APPROXIMATE MIN-MAX MPC FOR LINEAR HYBRID SYSTEMS

M. Ayala Botto [1] M. Silva J. Sá da Costa

Technical University of Lisbon, Instituto Superior Técnico
Department of Mechanical Engineering, GCAR
Avenida Rovisco Pais, 1049-001 Lisboa, Portugal
Phone: +351 21 841 90 28 Email: AyalaBotto@dem.ist.utl.pt

Abstract: This paper presents a Receding Horizon Control (RHC) strategy to robustly control linear hybrid systems subject to bounded additive disturbances. First, the open-loop finite horizon min-max optimal control problem associated with the RHC strategy is presented, defining a mixed-integer optimization problem. The goal of the min-max formulation is to minimize the worst-case cost of the performance index, while guaranteeing performance/safety constraints for all possible disturbance realizations. Then it will be proved that feasibility at the initial time as well as convexity arguments and set-invariance assumptions are sufficient conditions to guarantee robust stability and performance of the closed-loop system. Simulation results for a perturbed hybrid system will demonstrate the potential of the proposed robust optimal control design.

Keywords: hybrid systems, model predictive control control, min-max optimization, robust control, mixed logical dynamical systems

1. INTRODUCTION

The study of dynamical processes having continuous and discrete variables, denoted as hybrid systems, has recently seen a rapid development trigged by the spread of dynamical systems integrated with logical/discrete decision components and a market competition pressure to achieve fast and "optimal" designs. The research field of hybrid systems results from the interaction between the computer science and the control engineering communities, motivated by the strong impact on applications, e.g. in embedded systems, chemical and biotechnological processes, aerospace, manufacturing, robotics, automotive applications, etc. [Antsaklis, 2000]. In the industrial context, the synthesis of control schemes for hybrid systems is usually approached with heuristic rules, mainly driven by engineering insight and experience, with a consequently long design and verification process. Therefore, the development of new tools to design control/supervisory schemes for hybrid systems and to analyse their stability, safety and performance is of great importance. An hybrid system is characterized by a set of operating modes, each one evolving according to some time-driven dynamics. The switching between modes is regulated by events which may be controlled or uncontrolled. The control of the switching times and the choice among several feasible modes, gave rise to a new rich class of optimal control problems. Several researchers presented new techniques to obtain solutions for some subclasses of this new class of optimal control problems, e.g. [Bemporad and Morari, 1999, Branicky et al., 1998, Cassandras et al., 2001, Hedlund and Rantzer, 1999, Kerrigan and Mayne, 2002, Mayne and Rakovic, 2003]. Some of these techniques extend classical optimal control principles, while others apply dynamic programming techniques, as well as tools from computational geometry and optimization such as, parametric programming, convex and mixed integer optimization. However, virtually all techniques of optimal control of hybrid systems suffer from the

[1] This research was partially supported by the FCT project POCTI/EME/44336/2002.

well-known curse of dimensionality, i.e. the computational complexity to solve such problems increase exponentially with the dimension of the problem [Blondel and Tsitsiklis, 2000]. According to Cassandras and Gokbayrak [2002], the keys to the successful development of optimal control methods for hybrid systems are: a) seeking structural properties that allows the decomposition of such systems into simpler components, and b) making use of efficient numerical techniques. For instance some authors explored the possibility of hierarchical decomposition of some systems into a lower-level component representing physical processes characterized by time-driven dynamics and a higher-level component representing discrete events related to these physical processes. Other examples include the work on PWA systems [Kerrigan and Mayne, 2002], MLD systems [Bemporad and Morari, 1999], ELC systems [Heemels et al., 2000], and MMPS systems [de Schutter and van den Boom, 2001]. Recently Heemels et al. [2001] proved the equivalence between PWA, ELC, MMPS, and MLD systems, allowing to interchange analysis and synthesis tools among them. The choice of a suitable modelling framework is crucial since it reflects a trade-off between two conflicting criteria: the modelling power and the decisive power. The MLD modelling framework will be adopted in this paper since it allows to model systems described by interdependent physical laws (with linear dynamics), logic rules (if-then-else rules) and operating constraints. Another important characteristic of the MLD model is its optimization-oriented structure, allowing to "smoothly" extend existing optimal control methodologies developed for continuous-valued dynamics to the hybrid setting.

Achieving robust stability and/or performance when uncertainty is present in the system dynamics is a crucial subject on any control design. Robust stability is a serious concern in industrial Model Predictive Control (MPC) applications and is currently addressed, for the most part, through the use of extensive closed-loop simulation prior to implementation, relying on the control engineer to anticipate and test every important combination of plant dynamics and active constraints, leading to an expensive and time consuming task. Most studies on robustness consider unconstrained systems subject to small perturbations [Mayne et al., 2000]. However, when hard constraints on states and controls are present, it is necessary to ensure, in addition, that disturbances do not cause transgression of the constraints, which adds an extra level of complexity to the control design.

Recently, Silva et al. [2004] presented a novel procedure to extend the MLD framework for synthesizing robust optimal control inputs of constrained PWA systems subject to bounded additive input disturbances. The control sequence minimizes, on a finite time interval horizon, a nominal quadratic performance index guaranteeing that the mode of the system, at each time instant, is independent of the disturbances and that all safety/performance constraints are verified. The approach is based on the robust mode control concept, which imposes a restriction on the admissible control sequences. From a MPC point of view robustness is achieved by using *open-loop prediction* to estimate the worst-case effect of disturbances on the state trajectory, which is commonly referred in the literature as an open-loop MPC strategy. However the minimization of a nominal performance index is not suitable for solving the robust stability problem since the optimal value of the cost index is not a Lyapunov function. So, the minimization of a cost function that considers all possible realizations of the uncertainty is preferred. This MPC strategy is known as the *min-max* open-loop approach. Additionally, the inclusion of the uncertain behavior in the performance index also implies that the performance goal has to be "relaxed". So, instead of penalizing the distance from a desired point of convergence, the performance index penalizes the distance from a desired *set*. Therefore, the main contribution of this paper is the development of an MLD-based min-max procedure to obtain a stable and robust MPC algorithm for hybrid systems subject to bounded additive disturbances. As closed-loop stability is a fundamental property of any controller, the conditions for stability are also presented.

This paper has the following structure. In Section 2 the robust min-max optimal control problem is formulated. In Section 3 the receding horizon control algorithm is developed and the sufficient robust stability and performance conditions are presented. Section 4 presents the simulation results for the hybrid two tanks system subject to bounded additive disturbances. Finally, some conclusions are drawn in Section 5.

2. PROBLEM DEFINITION

Consider the following discrete-time PWA system subject to bounded additive exogenous disturbances, with known initial state, x_0:

$$x_{k+1} = A_i x_k + B_i u_k + e_i + W_i v_k, \begin{bmatrix} x_k \\ u_k \\ v_k \end{bmatrix} \in \Omega_i \quad (1)$$

where,

$$\Omega_i \triangleq \left\{ \begin{bmatrix} x \\ u \\ v \end{bmatrix} : \; F_i x_k + G_i u_k + J_i v_k \leq h_i \right\} \qquad (2)$$

where $u_k \in \mathbb{U} \subset \mathbb{R}^{n_u}$, $x_k \in \mathbb{X} \subset \mathbb{R}^{n_x}$ and $v_k \in \mathbb{V} \subset \mathbb{R}^{n_v}$ denote the input, state and disturbance at time k, respectively. The index i belongs to a finite set, i.e. $i \in \{1, \ldots, s\}$, and represents the system mode or the active partition. Each partition Ω_i is defined by a bounded intersection of half-spaces in the state+input+disturbance space, and so each Ω_i is convex polyhedron. A_i, B_i, W_i, F_i, G_i and J_i are real matrices of appropriate dimensions, h_i is a real vector, and e_i is the affine real vector, for all $i = 1, \ldots, s$. Moreover, $\Omega = \bigcup_{i=1}^{s} \Omega_i$, $\mathring{\Omega}_i \bigcap \mathring{\Omega}_j = \emptyset$, $\forall\, i \neq j$, where $\mathring{\Omega}_i$ denotes the interior of the polytope Ω_i, i.e. the partitions have disjoint interiors. For synthesis purposes the PWA system (1)–(2) is assumed to be subjected to a set of (possibly time-varying) safety and/or performance constraints, called the *operational constraints*, on the state+input+disturbance space, defined by:

$$\left[\begin{smallmatrix} x_k \\ u_k \\ v_k \end{smallmatrix}\right] \in \mathbb{C}_k \triangleq \left\{ \left[\begin{smallmatrix} x \\ u \\ v \end{smallmatrix}\right] : K_k x_k + L_k u_k + M_k v_k \leq n_k \right\} \quad (3)$$

where the disturbance $v_k \in \mathbb{V}$, with $\mathbb{V}$ assumed to be a polytope containing the origin, according to the typical unknown-but-bounded characterization of disturbances, with K_k, L_k and M_k being real matrices of appropriate dimensions, and n_k a real vector of appropriate dimensions. As shown by Bemporad and Morari [1999], the PWA system (1)–(2) can be expressed by the following MLD system, where the last inequality represents the operational constraints (3):

$$x_{k+1} = B_{z_x} z_x(k) + B_{z_u} z_u(k) + B_\delta \delta_k + B_{z_v} z_v(k) \quad (4a)$$

$$E_\delta^{c/d} \delta_k + E_x^{c/d} x_k + E_u^{c/d} u_k + E_v^{c/d} v_k \leq e^{c/d} \quad (4b)$$

$$E_{z_x}^{d/c} z_x(k) + E_{z_u}^{d/c} z_u(k) + E_\delta^{d/c} \delta_k + E_{z_v}^{d/c} z_v(k) + $$
$$+ E_x^{d/c} x_k + E_u^{d/c} u_k + E_v^{d/c} v_k \leq e^{d/c} \quad (4c)$$

$$E_x^{ctr} x_k + E_u^{ctr} u_k + E_v^{ctr} v_k \leq E^{ctr} \quad (4d)$$

where inequalities should be understood componentwise, $\delta_k \in \{0,1\}^s$ is an auxiliary vector that defines the mode i (or equivalently partition i) of the system ($\dim[\delta] = (s \times 1)$), so if mode i is active then $\delta_i(k) = 1$ and $\delta_j(k) = 0$, $\forall j \neq i$, $i, j \in \{1, \ldots, s\}$, and $z_{xi}(k) = \delta_i(k) x_k$ are auxiliary continuous variables ($\dim[z_{xi}] = \dim[x] = (n_x \times 1)$). Using the Kronecker product to abbreviate notation, $z_x(k) = \delta_k \otimes x_k$, ($\dim[z_x] = (s \cdot n_x \times 1)$), $z_u(k) = \delta_k \otimes u_k$ ($\dim[z_u] = (s \cdot m \times 1)$) and $z_v(k) = \delta_k \otimes v_k$ ($\dim[z_v] = (s \cdot p \times 1)$). Notice also that, $B_{z_x} \equiv [\, A_1\ A_2\ \ldots\ A_s \,]$, $B_{z_u} \equiv [\, B_1\ B_2\ \ldots\ B_s \,]$, $B_\delta \equiv [\, e_1\ e_2\ \ldots\ e_s \,]$, and $B_{z_v} \equiv [\, W_1\ W_2\ \ldots\ W_s \,]$. Due to the well-poseness of (4), i.e., due to the fact that all auxiliary variables are uniquely defined for all (x_k, u_k, v_k), the knowledge of the initial state, disturbances and control inputs is sufficient for simulating the dynamic behaviour of the system. However, the computation of the optimal control sequences based on the prediction of future states, assuming that disturbances are unknown but with known bounds, is a much harder task due to the fact that the predicted state x_k is set-valued and non-convex. Thus, the typical approach of using the extreme disturbance realizations [Scokaert and Mayne, 1998] can not be directly applied in this case.

2.1 The Open-Loop Min-Max Optimal Control Problem

Consider the constrained discrete-time PWA system subject to bounded additive exogenous disturbances defined in (1)–(2). The finite horizon min-max optimal control problem for the disturbed PWA system under operational constraints is defined as follows [Kerrigan and Maciejowski, 2003].

Problem 1. The PWA formulation
Given an initial state x_t at time t and a final time $t+N$, find (if it exists) the control sequence $\mathbf{u}(t) \equiv \mathbf{u}_t^{N-1} \equiv (u'(0|t), u'(1|t), \ldots, u'(N-1|t))'$ which (*i*) transfers the state from x_t to a given final set $\mathcal{X}_f \subseteq \mathbb{X}$, which contains a final (target) nominal equilibrium state x_f, and (*ii*) minimizes the performance index

$$V_N(x_t, \mathbf{u}) \triangleq \max_{\mathbf{v} \in \mathbb{V}^N} J_N(x_t, \mathbf{u}, \mathbf{v}) \quad (5)$$

where $J_N(x_t, \mathbf{u}, \mathbf{v})$ is given by:

$$\sum_{k=0}^{N-1} \min_{a(k|t) \in \mathcal{X}_f} \|x(k|t) - a(k|t)\|_{Q,l} + \|u(k|t) - u_f\|_{R,l} \quad (6)$$

subject to:

$$x(0|t) = x_t \quad (7a)$$

$$x(k+1|t) = A_i x(k|t) + B_i u(k|t) + e_i + W_i v(k|t)$$
$$\text{for } \left[\begin{smallmatrix} x(k|t) \\ u(k|t) \\ v(k|t) \end{smallmatrix}\right] \in \Omega_i \quad (7b)$$

$$\left[\begin{smallmatrix} x(k|t) \\ u(k|t) \\ v(k|t) \end{smallmatrix}\right] \in \mathbb{C},\ \forall v(k|t) \in \mathbb{V}, \text{for } k = 0, \ldots, N-1$$
$$\text{for } \mathbb{C} \triangleq \left\{ \left[\begin{smallmatrix} x \\ u \\ v \end{smallmatrix}\right] : Kx + Lu + Mv \leq n \right\} \quad (7c)$$

$$x(N|t) \in \mathcal{X}_f, \quad \forall v(.|t) \in \mathbb{V}, \quad (7d)$$

where $x(k|t)$ represents the state trajectory, $\|x\|_{Q,l}$ and $\|u\|_{R,l}$ the l-norm of vector x and u weighted with matrices Q and R, respectively, with Q and R being full column rank matrices, and u_f is the steady-sate equilibrium input when $x(k|t) = x_f$ and disturbances are not present.

Problem 1 minimizes the worst-case performance cost (5)–(6) and robustly guarantee constraints (7), at all time steps k. It also penalizes the distance from a given final state-set $\mathcal{X}_f$, while the state at the end of the horizon is not penalized. This structure and properties of the stage cost, terminal cost, and terminal state-set are important to achieve closed-loop robust asymptotic stability of the MPC controlled system. For converting the PWA-based optimal control Problem 1 into an MLD-based framework, it is necessary to convert this *infinite*-dimensional min-max optimization problem into a *finite*-dimensional one.

The robust mode control strategy described in [Silva et al., 2004] assures that the mode of the system is "certain" regardless of the disturbances over a fixed horizon. As a consequence, for each possible "mode trajectory" the system behaves as a linear (affine) system, though time-variant, and so convex state-sets are generated. Besides, since the stage cost (5) of Problem 1 is a convex function, the technique presented in [Scokaert and Mayne, 1998] to convert a min-max problem into an equivalent convex program based on the linearity of the dynamic model and convexity of the stage cost and disturbance can be adopted here. In view of this, consider the following infinite-dimensional min-max optimization problem, where $\mathcal{U}$ and $\mathcal{V}$ are convex polytopes, and function $L(.,.)$ is convex: $\min_{u \in \mathcal{U}} \max_{v \in \mathcal{V}} L(u, v)$. Consider also that $q \in \mathcal{Q}_v$ indexes all extreme realizations of v, i.e. those disturbances v that take values at the vertices of the polytope $\mathcal{V}$, which are denoted by v^q. As $L(.,.)$ is assumed convex relatively to v, the above infinite-dimensional optimization problem is equivalent to the following finite-dimensional one:

$$\min_{u\in\mathcal{U}}\max_{q\in\mathcal{Q}_v} L\left(u, v^q\right) \tag{8}$$

The previous step was obtained by knowing that the maximum of a convex function L over a convex set $\mathcal{V}$ is at one of the vertices of $\mathcal{V}$ (see e.g. [Boyd and Vandenberghe, 2004]). In turn, the optimization problem (8) is also equivalent to the convex program:

$$\min_{u,\gamma}\left\{\gamma \mid u\in\mathcal{U}\,, L\left(u,v^q\right)\le\gamma\,,\forall q\in\mathcal{Q}_v\right\} \tag{9}$$

Based on the previous technique, the min-max Problem 1 is now converted into an equivalent finite-dimensional minimizing one, however restricted by the robust mode condition and based on the MLD framework. Consider system dynamics, operational constraints, and the robust mode condition represented within the MLD framework, i.e. equations (4). The robust mode min-max optimal control problem equivalent to Problem 1 is defined as follows,

$$J_N\left(x_t\right)\triangleq \min_{\mathbf{u},\mathbf{z_u},\boldsymbol{\delta},\mathbf{z_x},\mathbf{z_v},\mathbf{a},\gamma} \gamma \tag{10}$$

subject to $\forall q\in\mathcal{Q}_v$:

$$\sum_{k=0}^{N-1}\{\|x^q(k|t)-a^q(k|t)\|_{Q,l}+\|u(k|t)-u_f\|_{R,l}\}\le\gamma \tag{11a}$$

$$\begin{aligned}\mathbf{E_u u}+\mathbf{E_{z_u} z_u}+\mathbf{E}_{\boldsymbol{\delta}}\boldsymbol{\delta}+\mathbf{E_{z_x} z_x}+\mathbf{E_{z_v} z_v}\le&\\ \le\mathbf{E_v v}+\mathbf{E}_{x_t}x_t+\mathbf{E}&\end{aligned} \tag{11b}$$

$$\mathbf{a}\in\mathcal{X}_f^{m.N}\,, \tag{11c}$$

$$\begin{aligned}&x^q(k+1|t)=B_{z_x}z_x^q(k|t)+B_{z_u}z_u(k|t)+\\ &+B_\delta\delta(k|t)+B_{z_v}z_v^q(k|t)\,,\ x^q(0|t)=x_t\end{aligned} \tag{11d}$$

where $\mathbf{u}\triangleq(u'(0|t),\ldots,u'(N-1|t))'$, $\mathbf{z_u}\triangleq({z_u}'(0|t),\ldots,{z_u}'(N-1|t))'$, $\boldsymbol{\delta}\triangleq(\delta'(0|t),\ldots,\delta'(N-1|t))'$, $\mathbf{z_x}\triangleq({z_x^1}'(0|t),\ldots,{z_x^1}'(N-1|t),\ldots,{z_x^m}'(N-1|t))'$, $\mathbf{z_v}\triangleq({z_v^1}'(0|t),\ldots,{z_v^1}'(N-1|t),\ldots,{z_v^m}'(N-1|t))'$, and the vector of parameters relative to the extreme disturbance realizations is given as $\mathbf{v}\triangleq({v^1}'(0|t),\ldots,{v^1}'(N-1|t),\ldots,{v^m}'(N-1|t))'$ with $m=p^N$, where p is the number of vertices of $\mathbb{V}$. Consider $l=\infty$ in (11a). As an intermediate result, notice that if the stage cost L is based on the ∞-norm (i.e. $\|\,x\,\|_\infty=\max_{1\le i\le n}|x_i|$) such that

$$\begin{aligned}&L(x(k|t),u(k|t))=\\ &=\|x(k|t)-a(k|t)\|_{Q,\infty}+\|u(k|t)-u_f\|_{R,\infty} && (12a)\\ &=\|Q(x(k|t)-a(k|t))\|_\infty+\|R(u(k|t)-u_f)\|_\infty && (12b)\end{aligned}$$

then the value of $\min_{u\in\mathcal{U},x\in\mathcal{X}}L(x,u)$ can be obtained by solving the following linear program:

$$\begin{aligned}&\min_{u(k|t),x(k|t),\alpha(k|t),\beta(k|t)}\{\alpha(k|t)+\beta(k|t)\}\\ &\text{subject to}\\ &-\mathbf{1}\alpha(k|t)\le Q(x(k|t)-a(k|t))\le\mathbf{1}\alpha(k|t)\\ &-\mathbf{1}\beta(k|t)\le R(u(k|t)-u_f)\le\mathbf{1}\beta(k|t)\end{aligned} \tag{13}$$

where $\alpha(k|t),\beta(k|t)\in\mathbb{R}$, $u(k|t)\in\mathcal{U}$, $x(k|t)\in\mathcal{X}$. Applying this technique, the MLD-based optimal control can be formulated as follows.

Problem 2. The MLD formulation with ∞-norm
Given an initial state x_t at time t and a final time $t+N$, find (if it exists) the control sequence $\mathbf{u}$, and the auxiliary variables $\boldsymbol{\delta}$, $\mathbf{z_u}$, $\mathbf{z_x}$, $\mathbf{z_v}$, $\mathbf{a}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and γ which (*i*) transfer the state from x_t to a given final set $\mathcal{X}_f$, that contains a final (target) nominal equilibrium state x_f and (*ii*) that minimizes the performance index

$$J_N\left(x_t,\mathbf{u},\mathbf{z_u},\boldsymbol{\delta},\mathbf{z_x},\mathbf{z_v},\mathbf{a},\boldsymbol{\alpha},\boldsymbol{\beta},\gamma\right)\quad\triangleq\quad\gamma \tag{14}$$

subject to $\forall q\in\mathcal{Q}_v$, $\forall\,k=0,\ldots,N-1$:

$$\begin{aligned}&x^q(0|t)=x(0|t)=x_t && (15a)\\ &x^q(k+1|t)=B_{z_x}z_x^q(k|t)+B_{z_u}z_u(k|t)+\\ &\qquad+B_\delta\delta(k|t)+B_{z_v}z_v^q(k|t) && (15b)\\ &E_{z_x}z_x^q(k|t)+E_{z_u}z_u(k|t)+E_\delta\delta(k|t)+E_{z_v}z_v^q(k|t)\le\\ &\qquad\le E_xx^q(k|t)+E_uu(k|t)+E_vv^q(k|t)+e && (15c)\\ &x^q(N|t)\in\mathcal{X}_f && (15d)\\ &-\mathbf{1}\alpha^q(k|t)\le Q\left(x^q(k|t)-a^q(k|t)\right)\le\mathbf{1}\alpha^q(k|t) && (15e)\\ &-\mathbf{1}\beta(k|t)\le R\left(u(k|t)-u_f\right)\le\mathbf{1}\beta(k|t) && (15f)\\ &\sum_{k=0}^{N-1}\alpha^q(k|t)+\beta(k|t)\le\gamma && (15g)\\ &a^q(k|t)\in\mathcal{X}_f\,,\ \forall q\in\mathcal{Q}_v^j && (15h)\end{aligned}$$

where $\mathbf{a}\triangleq({a^1}'(0|t),\ldots,{a^1}'(N-1|t),\ldots,{a^m}'(N-1|t))'$, $\boldsymbol{\alpha}\triangleq({\alpha^1}'(0|t),\ldots,{\alpha^1}'(N-1|t),\ldots,{\alpha^m}'(N-1|t))'$, $\boldsymbol{\beta}\triangleq(\beta'(0|t),\ldots,\beta'(N-1|t))'$, Q and R are full column rank matrices, and u_f is the steady-state equilibrium input for $x(k|t)=x_f$ when disturbances are not present.

Problem 2 defines a Mixed-Integer Linear Program (MILP). By substitution of equality (15b) into inequalities (15c)–(15e), $k=1\ldots N$, state variables are eliminated and a compact notation for Problem 2 is obtained:

$$\begin{aligned}&\min_{\boldsymbol{\sigma}}\ \gamma\\ &\text{subject to: }\boldsymbol{E}_\sigma\boldsymbol{\sigma}\le\boldsymbol{E}_{x_t}x_t+\boldsymbol{E}\end{aligned}$$

where, $\boldsymbol{\sigma}\triangleq\left(\mathbf{u}',{\mathbf{z_u}}',\boldsymbol{\delta}',{\mathbf{z_x}}',{\mathbf{z_v}}',\mathbf{a}',\boldsymbol{\alpha}',\boldsymbol{\beta}',\gamma\right)'$.

3. THE RECEDING HORIZON CONTROL STRATEGY

The solution of the open-loop min-max robust mode optimal control Problem 2 can be obtained using a Branch-&-Bound (B&B) based algorithm to solve the correspondent mixed-integer optimization. However, due to the min-max formulation with ∞-norm, each subproblem associated to the nodes of the B&B tree defines now a Linear Programming (LP) problem. Based on this solution a Receding Horizon Control (RHC) strategy can be implemented such that state-feedback is obtained. Therefore, consider the following Model Predictive Control algorithm.

Algorithm 1. **Model Predictive Control Algorithm**.

(1) Read state at time t, denoted by $x(t)$, and set $x(0|t) = x_t = x(t)$.
(2) Solve Problem 2 with the B&B based algorithm, and obtain the optimal input sequence $\mathbf{u}_t^*$.
(3) Apply the first component of $\mathbf{u}_t^*$ to the hybrid system, i.e. apply $u(t) = u^*(0|t)$.
(4) When sampling time is reached, $t=t+1$, go to 1.

So, based on the MLD prediction model, at each time step t the controller selects the first sample of the optimal sequence of future input actions through an online optimization procedure, which aims at minimizing a worst-case cost index, and enforces fulfilment of the constraints and of the robust mode condition for all possible disturbances. At the next sampling time, i.e. at time $t+1$, a new optimal sequence is evaluated to replace the previous one, providing the desired feedback control structure. However, the use of a finite-horizon implies particular attention to closed-loop stability.

3.1 *Robust Stability*

As the PWA/MLD system is subject to persistent disturbances the system must be steered to a target state-set. So, convergence to a final equilibrium state-set (or a desired reference trajectory tube) must be studied. Next, some important definitions to establish closed-loop robust stability are presented.

Definition 1. The pair (x_f, u_f) is said to be a *nominal equilibrium pair* of a completely well-posed time-invariant MLD system if there exists an equilibrium state $x_f \in \mathbb{R}^{n_x}$, equilibrium input $u_f \in \mathbb{R}^{n_u}$, and equilibrium auxiliary variables δ_f and z_f such that

$$x_f = A_x x_f + B_u u_f + B_z z_f + B_\delta \delta_f$$
$$E_x x_f + E_u u_f + E_z z_f + E_\delta \delta_f \leq e$$

is verified.

Notice that an equilibrium pair can be computed by solving a mixed integer program. Consider also the following definitions, which can be found in e.g. [Kerrigan and Mayne, 2002] and [Blanchini, 1999].

Definition 2. A set $\mathcal{X}_f$ is *robustly stable* iff, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x(0), \mathcal{X}_f) \leq \delta$ implies $d(x(i), \mathcal{X}_f) \leq \epsilon$, $\forall i \geq 0$ and all admissible disturbance sequences (where $d(z, \mathcal{Z}) \triangleq \min_{y\in\mathcal{Z}} \|z - y\|$, such that $\mathcal{Z} \subset \mathbb{R}^n$ and $\|.\|$ denotes any norm).

Definition 3. The set $\mathcal{X}_f$ is *robustly asymptotically (finite-time) attractive* with domain of attraction $\mathcal{X}$ iff for all $x(0) \in \mathcal{X}$, $d(x(i), \mathcal{X}_f) \to 0$ as $i \to \infty$ (there exists a time M such that $x(i) \in \mathcal{X}_f$, $\forall i \geq M$) for all admissible disturbance sequences.

Definition 4. The set $\mathcal{X}_f$ is *robustly asymptotically (finite-time) stable* with domain of attraction $\mathcal{X}$ iff it is robustly stable and robustly asymptotically (finite-time) attractive with domain of attraction $\mathcal{X}$.

Definition 5. The set $\mathcal{X}_f$ is *robustly positively invariant* for the system $x(k+1) = F(x(k), v(k))$ iff $\forall x(0) \in \mathcal{X}_f$ and $\forall v(k) \in \mathbb{V}$, the system behavior is such that $x(k) \in \mathcal{X}_f$, $\forall k \in \mathbb{N}$.

Definition 6. The set $\mathcal{X}_f$ is *robustly controlled invariant* for the system $x(k+1) = F(x(k), u(k), v(k))$ iff there exists a feedback control law $u(k) = \kappa_f(x(k))$ such that $\mathcal{X}_f$ is a robust positively invariant set for the closed-loop system $x(k+1) = F(x(k), \kappa_f(x(k)), v(k))$ and $u(k) \in \mathbb{U}$, $\forall x(k) \in \mathcal{X}_f$.

In order to prove robust stability of the PWA/MLD when the RHC strategy is applied, consider the following set of assumptions regarding the stage cost $L(.)$, the terminal cost $P(.)$, and the terminal state constraint $\mathcal{X}_f$.

Assumptions 1.

a) $L(x, u)$ is a convex function over $\mathbb{X} \times \mathbb{U}$ and there exists a $c > 0$ such that $L(x, u) \geq c\,(d(x, \mathcal{X}_f))$, $\forall (x, u) \in (\mathbb{X} \backslash \mathcal{X}_f) \times \mathbb{U}$.
b) The stage cost $L(x, u) = 0$ if $x \in \mathcal{X}_f$ and $u = u_f$.
c) The terminal cost $P(x) = 0$, $\forall x \in \mathbb{R}^{n_x}$.
d) The terminal state constraint $\mathcal{X}_f \subseteq \mathbb{X}$ is a compact convex polyhedron containing the final nominal state x_f in its interior.
e) If the nominal equilibrium pair (x_f, u_f) is such that $\begin{bmatrix} x_f \\ u_f \\ 0 \end{bmatrix} \in \Omega_j$ then $\begin{bmatrix} x \\ u_f \\ v \end{bmatrix} \in \Omega_j$, $\forall x \in \mathcal{X}_f, \forall v \in \mathbb{V}$.
f) The terminal state constraint $\mathcal{X}_f$ is robustly controlled invariant for $u = u_f \in \mathbb{U}$ and $\forall v \in \mathbb{V}$.

Based on the previous set of assumptions, the following theorem is presented.

Theorem 1.
Consider that Assumptions 1 hold for Problem 2, and that $\mathcal{X}_N$ is a non-empty set defined by all initial states x_t such that Problem 2 is feasible at time t. Then $\mathcal{X}_f$ is robustly asymptotically stable, with domain of attraction $\mathcal{X}_N$, for the closed-loop system when the MPC Algorithm 1 is applied.

The proof follows from considering standard Lyapunov arguments.

4. APPLICATION EXAMPLE

The approximate robust MPC strategy is applied to the perturbed two tanks hybrid system depicted in figure 1. For a complete description of the system characteristics see [Silva et al., 2004]. The control objective is to regulate levels h_1 and h_2 through the on-off bidirectional valve V_2 and flow $Q_4 = [0, Q_{4M}]$. The system is perturbed by a bounded disturbance flow Q_5. The PWA model of the system is obtained by first considering the partition of the state-space

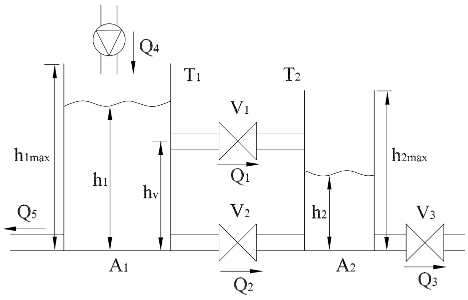

Fig. 1. The two tanks system.

(h_1, h_2) into four different regions, which have different dynamics dependent on the height of h_1, h_2, and h_v. Each one of the four state dependent partitions has different dynamics depending if valve V_2 is on or off, so the system is characterized by eight modes. At modes 1–4 valve V_2 is fully open, while at modes 5–8 it is completely closed. The optimal control problem to be solved is defined by the following parameters: $N = 3$, $Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $R = \left[\begin{smallmatrix} 100 & 0 \\ 0 & 100 \end{smallmatrix}\right]$, $x_0 = \left[\begin{smallmatrix} 1.0 \\ 0.5 \end{smallmatrix}\right]$, $x_f = \left[\begin{smallmatrix} 2.4655 \\ 1.8461 \end{smallmatrix}\right]$, $u_f = \left[\begin{smallmatrix} 0.0152 \\ 0.5 \end{smallmatrix}\right]$, where (x_f, u_f) is an equilibrium pair. The robust min-max MPC optimal input trajectory computed by Algorithm 1 resulted in the optimal state trajectory shown in figures 2 and 3, where $x_1{=}h_1$ and $x_2{=}h_2$. Notice that, as expected, the closed-loop converges to the robust invariant set, x_f.

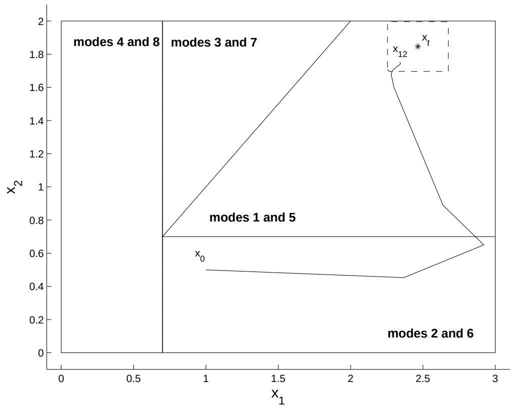

Fig. 2. State partition with optimal trajectory.

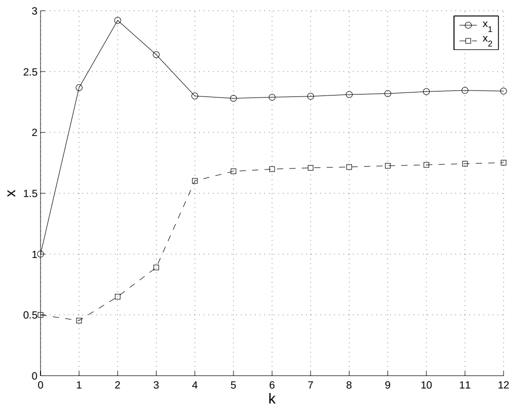

Fig. 3. State evolution.

5. CONCLUSIONS

This paper presented a receding horizon control algorithm for uncertain PWA/MLD systems subject to bounded additive disturbances that robustly guarantees stability and respect of operational constraints. A min-max formulation minimizes the worst-case cost of the performance index, while guaranteeing performance/safety constraints for all possible disturbance realizations. The conditions for robust stability and performance satisfaction were also presented. The algorithm was then validated in a simulation experiment of the perturbed hybrid two tanks system.

REFERENCES

P.J. Antsaklis. A brief introduction to the theory and applications of hybrid systems. *Proc. IEEE*, 88(7):879–886, 2000.

A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427, 1999.

F. Blanchini. Set invariance in control – a survey. *Automatica*, 35 (11):1747–1768, November 1999.

V. D. Blondel and J. N. Tsitsiklis. A survey of computational complexity results in systems and control. *Automatica*, 36(9): 1249–1274, 2000.

S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

M. S. Branicky, V. S. Borkar, and S. K. Mitter. A unified framework for hybrid control: model and optimal control theory. *IEEE Transactions on Automatic Control*, 43(1):31–45, 1998.

C.G. Cassandras and K. Gokbayrak. *Modeling, Control, and Optimization of Complex Systems*, chapter Optimal Control for Discrete Event and Hybrid Systems, pages 285–304. Kluwer, 2002.

C.G. Cassandras, D.L. Pepyne, and Y. Wardi. Optimal control of a class of hybrid systems. *IEEE Transactions on Automatic Control*, 46(3):398–415, 2001.

B. de Schutter and T.J.J. van den Boom. Model predictive control for max-min-plus-scaling systems. In *Proc. American Control Conference*, pages 319–324, Arlington, Virginia, USA, 2001.

S. Hedlund and A. Rantzer. Optimal control of hybrid systems. In *Proc. IEEE Conference on Decision and Control*, 1999.

W.P.M.H. Heemels, B. de Schutter, and A. Bemporad. Equivalence of hybrid dynamical models. *Automatica*, 37(7):1085–1091, 2001.

W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. Linear complementarity systems. *SIAM Journal on Applied Mathematics*, 60(4):1234–1269, 2000.

E.C. Kerrigan and J.M. Maciejowski. Robustly stable feedback min-max model predictive control. In *Proceedings of the 2003 American Control Conference*, Denver, Colorado, USA, June 2003.

E.C. Kerrigan and D.Q. Mayne. Optimal control of constrained, piecewise affine systems with bounded disturbances. In *Proc. IEEE Conference on Decision and Control*, Las Vegas, Nevada, USA, December 2002.

D.Q. Mayne and S. Rakovic. Model predictive control of constrained piecewise affine discrete-time systems. *International Journal of Robust and Nonlinear Control*, 13(3–4):261–279, 2003.

D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

P. O. M. Scokaert and D. Q. Mayne. Min-max feedback model predictive control. *IEEE Transations on Automatic Control*, 43 (8):1136–1142, 1998.

M.P. Silva, L. Pina, A. Bemporad, M. Ayala Botto, and J. Sá da Costa. Robust optimal control of linear hybrid systems: an mld approach. In *Proc. 6th Portuguese Conference on Automatic Control*, pages 208–213, Faro, Portugal, 2004.