al., 1997a]. The transport of a vector field τ_0 along the dynamics (1) is thus a vector field $\tau_1(\cdot, u)$ given by

$$\begin{aligned} \tau_1(\cdot,u) &= \left(\frac{\partial F(\cdot,u)}{\partial x}\tau_0(\cdot)\right)_{F^{-1}(\cdot,u)} = e^{-ad_{\underline{u}\mathcal{G}^0(\cdot,\underline{u})}}(Ad_{F_0}\tau_0(\cdot)) \\ &= (I + \sum_{p\geq 1}\frac{(-1)^p}{p!}ad^p_{\underline{u}\mathcal{G}^0(\cdot,\underline{u})})(Ad_{F_0}\tau_0(\cdot)) \\ &= Ad_{F_0}\tau_0(\cdot) - \sum_{i=1}^m u^i\ ad_{{}_iG^0_1}Ad_{F_0}\tau_0(\cdot) + \sum_{i=1}^m \\ &\sum_{j=1}^m \frac{u^i u^j}{2}(-ad_{{}_{ij}G^0_2} + ad_{{}_iG^0_1}ad_{{}_jG^0_1})Ad_{F_0}\tau_0(\cdot) + O(u^3) \end{aligned} \tag{3}$$

In the sequel, ${}_i\tilde{G}^0(z,u) := Ad_\phi {}_iG^0(x,u)$, while ${}_\eta\tilde{G}^0_k(z) := Ad_\phi\ {}_\eta G^0_k(x)$, $\forall k \geq 1$, $\eta = i_1, \cdots i_k$, will denote the vector fields under study in the coordinates $z = \phi(x)$.

In the multi ouput case, a first problem concerns the introduction of a suitable set of observability indices.

Definition 1. The observability index k_i, associated with the i-th output h_i is the first integer such that

$$d(h_i \circ F_0^{k_i}) \in \text{span}\{dh, \cdots d(h \circ F_0^{k_i - 1})\} \tag{4}$$

while for $1 \leq j \leq k_i - 1$,

$$d(h_i \circ F_0^j) \notin \text{span}\{dh, \cdots d(h \circ F_0^{j-1})\}. \tag{5}$$

From (4–5) the observability indices can be computed from the codistributions

$$\Omega_i = \text{span}\{dh \cdots d(h \circ F_0^i)\}, \quad i \geq 0.$$

In the following we will assume without any loss of generality $k_1 \geq \cdots \geq k_p$, since this can be easily achieved by reordering the output functions.

We will denote by $O_i = (dh_i^T, \cdots, d(h_i \circ F_0^{k_i-1})^T)^T$ and by $O = (O_1^T, \cdots, O_p^T)^T$. Finally

$$\begin{aligned}\mathcal{O} &= \{dh_i, \cdots, d(h_i \circ F_0^{k_i-1}), i = 1, \cdots, p\}\\ \mathcal{O}_i &= \mathcal{O} - \{d(h_i \circ F_0^{k_i-1})\}.\end{aligned}$$

Definition 2. The autonomous system

$$\begin{aligned}x(k+1) &= F_0(x(k))\\ y(k) &= h(x(k))\end{aligned}$$

is said to be locally strongly observable if there exist p observability indices $k_1, \cdots, k_p$ with $\sum_{i=1}^{p} k_i = n$ such that for any $x \in \mathcal{X}_0$, rank $O = n$.

Definition 3. A smooth vector field, f, defined on a manifold N is complete if the corresponding flow is defined on the whole Cartesian product $\mathbb{R} \times N$.

2.1 The single output case

Let $r_1(x)$ be the solution of

$$d(h \circ F_0^{i-1}) r_1 = \delta_{in} \text{ for } i = 1, \cdots n, \qquad (6)$$

with δ_{1n} the kronecker index, and denote by

$$r_i := Ad_{F_0} r_{i-1} = Ad_{F_0}^{i-1} r_1, \qquad i = 2, \cdots, n,$$

its iterated transport along $F_0(x)$.

Let us note that, under the strong observability condition, the vector fields $(r_1, \cdots, r_n)$ exist and are linearly independent since by construction they verify the equality

$$\begin{pmatrix} dh \\ \vdots \\ d(h \circ F_0^{n-2}) \\ d(h \circ F_0^{n-1}) \end{pmatrix} (r_1 \cdots r_n) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & * \\ 0 & \iddots & \iddots & \vdots \\ 1 & * & \cdots & * \end{pmatrix}.$$

Analogously, let $r_{i+1}(\cdot, u)$ be the transport of $r_i(\cdot)$ along the complete dynamics $F(\cdot, u)$ given by (3), with $r_{i+1}(\cdot, 0) = r_{i+1}(\cdot) := Ad_{F_0} r_i$. The following results hold true [Califano et al., 2003].

Theorem 1. The problem of the equivalence to the generalized observer form (2) is locally solvable if and only if on $\mathcal{X}_0$

A1) rank $O(\cdot) = n$
A2) $[r_1(\cdot),\ r_{i+1}(\cdot)] = 0$ for $i = (1, \cdots, n-1)$
A3) $\forall u \in \mathcal{U}_0$, $[r_j(\cdot),\ r_{i+1}(\cdot, u)] \equiv 0$ for $i = (1, \cdots, n-1)$, $j = (1, \cdots, n)$.

The coordinates change is such that

$$[J\phi(x)]^{-1} = (r_1(x) \cdots r_n(x)). \qquad (7)$$

Moreover in (2), $A(u) := A(0) = A$ if and only if

A4) $r_{i+1}(\cdot, u) = r_{i+1}(\cdot)$.

3. THE MULTI OUTPUT CASE

In the present section we will start to point out the main differences with respect to the single output case which do not allow a straightforward generalization of the single output case.

Once the observability indices are defined, let us consider for $i = 1, \cdots, p$, the vector fields $r_{i,1}$ (associated with the i–th output) which satisfy the following condition:

$$< d(h_l \circ F_0^{(j-1)}), r_{i,1} >= \delta_{li}\delta_{jk_i}, \qquad (8)$$

where $1 \leq l \leq p$, $1 \leq j \leq k_i$, and denote by $r_{j,l} := Ad_{F_0} r_{j,l-1} = Ad_{F_0}^{l-1} r_{j1}$ its iterated transport along F_0.

Unlike the single output case, the vector fields $r_{1,1} \cdots r_{p,1}$ thus defined are not univocally determined. In fact for a given set of observability indices $k_1 \geq \cdots \geq k_p$, while $r_{1,1}$ is univocally defined by the n independent conditions in (8), the same holds trueonly if all the observability indices are equal to k_1. In such a case a unique set of solutions can be computed from (8). The nilpotency of the associated distribution makes it possible to define the desired coordinates change. In the general case ($k_1 \geq k_2 \geq \cdots \geq k_p$) several solutions to (8) can be computed: as a matter of fact in the z–coordinates the set of all possible solutions to (8) is given by $r_{j,1} = \frac{\partial}{\partial z_{j,1}} + \sum_{i=1}^{j-1} \sum_{l=1}^{k_i - k_j} a_{j,l}(z) r_{j,l}$ where the coefficients $a_{j,l}$ are generic functions of z; only suitable properties of the fucntions $a_{j,l}$ ensure the necessary nilpotency property. It can be easily understood that if the $a_{j,l}$'s are constant nilpotency is achieved. With this in mind we stress that the costant coefficients $a_{j,l}$ can be computed arguing as follows.

Compute r_{11} solution to (8) and accordingly $r_{1,s}$, $s = 2, \cdots k_1$. For $j = 2, \cdots, p$, iteratively consider $z = \phi(x)$ such that in the new coordinates $\tilde{r}_{il}(z) = \frac{\partial}{\partial z_{il}}$, $l = 1, \cdots, k_i$ $i = 1, \cdots, j-1$. In the set of all possible solutions

$$a_{il}^j(z) = \int < \tilde{r}_{i,l}, [\tilde{r}_{j1}, \tilde{r}_{1,1}] > dz_{11} + a_{il}^j(0)$$

choose, for $i = 1, \cdots, j-1$, $l = 1 \cdots k_j - k_i$

$$a_{il}^j(z) = \int < \tilde{r}_{i,l}, [\tilde{r}_{j1}, \tilde{r}_{1,1}] > dz_{11} + a_{il}^j(0) = cost. \quad (9)$$

In conclusion, while (8) can be considered a generalization of (6), the additional property (9) is peculiar to the multi output case.

It is now possible to extend to the multi output case the results stated in the previous section.

Theorem 2. The problem of the equivalence to the generalized observer form (2) is locally solvable if and only if there exist observability indices $(k_1, \cdots, k_p)$ such that on $\mathcal{X}_0$

B1) $\text{rank } O(\cdot) = n$
B2) $\text{span}\mathcal{O}_i(\cdot) = \text{span}\{\mathcal{O}_i(\cdot) \cap \mathcal{O}(\cdot)\}$
B3) $[r_{i,1}(\cdot), r_{j,l}(\cdot)] = 0 \quad i,j \in [1,p], \quad 1 \le l \le k_j$
B4) $\forall u \in \mathcal{U}_0,\ [r_{i,s}(\cdot), r_{l,j+1}(\cdot, u)] = 0,\ s = 1, \cdots, k_i,\ j = 1 \cdots k_l - 1,\ i,j \in [1,p]$

The coordinates change is such that

$$[J\phi(x)]^{-1} = (r_{1,1} \cdots r_{1,k_1}, \cdots, r_{p,1} \cdots r_{p,k_p})(x). \quad (10)$$

Moreover in (2), $A(u) := A(0) = A$ if and only if

B5) $r_{j,i+1}(\cdot, u) = r_{j,i+1}(\cdot)$ for $i = 1, \cdots k_i - 1$ and $j = 1, \cdots, p$.

Proof Necessity. Assume that the system is in the observer form (2), then conditions B1) and B2) are satisfied. By construction the vector fields $r_{ij} = \frac{\partial}{\partial z_{ij}}$ satisfy B3) and (8). Since $F(\cdot, u) = A(u)z + \Psi(y, u)$, then for $j = 1, \cdots, k_l - 1$, $l = 1, \cdots, p$,

$$\frac{\partial F(\cdot, u)}{\partial z_{l,j}} = r_{l,j+1}(\cdot, u)|_{F(\cdot,u)}\, A(u)\frac{\partial}{\partial z_{l,j}} = A_{l,j}(u).$$

It follows that, for $j = (1, \cdots, k_l - 1)$, $l = (1, \cdots, p)$, and $s = (1, \cdots, k_i)$, $i = (1, \cdots, p)$, $u \in \mathcal{U}_0$, $[r_{i,s}(\cdot), r_{l,j+1}(\cdot, u)] = 0$ which proves B4).

Finally if $A(u) := A(0) = A$ then

$$_{i_1}G^0(\cdot, u) = {}_{i_1}G^0(z_{11}, \cdots, z_{p1}, u),$$

so that $\forall k > 1$ and $\forall j \in [1, p]$

$$[r_{j,i+1}, {}_\eta G^0_k] = 0,\ i \in [1, k_j - 1],\ \forall \eta = i_1 \cdots i_k$$

i.e., according to (3), B4) is verified. These conditions are invariant under coordinates change, so that the result follows.

Sufficiency. Condition B1) and B2) ensure the existence of p vector fields $r_{1,1}, \cdots r_{p,1}$ satisfying the set of algebraic equations (8). Moreover the n vector fields $r_{i,1} \cdots r_{i,k_i}$ for $i = 1, \cdots, p$, are linearly independent. In fact denoting by $R(x) = (r_{1,1} \cdots r_{1,k_1} \cdots r_{p,1} \cdots r_{p,k_p})$, it is sufficient to note that

$$O(x)\, R(x) =$$

$$= \begin{pmatrix}
0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & \\
\vdots & & \iddots & \iddots & * & \vdots & & \iddots & 0 & \\
\vdots & \iddots & \iddots & \iddots & & \vdots & \iddots & \iddots & * & \\
0 & \iddots & \iddots & & \vdots & 0 & \iddots & \iddots & * & \\
1 & * & \cdots & \cdots & * & 0 & * & * & * & \\
0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \\
\vdots & & \iddots & \iddots & * & \vdots & \iddots & \iddots & * & \\
\vdots & \iddots & \iddots & \iddots & * & 0 & \iddots & * & \vdots & \\
0 & 0 & * & * & * & 1 & * & \cdots & * & \\
 & & & & & & & & & \ddots
\end{pmatrix}$$

According to B3) the distribution $R(x)$ is nilpontent at the first order since for any two vector fields $r_{l,s}, r_{i,j}$, assuming $s \le j$ we have $[r_{l,s}, r_{i,j}] = Ad^{s-1}_{F_0}[r_{l,1}, r_{i,j-s+1}] = 0$. Consequently, there exists a coordinates change $z = \phi(x)$ satisfying (10). In the new coordinates $\tilde{r}_{i,j}(\cdot) = Ad_\phi r_{i,j}(\cdot)$ and $\tilde{R}(\cdot) = Ad_\phi R(\cdot) = I_{n\times n}$. Let us now consider the output functions which in the new coordinates read $y_i = (h_i \circ \phi^{-1}(z))$, $i = 1, \cdots p$.

$$\frac{\partial(h_i \circ \phi^{-1})}{\partial z_{l,j}} = \left[\frac{\partial h(x)}{\partial x}\left.\frac{\partial \phi^{-1}}{\partial z_{l,j}}\right|_\phi\right]_{\phi^{-1}} = \left[\frac{\partial h_i(x)}{\partial x} r_{l,j}\right]_{\phi^{-1}}$$

which, on the basis of (8), implies that $y_i = z_{i,k_i}$. Let us now consider the drift term $\tilde{F}_0(z) = \phi \circ F_0 \circ \phi^{-1}$. Standard computations show that

$$\frac{\partial \tilde{F}_0(z)}{\partial z_{l,j}} = \left[\frac{\partial \phi}{\partial x} r_{l,j+1}\right]_{F_0\circ\phi^{-1}} = \tilde{r}_{l,j+1}(\cdot)|_{\tilde{F}_0}$$

which proves that

$$\tilde{F}_0(z) = \begin{pmatrix}
\psi^1_1(z_{1,k_1} \cdots z_{p,k_p}) \\
z_{1,1} + \psi^1_2(z_{1,k_1} \cdots z_{p,k_p}) \\
\vdots \\
z_{1,k_1-1} + \psi^1_{k_1}(z_{1,k_1} \cdots z_{p,k_p}) \\
\vdots \\
\psi^p_1(z_{1,k_1} \cdots z_{p,k_p}) \\
z_{p,1} + \psi^p_2(z_{1,k_1} \cdots z_{p,k_p}) \\
\vdots \\
z_{p,k_p-1} + \psi^p_{k_p}(z_{1,k_1} \cdots z_{p,k_p})
\end{pmatrix} \quad (11)$$

$$= Az + \psi_0(y)$$

In the new coordinates, since B4) is invariant under coordinates change, one verifies that for $l, i = (1, \cdots, p)$ and $s = (1, \cdots, k_i)$, $j = (1, \cdots, k_l - 1)$,

$$[\tilde{r}_{i,s}, \tilde{r}_{l,j+1}(\cdot, u)] = \frac{\partial \tilde{r}_{l,j+1}(\cdot, u)}{\partial z_{i,s}} = 0,$$

i.e. $\tilde{r}_{l,j+1}(\cdot, u) := A_{l,j}(u)$. Moreover, by definition, for $j = (1, \cdots, k_l - 1)$, $l = 1, \cdots, p$,

$$\frac{\partial \tilde{F}(\cdot, u)}{\partial z}\tilde{r}_{l,j}(\cdot) = \tilde{r}_{l,j+1}(\tilde{F}(\cdot, u), u) = A_{l,j}(u) \quad (12)$$

so that $\tilde{F}(\cdot, u) = A(u)z + \Psi(y, u)$, with $A(u)$ such that $A(u)\frac{\partial}{\partial z_{i,s}} = A_{i,s}(u)$, for $s = (1, \cdots, k_i - 1)$, $i = 1, \cdots, p$.

Finally if B5) is verified then according to (12), $l = 1 \cdots p$ and $j = 2, \cdots, k_l$, we have that

$$\left[\frac{\partial \tilde{F}(\cdot, u)}{\partial z}\tilde{r}_{l,j}(\cdot)\right]_{\tilde{F}^{-1}(\cdot,u)} = \tilde{r}_{l,j+1}(\cdot) = A_{l,j}$$

which ends the proof. ◁

3.1 A global result

In the present section we will state the equivalent global version of Theorem 2. To this end let us preliminarily note that assuming local Lipschitz continuity leads to the following result.

Lemma 1. Let $r_{1,1}(\cdot)$ and $r_{1,2}(\cdot) := Ad_{F_0} r_{1,1}(\cdot)$ be locally Lipschitz continuous vector fields. Then $r_{1,2}(\cdot)$ is complete if and only if $r_{1,1}(\cdot)$ is complete.

Proof From [Angeli et al., 1999] $r_{1,1}(\cdot)$ is forward complete if and only if there exists a proper and smooth positive definite function $V_1(x)$, such that $\forall x$, $\frac{\partial V_1(x)}{\partial x} r_{1,1}(x) < V_1(x)$. Due to the invertibility of the drift, $\forall x$ we have

$$\frac{\partial V_1(x)}{\partial x} r_{1,1}(x) = \frac{\partial V_1(F_0^{-1} \circ F_0)}{\partial x} r_{1,1}(x) = \left(\frac{\partial V_1(F_0^{-1})}{\partial x} r_{1,2}(x)\right)\bigg|_{F_0} < V_1(x)$$

i.e. $\frac{\partial V_1(F_0^{-1})}{\partial x} r_{1,2}(x) < V_1(F_0^{-1})$. Since $V_1(F_0^{-1})$ is still a proper and smooth positive definite function, the forward completeness of $r_{1,1}\cdot)$ implies the forward completeness of $r_{1,2}(\cdot)$. The same arguments prove that $r_{1,2}(\cdot)$ is also backward complete. Conversely, suppose that $r_{1,2}(\cdot)$ is complete while $r_{1,1}(\cdot)$ is not. Then there would exist a function $V_2(x)$ such that $\frac{\partial V_2(x)}{\partial x} r_{1,2}(x) < V_2(x)$, $\forall x$, i.e. $\frac{\partial V_2(F_0)}{\partial x} r_{1,1}(x) < V_2(F_0) \quad \forall x$, where $V_2(F_0)$ is still proper and smooth positive definite, so that $r_{1,1}(\cdot)$ is necessarily forward complete. Since the same arguments prove the backward completeness, $r_{1,1}(\cdot)$ must be complete. ◁

Based on the previous result we have,

Theorem 3. Assume that for $i = 1, \cdots, p$ the vector fields $r_{i,1}(x)$ are locally Lipschitz continuous. Then the problem of the equivalence to the generalized observer form (2) is globally solvable if and only if there exist observability indices $(k_1, \cdots, k_p)$, such that

i) Conditions B1), B2), B3) and B4) hold for all $x \in \mathbb{R}^n$
ii) The vector fields $r_{1,1}(x) \cdots r_{p,1}(x)$ are complete.

Proof We will prove only the sufficiency since the necessity is straightforward. Since B1) and B2) hold for all x, the vector fields $r_{i,j}(x)$, $i = 1, \cdots, p$, $j = 1, \cdots k_i$ are defined for all x. Moreover according to Lemma 1 condition ii) together with the assumption of local Lipschitz continuity implies that the vector fields $r_{i,j}(x)$, $i = 1, \cdots, p$, $j = 1, \cdots k_i$ are complete and since B3) holds for all x, (10) is a global diffeomorphism [Isidori, 1995]. Using the same arguments as in Theorem 2 we can then show that $F_0(z) = Az + \psi_0(y)$ and for $j = 1, \cdots, p$, $y_j = z_{j,k_j}$. As for the control dependent part, since B4) holds for all x, and due to the analyticity of $F(\cdot, u)$, we have that for $j = 1, \cdots, p$, $i = 1 \cdots k_j - 1$, $\frac{\partial \tilde{F}(\cdot, u)}{\partial z_{j,i}} = A_{j,i}(u)$ a nonlinear analytic function in u. As a consequence

$$\begin{aligned}\tilde{F}(\cdot, u) &= \int_0^{z_{1,1}} \frac{\partial \tilde{F}(\cdot, u)}{\partial z_{1,1}} dz_{1,1} + \tilde{F}(0, z_{1,2} \cdots, z_{p,k_p}, u) \\ &= \sum_{j=1}^{p} \sum_{i=1}^{k_j - 1} \int_0^{z_{j,i}} A_{j,i}(u) dz_{j,i} \\ &\quad + \tilde{F}(0, \cdots, z_{1,k_1}, \cdots, 0, \cdots, z_{p,k_p}, u) \\ &= A(u)z + \psi(y, u)\end{aligned}$$

Finally if B5) holds for all x then reasoning as in the proof of Theorem 2 we have that $A_{j,i}(u) = A_{j,i}$ and the result follows immediately. ◁

Remark. It is straightforward to understand that the condition

ii)' the vector fields $r_{1,1}, \cdots, r_{1,k_1}, r_{p,1}, \cdots, r_{p,k_p}$ are complete,

can take the place of ii) in Theorem 3 if the assumption of Lipschitz continuity is removed. ◁

CONCLUSIONS

In the present paper we have given necessary and sufficient conditions for the equivalence under coordinates change to observer canonical forms. The proposed approach allows to take into account nonlinear output transformations also as shown in [Califano et al., 2003] for the single output case. This concerns future work.

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