STABILITY RESULTS FOR NETWORKED CONTROL SYSTEMS SUBJECT TO PACKET DROPOUTS

Christopher M. Kellett * **Iven M.Y. Mareels** ** **Dragan Nešić** **

* *The Hamilton Institute, National University of Ireland, Maynooth, Co. Kildare, Ireland*
** *Department of Electrical and Electronic Engineering, University of Melbourne, Victoria 3052, Australia*

Abstract: Stability and performance of networked control systems has been a recent area of interest in the control literature. The inclusion of a shared communication network between plant and controller inevitably leads to occasional random data loss. We provide novel results relating to the probability of such a system being globally asymptotically stable or input to state stable.

Keywords: Networked Control Systems, Packet Dropouts, Switched Systems

1. INTRODUCTION

Networked control systems are systems where the feedback loop is closed through a communication network in which information from source to destination is transmitted in finite packets. More often than not, packet based communication network protocols cannot guarantee that the transmit delay between source and destination is fixed or overbounded by a known limit nor can it be guaranteed that each packet will reach its destination. Indeed, most packet based communication networks work on a best effort basis, with very few performance guarantees. The performance and stability of such systems has received a great deal of attention in the recent control literature (see Walsh et al. (2002), Zhang et al. (2001), Tipsuwan and Chow (2003) and the references therein).

Motivated by this observation, we first consider the stability of a simple feedback situation modeled as follows

$$x_{k+1} = \alpha_k f_g(x_k) + (1 - \alpha_k) f_s(x_k) \qquad (1)$$

Here α_k is either 1, which corresponds to the situation that the feedback loop is closed (communication network delivering the packets regularly); or 0 which corresponds to the situation that the feedback loop is not closed (communication network dropping packets). When $\alpha_k = 1$ the system state x_k progresses as $x_{k+1} = f_g(x_k)$, a well behaved system for which it is assumed that the origin is (globally) asymptotically stable. If $\alpha_k = 0$ the system state progresses as $x_{k+1} = f_s(x_k)$, which is assumed to be the uncontrolled, not so well behaved system, for which the origin is still an equilibrium, but perhaps not a stable one.

In the sequel, the sequence α_k is modeled as a stochastic ergodic process. Its mean $\bar{\alpha}$ is a measure for the average performance of the communication network. The larger $\bar{\alpha}$ is, the better the communication network performs. Modeling a communication network in this manner is acceptable, and indeed most studies of IP based networks characterize network quality of service in a similar manner. In many ways this is a minimal model, as no other assumptions or knowledge about the α_k sequence will be required other than assuming it is an ergodic sequence and that its mean is known (or a bound for its mean is known). The first main result provides sufficient conditions under which almost any state sequence convergences to the trivial state.

Our work can be interpreted as a stochastic variation of the model exhibited in Hassibi et al. (1999), that could also be used to describe a networked control situation (as done in Zhang et al. (2001)). The present model only considers two possible states for the communication network unlike the work in Hassibi et al. (1999) which allows for a more comprehensive set of events, but it will transpire that the ideas presented here can easily be extended to the case where the system's transition dynamics are switched by a (finite state) Markov process. In fact, Theorem 1 recovers the result of Hassibi et al. (1999), but in a stochastic framework. Unfortunately, the result of Hassibi et al. (1999) provides no information on the transient behavior of trajectories. Theorem 2 provides a probabilistic statement related to the transient behavior of solutions of (1).

As a further extension of the stability consideration, we are also interested in studying systems that are subject to disturbances. Considering this situation gives us a better handle on performance issues in the control loop. Networked control systems subject to disturbances have been studied by Nešić and Teel (2003). To account for random data loss, we consider the system

$$x_{k+1} = \alpha_k f_g(x_k, d_k) + (1 - \alpha_k) f_s(x_k, d_k), \quad (2)$$

where we use the same conventions as before for the α_k process. The sequence d_k is a bounded sequence representing the disturbance acting on the system. As before, we assume $f_g(0,0) = f_s(0,0) = 0$. Theorem 3 provides a probabilistic statement on satisfying an input to state stability bound.

The paper is organized as follows: in Section 2 we present the specifics of our results. Sections 3, 4, and 5 contain the proofs of the main results.

2. RESULTS

We will denote the nonnegative integers by $\mathbb{Z}_{\geq 0}$. We will denote solutions of (1) from initial condition $x \in \mathbb{R}^n$ at time $k \in \mathbb{Z}_{\geq 0}$ by $\phi(k, x)$. Similarly, solutions of (2) from an initial condition $x \in \mathbb{R}^n$ at time $k \in \mathbb{Z}_{\geq 0}$, subject to the disturbance sequence $\{d_j\}_{j=0}^{k-1}$ will be denoted by $\phi(k, x, d)$.

We recall that a function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ ($\rho \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. We say that $\rho \in \mathcal{K}_\infty$ if, in addition to belonging to class-$\mathcal{K}$, the function is unbounded. A function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{L}$ if it is continuous, nonincreasing, and $\lim_{t\to\infty} \varphi(t) = 0$. Finally, a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if it is of class-$\mathcal{K}$ in its first argument and of class-$\mathcal{L}$ in its second argument.

Assumption 1. (1) There exist functions $\rho_1, \rho_2 \in \mathcal{K}_\infty$, $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and a constant $\lambda \in [0, 1)$ such that, for all $x \in \mathbb{R}^n$

$$\rho_1(|x|) \leq V(x) \leq \rho_2(|x|), \quad \text{and} \quad (3)$$

$$V(f_g(x)) \leq \lambda V(x). \quad (4)$$

(2) There exists a constant $L > 1$ such that, for all $x \in \mathbb{R}^n$

$$V(f_s(x)) \leq L V(x). \quad (5)$$

(3) The sequence $\{\alpha_k\}$ is ergodic. Let $\bar{\alpha} \in \mathbb{R}$ satisfy, almost surely,

$$\bar{\alpha} = \lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} \alpha_k. \quad (6)$$

Note that with α_k ergodic and taking values in $\{0, 1\}$, we necessarily have that $\bar{\alpha} \in (0, 1)$. (In particular, $\bar{\alpha} \neq 1$ and $\bar{\alpha} \neq 0$.)

Remark 1. We see that the function $V(\cdot)$ is a Lyapunov function demonstrating global asymptotic stability of the origin for the equation $x_{k+1} = f_g(x_k)$. In fact, the existence of a smooth function $V(\cdot)$ satisfying (3) and (4) follows from global asymptotic stability of the origin for $x_{k+1} = f_g(x_k)$ and continuity of $f_g(\cdot)$ (see Kellett and Teel (2004)).

We also see that an $L > 1$ satisying (5) exists so long as $f_s(x)$ is defined for all $x \in \mathbb{R}^n$. □

Theorem 1. Suppose Assumption 1 holds and

$$\lambda^{\bar{\alpha}} L^{1-\bar{\alpha}} < 1. \quad (7)$$

Then almost every solution of (1) converges to the origin as $k \to \infty$.

Remark 2. We observe that (7) is identical to the condition in Hassibi et al. (1999). Restricting Hassibi et al. (1999) to two subsystems, they consider two systems

$$x_{k+1} = f_g(x_k), \quad \text{and} \quad x_{k+1} = f_s(x_k)$$

which govern the overall system behaviour with a certain amount of time r_g in the good system and a certain amount of time r_s in the other. Their requirement is then that there exist a function V and constants $\alpha_g, \alpha_s > 0$ such that

$$V(f_g(x_k)) \leq \alpha_g^{-2} V(x_k), \quad \text{and}$$
$$V(f_s(x_k)) \leq \alpha_s^{-2} V(x_k)$$

and $\alpha_g^{r_g}\alpha_s^{r_s} > 1$. That these conditions yield the same result follows from taking

$$\alpha_g^{-2} = \lambda, \quad \alpha_s^{-2} = L, \quad r_s = 1 - r_g, \quad \text{and} \quad r_g = \bar{\alpha}.$$

Note that the restriction to two subsystems is simply for the sake of an easy comparison. □

In addition to the convergence acheived in Theorem 1, by slightly restricting (7), we obtain a probability bound on the instantaneous behaviour of the solutions to (1).

Theorem 2. Suppose Assumption 1 holds and that there exists a constant $\delta > 0$ such that

$$\lambda^{\bar{\alpha}} L^{1-\bar{\alpha}} < e^{-\delta}. \tag{8}$$

Then there exists a constant $\eta > 0$ such that, for every $\varepsilon > 0$ there exists a function $\beta_\varepsilon \in \mathcal{KL}$ such that, for all $k \in \mathbb{Z}_{\geq 0}$

$$\Pr\left\{|\phi(k,x)| > \beta_\varepsilon(|x|,k)\right\} \leq \min\left\{\varepsilon, e^{-\eta k}\right\}. \tag{9}$$

The characterization of global asymptotic stability via $\mathcal{KL}$ bounds has become increasingly common in the literature. The result of Theorem 2 then states that the probability of exceeding a $\mathcal{KL}$-bound, under appropriate conditions, is vanishingly small as time goes to infinity. Alternatively, the probability of the sequence satisfying the given KL-bound is arbitrarily close to one for all time and goes to one as time goes to infinity.

We would like a similar result for switching between an input to state stable (ISS) system and one which is not. However, such a result is not possible. Particularly, we cannot expect that the probability of satisfying an ISS bound will go to one. There will always be some residual probability that trajectories exceed any given bound. Consider, for instance, that the state has converged to the ISS disturbance level; i.e.,

$$|\phi(k,x,d)| \leq \gamma(||d||).$$

If the system simultaneously experiences several sequential large disturbances *and* is operating in the "bad" (non-ISS) mode when those disturbances arrive, the state will exceed any bound set a priori. As a consequence, rather than expecting a probabilistic bound such as (cf. (9))

$$\Pr\left\{|\phi(k,x,d)| \leq \beta(|x|,k) + \gamma(||d||)\right\} \geq 1 - e^{-\eta k}$$

we expect to get a bound such as

$$\Pr\left\{|\phi(k,x,d)| \leq \beta(|x|,k) + \gamma(||d||)\right\} \geq 1 - e^{-\eta k} - \epsilon$$

where $\epsilon > 0$. Furthermore, we expect there will be some sort of trade-off between the size of the ISS gain γ and the parameter ϵ.

In order to deal with (2), we need to modify our assumption.

Assumption 2. (1) There exist functions $\rho_1, \rho_2 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and a constant $\lambda \in [0,1)$ such that, for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$

$$\rho_1(|x|) \leq V(x) \leq \rho_2(|x|), \quad \text{and} \tag{10}$$

$$V(f_g(x,d)) \leq \lambda V(x) + \sigma(|d|). \tag{11}$$

(2) There exists a constant $L > 1$ such that, for all $x \in \mathbb{R}^n$

$$V(f_s(x,d)) \leq LV(x) + \sigma(|d|). \tag{12}$$

(3) The sequence $\{\alpha_k\}$ is ergodic. Let $\bar{\alpha} \in \mathbb{R}$ satisfy, almost surely,

$$\bar{\alpha} = \lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} \alpha_k. \tag{13}$$

Theorem 3. Suppose that Assumption 2 holds and that there exists a constant $\delta > 0$ such that

$$\lambda^{\bar{\alpha}} L^{1-\bar{\alpha}} < e^{-\delta}. \tag{14}$$

Then there exists a constant $\eta > 0$ such that, for every $\varepsilon_1, \varepsilon_2 > 0$, there exist functions $\beta_{\varepsilon_1} \in \mathcal{KL}$, $\gamma_{\varepsilon_2} \in \mathcal{K}$ such that, for all $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \Pr\left\{|\phi(k,x,d)| > \beta_{\varepsilon_1}(|x|,k) + \gamma_{\varepsilon_2}(||d||)\right\} \\ \leq \min\left\{\varepsilon_1, e^{-\eta k}\right\} + \varepsilon_2. \end{aligned} \tag{15}$$

3. PROOF OF THEOREM 1

To show that (7) indeed gives convergence requires the following observation:

$$\begin{aligned} \sum_{k=0}^{T} (\alpha_k \log\lambda \; + \; (1-\alpha_k)\log L) \\ = \sum_{k=0}^{T} \log\left(\alpha_k\lambda + (1-\alpha_k)L\right). \end{aligned} \tag{16}$$

Now, (7) implies $\bar{\alpha}\log\lambda + (1-\bar{\alpha})\log L < 0$ so that

$$\log\lambda\left(\lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}\alpha_k\right) + \log L\left(1 - \left(\lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}\alpha_k\right)\right) < 0.$$

Appealing to (16) we can rewrite this as

$$\lim_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T}\log\left(\alpha_k\lambda + (1-\alpha_k)L\right) < 0$$

which implies that

$$\lim_{T\to\infty}\prod_{k=0}^{T}\left(\alpha_k\lambda + (1-\alpha_k)L\right) = 0. \tag{17}$$

We observe that

$$
\begin{aligned}
V(\phi(k+1,x)) &= \\
&V\left(\alpha_k f_g(\phi(k,x)) + (1-\alpha_k) f_s(\phi(k,x))\right) \\
&= \alpha_k V(f_g(\phi(k,x))) + (1-\alpha_k) V(f_s(\phi(k,x))) \\
&\le \alpha_k \lambda V(\phi(k,x)) + (1-\alpha_k) L V(\phi(k,x)) \\
&= \left(\alpha_k \lambda + (1-\alpha_k) L\right) V(\phi(k,x)) \\
&= \lambda^{\alpha_k} L^{1-\alpha_k} V(\phi(k,x)). \qquad (18)
\end{aligned}
$$

That $|\phi(k,x)| \to 0$ then follows from the upper and lower bounds in (3), (17), and (18). ∎

4. PROOF OF THEOREM 2

We will use the following simplified version of Hoeffding's inequality (see (Vidyasagar, 2003, pg. 26)):

Lemma 1. Suppose $\alpha_0, \ldots, \alpha_{k-1}$ form an ergodic sequence taking values in $\{0,1\}$ with mean $\bar{\alpha}$. Then, for any $\varepsilon > 0$ and $k \in \mathbb{Z}_{\ge 1}$ we have

$$
\Pr\left\{\sum_{i=0}^{k-1}(\bar{\alpha} - \alpha_i) \ge \varepsilon k\right\} \;\le\; \exp\left[-\frac{\varepsilon^2}{2}k\right]. \qquad (19)
$$

Remark 3. We note that the proof of Hoeffding's Inequality goes through, without modification, if we reverse the order of $\bar{\alpha} - \alpha_i$; i.e., if we instead consider summing the differences $\alpha_i - \bar{\alpha}$.

We mention that the following proof is not valid for $\lambda = 0$. A similar proof is available for the case where $\lambda = 0$, which we omit due to space constraints.

From (18) we see that the Lyapunov function evolves as

$$
V(\phi(k,x)) \le V(x) \prod_{j=0}^{k-1} \left(\lambda^{\alpha_j} L^{1-\alpha_j}\right). \qquad (20)
$$

Let $\eta \in \mathbb{R}_{>0}$, $k^* \in \mathbb{Z}_{\ge 0}$, and $M \in \mathbb{R}_{\ge 0}$ be defined as

$$
\eta := \frac{1}{2}\left(\frac{\delta}{\log(\lambda/L)}\right)^2, \qquad (21)
$$

$$
k^* := \min\left\{k \in \mathbb{Z}_{\ge 0} : k \ge -\frac{\log \varepsilon}{\eta}\right\}, \quad \text{and} \qquad (22)
$$

$$
M := \left(e^{\delta}\left(\frac{\lambda}{L}\right)^{\bar{\alpha}}\right)^{-k^*}. \qquad (23)
$$

We let $\Psi := e^{\delta}\lambda^{\bar{\alpha}} L^{1-\bar{\alpha}}$. Define the function $\beta_\varepsilon \in \mathcal{KL}$ as

$$
\beta_\varepsilon(s,k) := \rho_1^{-1}\left(\Psi^k M \rho_2(s)\right). \qquad (24)
$$

We see that, for all $k \ge 0$

$$
\begin{aligned}
&\Pr\{|\phi(k,x)| > \beta_\varepsilon(|x|,k)\} = \\
&\quad \Pr\left\{|\phi(k,x)| > \rho_1^{-1}\left(\Psi^k M \rho_2(|x|)\right)\right\} \\
&\le \Pr\left\{|\phi(k,x)| > \rho_1^{-1}\left(\Psi^k M V(x)\right)\right\} \\
&= \Pr\left\{\rho_1(|\phi(k,x)|) > \Psi^k M V(x)\right\} \\
&\le \Pr\left\{V(\phi(k,x)) > \Psi^k M V(x)\right\}. \qquad (25)
\end{aligned}
$$

We will consider two cases: $k < k^*$ and $k \ge k^*$. First the case where $k < k^*$. From the definition of M (23) and the constraint (8) we observe that, for $k < k^*$,

$$
\Psi^k M = \left(e^{\delta}\lambda^{\bar{\alpha}} L^{1-\bar{\alpha}}\right)^{k-k^*} L^{k^*} \ge L^{k^*}.
$$

Therefore, continuing (25), for $k < k^*$ we have

$$
\begin{aligned}
&\Pr\left\{V(\phi(k,x)) > \Psi^k M V(x)\right\} \\
&\qquad \le \Pr\left\{V(\phi(k,x)) > L^{k^*} V(x)\right\}.
\end{aligned}
$$

However, (20) implies that

$$
V(\phi(k,x)) \le L^k V(x), \quad \forall k \ge 0
$$

and, consequently,

$$
\Pr\left\{V(\phi(k,x)) > L^{k^*} V(x)\right\} = 0, \quad \forall k < k^*.
$$

In other words, for all $k < k^*$,

$$
\Pr\{|\phi(k,x)| > \beta_\varepsilon(|x|,k)\} = 0. \qquad (26)
$$

We now turn to the case where $k \ge k^*$. We note that,

$$
M = \left(\frac{L}{e^{\delta}\lambda^{\bar{\alpha}} L^{1-\bar{\alpha}}}\right)^{k^*} \ge 1.
$$

Returning to (25), we may write, for all $k \ge 0$

$$
\begin{aligned}
&\Pr\left\{V(\phi(k,x)) > \Psi^k M V(x)\right\} \\
&\le \Pr\left\{V(\phi(k,x)) > \Psi^k V(x)\right\} \\
&\le \Pr\left\{V(x) \prod_{j=0}^{k-1}\left(\lambda^{\alpha_j} L^{1-\alpha_j}\right) > \Psi^k V(x)\right\} \\
&= \Pr\left\{V(x)\left(\prod_{j=0}^{k-1}\left(\lambda^{\alpha_j} L^{1-\alpha_j}\right) - \Psi^k\right) > 0\right\}. \quad (27)
\end{aligned}
$$

Since $V(\cdot)$ is positive definite, we see that the above expression being greater than zero is equivalent to the question of whether or not

$$
\prod_{j=0}^{k-1}\left(\lambda^{\alpha_j} L^{1-\alpha_j}\right) > \prod_{j=0}^{k-1}\left(e^{\delta}\lambda^{\bar{\alpha}} L^{1-\bar{\alpha}}\right)
$$

Taking logarithms on both sides we see the above is equivalent to

$$
\sum_{j=0}^{k-1}(\bar{\alpha} - \alpha_j) > -\frac{\delta k}{\log(\lambda/L)}. \qquad (28)
$$

Note that, since $\frac{\lambda}{L} < 1$, the right hand side of (28) is positive. Therefore, appealing to Hoeffding's Inequality (Lemma 1), we see that

$$\Pr\left\{\sum_{j=0}^{k-1}(\bar{\alpha}-\alpha_j) > -\frac{\delta k}{\log(\lambda/L)}\right\} \le e^{-\eta k}. \quad (29)$$

Therefore, combining (27) and (29), we see that

$$\Pr\{|\phi(k,x)| > \beta_\varepsilon(|x|,k)\} \le e^{-\eta k}.$$

Now, with (22), we see that, for $k \ge k^*$,

$$\Pr\{|\phi(k,x)| > \beta_\varepsilon(|x|,k)\} \le e^{-\eta k} \le e^{-\eta k^*} \le \varepsilon. \quad (30)$$

Therefore, combining (26) and (30) we have, for all $k \in \mathbb{Z}_{\ge 0}$,

$$\Pr\{|\phi(k,x)| > \beta_\varepsilon(|x|,k)\} \le \min\left\{\varepsilon, e^{-\eta k}\right\}.$$

5. PROOF OF THEOREM 3

5.1 Definitions

We reuse many of the ideas from the proof of Theorem 2. Let $\eta \in \mathbb{R}_{>0}$, $k^* \in \mathbb{Z}_{\ge 0}$, and $M \in \mathbb{R}_{\ge 0}$ be defined as

$$\eta := \frac{1}{2}\left(\frac{\delta}{\log(\lambda/L)}\right)^2, \quad (31)$$

$$k^* := \min\left\{k \in \mathbb{Z}_{\ge 0} : k \ge -\frac{\log \varepsilon_1}{\eta}\right\}, \quad \text{and} \quad (32)$$

$$M := \left(e^{\delta}\left(\frac{\lambda}{L}\right)^{\bar{\alpha}}\right)^{-k^*}. \quad (33)$$

As before, we let $\Psi := e^{\delta}\lambda^{\bar{\alpha}}L^{1-\bar{\alpha}}$. Define the function $\beta_{\varepsilon_1} \in \mathcal{KL}$ as

$$\beta_{\varepsilon_1}(s,k) := \rho_1^{-1}\left(2\Psi^k M\rho_2(s)\right). \quad (34)$$

Let $c > 0$ be

$$-2\frac{(\log(\lambda/L))^2}{\delta}\ \log\left(\varepsilon_2(1-e^{-\eta})\right) \quad (35)$$

and define $\gamma_{\varepsilon_2} \in \mathcal{K}$ as

$$\gamma_{\varepsilon_2}(s) := \rho_1^{-1}\left(\frac{e^c}{1-\Psi}\ \sigma(s)\right), \quad \forall s \ge 0. \quad (36)$$

5.2 Divide and Conquer

We now calculate

$$\Pr\{|\phi(k,x,d)| > \beta_{\varepsilon_1}(|x|,k) + \gamma_{\varepsilon_s}(||d||)\}. \quad (37)$$

We first observe that

$$\frac{1}{2}\rho_1\left(\beta_{\varepsilon_1}(|x|,k)\right) \ge \Psi^k MV(x), \quad \text{and} \quad (38)$$

$$\frac{1}{2}\rho_1\left(\gamma_{\varepsilon_2}(||d||)\right) = \frac{e^c}{1-\Psi}\sigma(||d||). \quad (39)$$

It is easy to see that, for any $\rho \in \mathcal{K}_\infty$

$$\rho(s_1+s_2) \ge \frac{1}{2}\rho(s_1) + \rho(s_2), \quad \forall s_1, s_2 \ge 0. \quad (40)$$

Therefore, using (38), (39), (40), and the bounds (10), we may bound (37) as

$$\begin{aligned}
&\Pr\{|\phi(k,x,d)| > \beta_{\varepsilon_1}(|x|,k) + \gamma_{\varepsilon_2}(||d||)\} \\
&= \Pr\{\rho_1(|\phi(k,x,d)|) > \rho_1\left(\beta_{\varepsilon_1}(|x|,k) + \gamma_{\varepsilon_2}(||d||)\right)\} \\
&\le \Pr\{V(\phi(k,x,d)) > \rho_1\left(\beta_{\varepsilon_1}(|x|,k) + \gamma_{\varepsilon_2}(||d||)\right)\} \\
&\le \Pr\left\{V(\phi(k,x,d)) > \Psi^k MV(x) + \frac{e^c}{1-\Psi}\sigma(||d||)\right\} \\
&= \Pr\Big\{V(\phi(k,x,d)) - \Psi^k MV(x) \\
&\qquad - \frac{e^c}{1-\Psi}\sigma(||d||) > 0\Big\}. \qquad (41)
\end{aligned}$$

We observe that the ISS Lyapunov function evolves as

$$V(\phi(k,x,d)) \le V(x)\prod_{j=0}^{k-1}(\lambda^{\alpha_j}L^{1-\alpha_j}) + \sum_{j=0}^{k-1}\left(\prod_{i=j+1}^{k-1}\lambda^{\alpha_i}L^{1-\alpha_i}\right)\sigma(|d_j|) \quad (42)$$

We further observe that

$$\begin{aligned}
\frac{1}{1-\Psi}\sigma(||d||) &\ge \frac{1-\Psi^k}{1-\Psi}\sigma(||d||) \\
&= \sigma(||d||)\sum_{j=0}^{k-1}\Psi^j \\
&\ge \sum_{j=0}^{k-1}\sigma(|d_j|)\left(\prod_{i=j+1}^{k-1}\Psi\right) \quad (43)
\end{aligned}$$

Since $V(\cdot)$ is positive definite and $\sigma \in \mathcal{K}$, we see that a condition guaranteeing positivity of the quantity in (41) is given by the two conditions

$$\prod_{j=0}^{k-1}(\lambda^{\alpha_j}L^{1-\alpha_j}) \ge \prod_{j=0}^{k-1}(e^{\delta}\lambda^{\bar{\alpha}}L^{1-\bar{\alpha}}), \quad \text{and} \quad (44)$$

$$\sum_{j=0}^{k-1}\left(\prod_{i=j+1}^{k-1}\lambda^{\alpha_i}L^{1-\alpha_i}\right) \ge \sum_{j=0}^{k-1}e^c\left(\prod_{i=j+1}^{k-1}e^{\delta}\lambda^{\bar{\alpha}}L^{1-\bar{\alpha}}\right). \quad (45)$$

Using arguments similar to those used to prove Theorem 2, we may obtain

$$\Pr\left\{\prod_{j=0}^{k-1}(\lambda^{\alpha_j}L^{1-\alpha_j}) \ge \prod_{j=0}^{k-1}(e^{\delta}\lambda^{\bar{\alpha}}L^{1-\bar{\alpha}})\right\} \le \min\left\{\varepsilon_1, e^{-\eta k}\right\}. \quad (46)$$

We now turn our attention to (45).

5.3 Bounding Equation (45)

We see that a conservative bound for (45) would follow from requiring that each term be positive.

Let $\ell = i - j - 1$. Then, changing the limits on the product and taking logarithms on both sides, we see that we require

$$\sum_{\ell=0}^{k-j-2} \left(\alpha_{\ell+j+1} \log \lambda + (1 - \alpha_{\ell+j+1}) \log L\right) \geq c + \sum_{\ell=0}^{k-j-2} \left(\delta + \bar{\alpha} \log \lambda + (1 - \bar{\alpha}) \log L\right).$$

Rearranging terms this becomes

$$\sum_{\ell=0}^{k-j-2} (\bar{\alpha} - \alpha_{\ell+j+1}) \geq \frac{c + \delta(k-j-1)}{\log(\lambda/L)}.$$

We use Hoeffding's Inequality and the definition of c in (35) to write

$$\begin{aligned}
&\Pr\left\{ \sum_{\ell=0}^{k-j-2} (\bar{\alpha} - \alpha_{\ell+j+1}) > \frac{c + \delta(k-j-1)}{\log(\lambda/L)} \right\} \\
&\leq \exp\left[-\frac{1}{2(k-j-1)} \left(\frac{c + \delta(k-j-1)}{\log(\lambda/L)} \right)^2 \right] \\
&\leq \exp\left[-\frac{c\delta}{2(\log(\lambda/L))^2} \right] \exp\left[-\eta(k-j-1)\right] \\
&= \varepsilon_2(1 - e^{-\eta}) e^{-\eta(k-j-1)}.
\end{aligned}$$

In other words,

$$\Pr\left\{ \prod_{i=j+1}^{k-1} \lambda^{\alpha_i} L^{1-\alpha_i} > e^c \prod_{i=j+1}^{k-1} e^{\delta} \lambda^{\bar{\alpha}} L^{1-\bar{\alpha}} \right\} \leq \varepsilon_2(1 - e^{-\eta}) e^{-\eta(k-j-1)}.$$

Consequently, we may write

$$\begin{aligned}
\Pr&\left\{ \sum_{j=0}^{k-1} \left(\prod_{i=j+1}^{k-1} \lambda^{\alpha_i} L^{1-\alpha_i} \right) > \sum_{j=0}^{k-1} e^c \left(\prod_{i=j+1}^{k-1} \Psi \right) \right\} \\
&\leq \sum_{j=0}^{k-1} \varepsilon_2(1 - e^{-\eta}) e^{-\eta(k-j-1)} \\
&= \varepsilon_2(1 - e^{-\eta}) \frac{1 - e^{\eta k}}{1 - e^{-\eta}} \leq \varepsilon_2. \qquad (47)
\end{aligned}$$

It is easy to see that, for fixed $A, B > 0$

$$\Pr\{a + b \geq A + B\} \leq \Pr\{a \geq A\} + \Pr\{b \geq B\}.$$

Therefore, combining (41), (46), and (47) we obtain (15). □

REFERENCES

Arash Hassibi, Stephen P. Boyd, and Jonathan P. How. Control of asynchronous dynamical systems with rate constraints on events. In *Proceedings of the 38th Conference on Decision and Control*, December 1999.

Christopher M. Kellett and Andrew R. Teel. Smooth Lyapunov functions and robustness of stability for difference inclusions. *Systems and Control Letters*, 52(5):395–405, 2004.

Dragan Nešić and Andrew R. Teel. $\mathcal{L}_p$ stability of networked control systems. In *Proceedings of the 39th IEEE Conference on Decision and Control*, Maui, Hawaii USA, December 2003.

Yodyium Tipsuwan and Mo-Yuen Chow. Control methodologies in networked control systems. *Control Engineering Practice*, 11:1099–1111, 2003.

M. Vidyasagar. *Learning and Generalization: With Applications to Neural Networks.* Springer, 2003.

G. C. Walsh, H. Ye, and L. G. Bushnell. Stability analysis of networked control systems. *IEEE Transactions on Control Systems Technology*, 10(3):438–446, 2002.

Wei Zhang, Michael S. Branicky, and Stephen M. Phillips. Stability of networked control systems. *Control Systems Magazine*, 21(1):84–99, 2001.