CONTROLLERS FOR AMPLITUDE LIMITED MODEL ERROR MODELS

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Abstract: In this paper, systems where information about model accuracy is contained in a model error model are considered. The validity of such a model is typically restricted to input signals that are limited in amplitude. It is then natural to require the same amplitude restriction when designing controllers. The resulting implications for controller design are investigated in both the continuous and the discrete time case. *Copyright © 2005 IFAC*

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1. INTRODUCTION

Much of control theory centers around the problem of designing a regulator for a system based on an approximate and uncertain model. There are many useful, classical, results for the case when the model error and uncertainty can be expressed as bounds on the model's Nyquist curve, thus (implicitly) assuming that the true system is linear, e.g. (Zhou *et al.*, 1996).

The common case where the true system may be nonlinear and the design is based on a linear model has also been treated extensively. Among many relevant references we may mention (Yakubovich, 1964; Megretski and Rantzer, 1997; Vidyasagar, 1993).

In (Glad *et al.*, 2004) the following situation was considered. A nominal model $\mathcal{S}$ of the system is constructed, e.g. using system identification. Such a model cannot exactly describe the system and thus there will be model errors and disturbances that give discrepancies from the model's predictions. The errors or residuals are denoted by

$$\varepsilon(t) = y(t) - y_s(t) \tag{1}$$

where y_s is the output of the model $\mathcal{S}$. To be able to say something about the quality of a controller based

on the nominal model $\mathcal{S}$, we must assume something about these residuals.
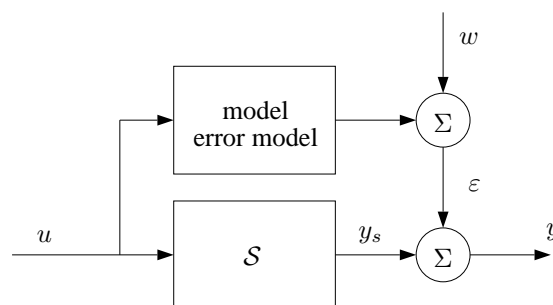

Fig. 1. Block diagram of the model error model.

A typical assumption, illustrated in Figure 1, is that ε has one part that originates from the input u (a "model error") and one part that is due to disturbances $w(t)$:

$$\varepsilon(t) = g_t(u^t) + w(t) \tag{2}$$

Here $u^t = \{u(\tau), \ \tau \le t\}$ and g_t is a representation of the model error model. Much work has been devoted to the problem of separating the two effects in (2), e.g. (Goodwin *et al.*, 1992; Ljung, 1999; Smith and Doyle, 1992; Smith and Dullerud, 1996; Hakvoort and van den Hof, 1997; Bombois *et al.*, 2000). Often the model error is assumed to be linear (i.e., assuming the true system to be linear):

$$g_t(u^t) = \tilde{G}(q)u(t) \qquad (3)$$

which allows some conventional model validation techniques to be used, e.g. (Ljung and Guo, 1997), or more sophisticated Toeplitz operator methods, e.g. (Smith and Dullerud, 1996) and (Poolla *et al.*, 1994). In that case, the error model naturally becomes global in the amplitude of u.

If the unmodeled part of the system is nonlinear, it will often have a large gain if a gain definition without an offset term is used. Hence, we propose to use an affine version of the gain definition. This has been done previously in the nonlinear control literature, see e.g. (Vidyasagar, 1993). We use a power gain definition similar to (Dower and James, 1998; Dower, 2000) and say that the nonlinear system has gain β with offset α if there are positive constants α and β such that

$$||\varepsilon||_T \le \alpha\sqrt{T} + \beta||u||_T \qquad (4)$$

for all positive T. Here $||\cdot||_T$ denotes the following truncated norm:

$$||z||_T^2 = \int_0^T z(t)^T z(t) \ dt$$

The form of the term $\alpha\sqrt{T}$ makes it possible to model a component in g_t or w that is unknown but has a known upper bound.

In (Glad *et al.*, 2004) linear models were used to represent $\mathcal{S}$. Here we introduce more general nonlinear models of the form

$$\dot{x}_s = f_s(x_s, u), \quad y_s = h_s(x_s, u), \quad x_s(0) = x_o \qquad (5)$$

Since nonlinear models might have drastically different behavior in different parts of the state space, we have also introduced the initial state in the model. This means that the model error model has to take into account uncertainties resulting from imprecise knowledge of the initial state. It is then natural to replace the term $\alpha\sqrt{T}$ in (4) with a general function $\alpha(T)$. As was argued in (Glad *et al.*, 2004), it is often natural to assume that the model error model is valid only in a certain amplitude range for the input u. In particular we have this situation when the model is identified from data, and the validation has only been performed for certain classes of inputs. We will investigate a fairly general form of input constraints and assume that $u(t)$ belongs to a certain set U for every t. The model error model is then of the form

$$u(t) \in U, \ t \le T \ \Rightarrow ||\varepsilon||_T \le \alpha(T) + \beta||u||_T \qquad (6)$$

2. GAIN REQUIREMENTS

Suppose a design of a controller is done for the nominal model $\mathcal{S}$. This controller would compute a control u from the output y in such a way that $u(t) \in U$ is satisfied. This controller together with the model $\mathcal{S}$

would give the control u for an input ε. Suppose this design satisfies

$$||u||_T \le \alpha_c(T) + \beta_c||\varepsilon||_T \qquad (7)$$

Then (7) and (6) together give an estimate of small gain theorem type:

$$\begin{aligned}||u||_T &\le \alpha_c(T) + \beta_c||\varepsilon||_T \\ &\le \alpha_c(T) + \beta_c(\alpha(T) + \beta||u||_T) \quad (8)\end{aligned}$$

If $\beta_c\beta < 1$ we have the estimate

$$||u||_T \le \frac{1}{1 - \beta_c\beta}(\alpha_c(T) + \beta_c\alpha(T))$$

3. CONTROLLER DESIGN

Consider the control problem as described by figure 2. Here y is the measured signal and z represents
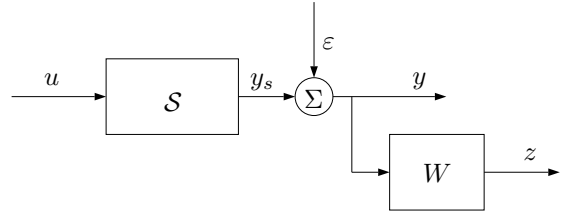

Fig. 2. Control problem

the controlled output, i.e., the goal of the control is to keep z small, despite the influence of ε. In linear H_∞ design, the W block would be used to shape the sensitivity function. We assume that the W block has a representation

$$\dot{x}_w = f_w(x_w) + g_w(x_w)y, \quad z = h_w(x_w) \qquad (9)$$

The system model $\mathcal{S}$ together with W is then given by

$$\begin{aligned}\dot{x} &= f(x, u) + n(x)\varepsilon, \quad u \in U \\ y &= h(x, u) + \varepsilon \qquad (10)\\ z &= m(x)\end{aligned}$$

where $x = [x_s^T, \ x_w^T]^T$ and

$$\begin{aligned}f(x, u) &= \begin{bmatrix} f_s(x_s, u) \\ f_w(x_w) + g_w(x_w)h_s(x_s, u) \end{bmatrix}, \\ n(x) &= \begin{bmatrix} 0 \\ g_w(x_w) \end{bmatrix}, \qquad (11)\\ h(x, u) &= h_s(x_s, u), \\ m(x) &= h_w(x_w).\end{aligned}$$

Note that (10) is only assumed to be valid for $u \in U$.

To get good control despite the influence of the model error ε we try to design a controller with $u \in U$ such that the gain from ε to the interesting variables is small. One way of doing that is to use the criterion

$$J_T = \int_0^T (z^T z + u^T u - \gamma^2 \varepsilon^T \varepsilon)dt$$

If we can find a control law such that $J_T \le 0$, we have achieved the inequality

$$||z||_T^2 + ||u||_T^2 \le \gamma^2||\varepsilon||_T^2$$

Obviously it is desirable to achieve this inequality for a value of γ that is as small as possible.

Using standard nonlinear H_∞ techniques (van der Schaft, 1992), J_T can be rewritten by subtracting $V(x(0))$ and using the following equality,

$$V(x(0)) = V(x(T)) - \int_0^T \frac{d}{dt} V(x(t)) \, dt,$$

where V is an arbitrary continuously differentiable function. If V is postulated to be positive semidefinite, one gets

$$J_T - V(x(0)) \leq$$
$$\int_0^T (z^T z + u^T u - \gamma^2 \varepsilon^T \varepsilon + V_x(f + n\varepsilon)) \, dt =$$
$$\int_0^T (z^T z + u^T u + V_x f + \frac{1}{4\gamma^2} V_x n n^T V_x^T$$
$$- \gamma^2 (\varepsilon - \frac{1}{2\gamma^2} n^T V_x^T)^T (\varepsilon - \frac{1}{2\gamma^2} n^T V_x^T)) \, dt$$
$$\leq \int_0^T (z^T z + u^T u + V_x f + \frac{1}{4\gamma^2} V_x n n^T V_x^T) \, dt$$

This leads to the following proposition.

Proposition 1. Suppose that there is a continuously differentiable positive semidefinite function V and a control law $u = k(x)$ such that $u \in U$ and the Hamilton-Jacobi inequality

$$m^T m + u^T u + V_x f + \frac{1}{4\gamma^2} V_x n n^T V_x^T \leq 0 \quad (12)$$

is satisfied for all x. Then the control law $u = k(x)$ gives a closed loop system that satisfies

$$||z||_T^2 + ||u||_T^2 \leq V(x(0)) + \gamma^2 ||\varepsilon||_T^2 \quad (13)$$

where $z = m(x)$.

So far the nominal design has been considered. Now it is possible to investigate what happens when the real system is controlled using the model error model.

Proposition 2. Let the controller of Proposition 1 be used for a system with model error model (6) and assume that $\gamma \beta < 1$. Then the closed loop dynamics, including the model error model, satisfies

$$||z||_T^2 \leq V(x(0)) + \frac{\gamma^2 \alpha^2(T)}{1 - \gamma^2 \beta^2} \quad (14)$$

Proof.

$$||z||_T^2 + ||u||_T^2$$
$$\leq V(x(0)) + \gamma^2 (\alpha(T) + \beta ||u||_T)^2 \quad (15)$$

Completing squares gives

$$||z||_T^2 + (1 - \gamma^2 \beta^2)(||u||_T - \frac{\gamma^2 \alpha(T) \beta}{1 - \gamma^2 \beta^2})^2$$
$$\leq V(x(0)) + \gamma^2 \alpha^2(T) + \frac{\gamma^4 \alpha^2(T) \beta^2}{1 - \gamma^2 \beta^2} \quad (16)$$

and deleting the positive term on the left hand side gives the result. $\square$

The result in Proposition 2 is that $||z||_T^2$ has an upper bound which is proportional to the square of the term $\alpha(T)$ in the model error model (6). The first term, $V(x(0))$, in (14) gives a bound on $||z||_T^2$ that holds if the nominal model is true ($\alpha = 0$ and $\beta = 0$), i.e., if there are no disturbances. If $\alpha(T) = \alpha_0 \sqrt{T}$ like in (4), (14) can be used to show that the average power of z will be bounded. Furthermore, for a fixed value of β, the upper bound on $||z||_T^2$ in (14) will become smaller if (12) can be solved for a smaller value of γ without changing $V(x(0))$.

4. ESTIMATING THE STATE

The control law $u = k(x)$ used in the previous section requires feedback from the full state x of the model. In principle we would thus require a state estimator whose dynamics would further complicate the analysis. However, the specific structure of (10), (11) makes it possible to use a particular approach. Rewriting (5) we have

$$\dot{x} = f(x, u) + n(x)(y - h(x, u)) \quad (17)$$

This dynamic system is its own observer, so in principle we could use the following observer-feedback configuration

$$\dot{\hat{x}} = f(\hat{x}, k(\hat{x})) + n(\hat{x})(y - h(\hat{x}, k(\hat{x}))) \quad (18)$$
$$u = k(\hat{x}) \quad (19)$$

In linear H_∞-design, this approach is well known, see e.g. (Glad and Ljung, 2000). In fact, writing out the sub-models of (18), we have

$$\dot{\hat{x}}_s = f_s(\hat{x}_s, u) \quad (20)$$
$$\dot{\hat{x}}_w = f_w(\hat{x}_w) + g_w(\hat{x}_w)y \quad (21)$$

Since the initial conditions of x_s and x_w are known, this is a pure simulation that will give arbitrarily small errors in x_s and x_w, provided the model $\mathcal{S}$ and the system W can be simulated with sufficient numerical accuracy.

5. AN EXAMPLE

To solve nonlinear H_∞ problems is a subject in itself and we will confine ourselves to a simple example. We assume that the nominal model is just a static gain:

$$y_s = u$$

All the dynamic behavior is thus incorporated into the model error model that is assumed to be valid for $|u| \leq 1$. We aim at a design where the sensitivity function when operating in the linear region is given by

$$S(s) = \frac{1}{b}(1 + s)$$

where b denotes the desired attenuation of disturbances at low frequencies. The sub-system W is then given by the transfer function

$$\frac{b}{s+1}$$

and the complete system given by $\mathcal{S}$ and W is

$$\dot{x} = -x + bu + b\varepsilon, \quad z = x$$

The Hamilton-Jacobi inequality becomes

$$x^2 - V_x x + V_x bu + u^2 + \frac{b^2}{4\gamma^2}V_x^2 \leq 0$$

In the region $|x| \leq 1/b$ the inequality is satisfied by

$$V = x^2, \quad u = -bx$$

provided γ is chosen as

$$\gamma^2 \geq \frac{1}{1 + 1/b^2} \tag{22}$$

For $|x| > 1/b$ the control becomes saturated, $u = -\text{sign}(x)$. Using the minimal value of γ in (22),

$$\gamma^2 = \frac{1}{1 + 1/b^2}, \tag{23}$$

the Hamilton-Jacobi inequality is

$$x^2 - V_x x - \text{sign}(x)V_x b + 1 + \frac{b^2 + 1}{4}V_x^2 \leq 0$$

Unfortunately, for this choice of γ and a quadratic V, it is impossible to satisfy this inequality for all $|x| > 1/b$. One possible remedy is to redefine m as

$$m(x) = \begin{cases} x & |x| \leq \dfrac{1}{b} \\ \dfrac{\sqrt{2bx - 1}}{b} & x > \dfrac{1}{b} \\ -\dfrac{\sqrt{-2bx - 1}}{b} & x < -\dfrac{1}{b} \end{cases} \tag{24}$$

With this redefinition of m we reduce the penalty for large amplitudes, realizing that it is not reasonable to require the same amount of disturbance rejection when the control saturates. With this definition of m, the following V is a solution

$$V = \begin{cases} x^2 & |x| \leq \dfrac{1}{b} \\ \dfrac{2bx - 1}{b^2} & x > \dfrac{1}{b} \\ -\dfrac{2bx + 1}{b^2} & x < -\dfrac{1}{b} \end{cases}$$

and the corresponding control is

$$u = \begin{cases} -bx & |x| \leq \dfrac{1}{b} \\ -\text{sign}(x) & |x| > \dfrac{1}{b} \end{cases}$$

From (23), it follows that a large b, corresponding to a large desired disturbance rejection, leads to a value of γ close to one, giving restrictions to the possible values of β satisfying $\beta\gamma < 1$. Intuitively this is not surprising: we need a more accurate model to build a regulator giving more disturbance rejection.

Instead of redefining $m(x)$ like in (24), a larger γ can be used. With $\gamma = b$, the Hamilton-Jacobi inequality is

$$x^2 - V_x x - \text{sign}(x)V_x b + 1 + \frac{1}{4}V_x^2 \leq 0 \tag{25}$$

when $|x| > 1/b$ and $u = -\text{sign}(x)$. Using $V = x^2$ in (25) gives

$$x^2 - 2x^2 - 2b|x| + 1 + x^2 = 1 - 2b|x| \leq 0$$

and this inequality is satisfied for all x with $|x| > 1/b$. Hence, by using a larger value of γ in the Hamilton-Jacobi inequality, a quadratic V (and a linear $m(x)$) can be used for all x. However, a larger value of γ puts harder restrictions on β for Proposition 2 to be applicable. On the other hand, a redefinition of the function $m(x)$ changes the relation between z and x and thus the interpretation of the bound (14) in terms of the state variable.

6. THE DISCRETE TIME CASE

The results presented so far have been for continuous time systems but similar results can be formulated in the discrete time case too. We will here briefly present the discrete time version of the nonlinear control design problem in Proposition 1 since it differs from its continuous time counterpart in some ways. The use of discrete time models for control design is interesting since many system identification methods produce such models. The simple model error model

$$u(t) \in U, \, t \leq N \Rightarrow ||\varepsilon||_N \leq \alpha(N) + \beta||u||_N \tag{26}$$

corresponds to the model error model (6) used previously in the continuous time case. Here, α is a function with $\alpha(N) \geq 0$ for all N, $\beta \geq 0$ is a constant and $|| \cdot ||_N$ denotes the truncated norm

$$||z||_N^2 = \sum_{t=0}^{N} z(t)^T z(t)$$

Assume that the nominal discrete time model $\mathcal{S}$, that can be used when $u(t) \in U$, together with a discrete time filter W can be written in state-space form as

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) + n(x(t))\varepsilon(t) \\ y(t) &= h(x(t), u(t)) + \varepsilon(t) \\ z(t) &= m(x(t)) \end{aligned} \tag{27}$$

Let

$$\begin{aligned} J_N = \sum_{t=0}^{N} (m(x(t))^T m(x(t)) + u(t)^T u(t) \\ - \gamma^2 \varepsilon(t)^T \varepsilon(t)) \end{aligned}$$

and consider the problem of finding $u(t) = k(x(t))$ such that $J_N \leq 0$. The following proposition is a discrete time version of Proposition 1 and can be useful in some applications.

Proposition 3. Consider the model (27) and suppose that there is a positive semidefinite function $V(x(t))$ and a control law $u = k(x)$ such that $u \in U$ and

$$m(x)^T m(x) + k(x)^T k(x) - \gamma^2 \varepsilon^T \varepsilon \\ + V(f(x, k(x)) + n(x)\varepsilon) - V(x) \leq 0 \qquad (28)$$

for all x and for all ε. Then the following inequality holds when $u(t) = k(x(t))$

$$||z||_N^2 + ||u||_N^2 \leq V(x(0)) + \gamma^2 ||\varepsilon||_N^2 \qquad (29)$$

for every signal ε.

Proof. Since the equality

$$V(x(0)) - V(x(N+1)) \\ + \sum_{k=0}^{N} (V(x(k+1)) - V(x(k))) = 0$$

holds for any function V, we have that

$$J_N - V(x(0)) \\ = \sum_{t=0}^{N} (m(x(t))^T m(x(t)) + k(x(t))^T k(x(t)) \\ - \gamma^2 \varepsilon(t)^T \varepsilon(t) + V(f(x(t), k(x(t))) + n(x(t))\varepsilon(t)) \\ - V(x(t))) - V(x(N+1)) \leq 0$$

where we have used (28) and the fact that V is positive semidefinite in the last inequality. The fact that

$$J_N - V(x(0)) \leq 0$$

implies that (29) holds for every ε. $\qquad \square$

For the discrete time robust control design problem in Proposition 3 to be solved, the inequality (28) must hold for all x and all ε. Hence, this problem might be harder than the corresponding continuous time problem in Proposition 1. However, if V is assumed to be a quadratic function, $V(x) = x^T P x$, for some choice of a symmetric, positive semidefinite matrix P, ε can be eliminated from (28). In this case, the two control design problems are more similar. Using a quadratic V such that $Q = (\gamma^2 I - n^T P n)$ is positive definite, we have

$$z^T z + u^T u - \gamma^2 \varepsilon^T \varepsilon + f^T P f + f^T P n \varepsilon \\ + \varepsilon^T n^T P f + \varepsilon^T n^T P n \varepsilon - x^T P x \\ = z^T z + u^T u + f^T P f \\ + f^T P n Q^{-1} n^T P f - x^T P x \\ - (\varepsilon - Q^{-1} n^T P f)^T Q (\varepsilon - Q^{-1} n^T P f) \\ \leq z^T z + u^T u + f^T P f \\ + f^T P n Q^{-1} n^T P f - x^T P x$$

Hence, the condition (28) in Proposition 3 is satisfied if a positive semidefinite matrix P can be found such that $Q(x, P) = \gamma^2 I - n(x)^T P n(x)$ is positive definite and

$$m(x)^T m(x) + k(x)^T k(x) \\ + f(x, k(x))^T P f(x, k(x)) \\ + f(x, k(x))^T P n(x) Q(x, P)^{-1} n(x)^T P f(x, k(x)) \\ - x^T P x \leq 0$$

for all x.

The result in Proposition 3 implies that it is possible to derive an upper bound on $||z||_N^2$ and $||u||_N^2$ which size depends on how large the disturbance ε is. This upper bound can be used to prove the following proposition, which is a discrete time version of Proposition 2.

Proposition 4. Let the controller of Proposition 3 be used for a system with model error model (26) and assume that $\gamma\beta < 1$. Then the closed loop dynamics, including the model error model, satisfies

$$||z||_N^2 \leq V(x(0)) + \frac{\gamma^2 \alpha^2(T)}{1 - \gamma^2 \beta^2} \qquad (30)$$

Proof. Analogous to the proof of Proposition 2. $\qquad \square$

Just like the corresponding continuous time result, Proposition 4 shows that $||z||_N^2$ has an upper bound which depends on the size of $\alpha(T)$ in the model error model (26). Furthermore, (30) shows how this upper bound depends on γ.

7. CONCLUSIONS

In this paper, it has been shown how the framework of model error models can be tied together with nonlinear H_∞ techniques to get a systematic design procedure based on identified models. The results have been presented both for continuous and discrete time systems. It is interesting to note that the presence of a set U of allowed control signal values gives great flexibility in the modeling, model validation and design. The set U might for instance contain only a finite number of points, corresponding to actuators of on-off type. The model error model needs then be valid only for those controls.

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