PHYSICAL CONSISTENCY OF THE HYSTERETIC BOUC-WEN MODEL ⋆

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Abstract: This paper deals with the determination of the conditions for which the hysteretic Bouc-Wen model is consistent with the physical phenomenon it is supposed to represent. In particular, we derive the conditions on the model parameters under which the model is stable, asymptotically dissipative and verifies the hysteretic property. These conditions are in the form of algebraic inequalities to be verified by the Bouc-Wen model parameters.

Keywords: Black-box modelling, physical properties, mathematical analysis, Bouc-Wen model.

1. INTRODUCTION

To obtain a mathematical description of physical systems, we may use the laws of physics to derive appropriate models. For example, we use Newton's laws in mechanical systems and Maxwell's laws in electromagnetic systems. The models obtained using the laws of physics often describe the behavior of the true systems with a reasonable accuracy. However, in many cases of practical relevance, deriving models using the laws of physics is not done either because the laws governing the behavior of the systems are not known or because the obtained models are too complex to be used meaningfully. Black-box modelling is an alternative to physical modelling as it uses simpler mathematical models that are supposed to approximate the behavior of the true process for a relevant range of input signals. The black-box modelling consists in choosing a structure of the model that does not come necessarily from the laws of physics. Then, using experimental input-output data, the parameters of the chosen model are tuned to mach the behavior of the experimental data. Once obtained a black-box model with numerical values of its parameters, it is necessary to validate it. The validation test consists generally in exciting the true process with an input test signal, and comparing the obtained experimental output with the one delivered by the model for the same input. If the two outputs are close in some appropriate sense for a relevant class of test inputs, then the model is claimed to be a "good" approximation of the physical system to be modelled.

In the case of black-box modelling, the parameters of the model may not have a physical meaning. For this reason, it may happen that the model matches a finite number of experimental input-output data without sharing some general physical properties with the real systems. Moreover, the set of parameters that describes the model input-output relationship may not be unique. This means that the experimental validation of the black-box models may not be enough to guarantee that these models represent adequately the physical behavior of the real system. Previous to this experimental validation, we should carry out a physical validation. This consists in using

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mathematical analysis to derive conditions to ensure a unique black-box model representation that shares some general physical properties with the true system. It is only when this step of physical validation has been completed that the step of experimental validation should take place.

In this paper we perform this physical validation on the so-called Bouc-Wen model (Wen, 1976) which is used in structural and mechanical engineering as a simple and effective model for a hysteretic behavior (Foliente, 1995). In this paper we use mathematical analysis in order to derive conditions on the Bouc-Wen model so that it is stable, dissipative, has a hysteretic behavior and is described by a unique set of parameters. We show that the physical validation process leads to a relevant information about the model parameters range.

2. THE BOUC-WEN MODEL

Consider a physical system with a hysteretic component that can be represented by a map $x(t) \mapsto \Phi_s(x)(t)$, which is referred to as the "true" hysteresis. The so-called Bouc-Wen model represents the true hysteresis in the form (Wen, 1976):

$$\Phi_{BW}(x)(t) = \alpha k x(t) + (1-\alpha) D k z(t) \quad (1)$$
$$\dot{z} = D^{-1}\left(A\dot{x} - \beta|\dot{x}|\,|z|^{n-1}z - \gamma\dot{x}|z|^{n}\right) \quad (2)$$

where $\dot{z}$ denotes the time derivative, $n \geq 1$, $D > 0$, $k > 0$ and $0 < \alpha < 1$. This model was originally developed in the context of mechanical systems in which x is a displacement and Φ_s is a restoring force (Bouc, 1967). It represents the hysteretic force $\Phi_s(x)(t)$ as the superposition of an elastic component $\alpha k x$ and a purely hysteretic component $(1-\alpha)kDz$, in which $D > 0$ is the yield constant displacement and $\alpha \in (0,1)$ is the post to pre-yielding stiffness ratio. The hysteretic part involves a nondimensional auxiliary variable z which is the solution of the nonlinear first order differential equation (2). In this equation, A, β and γ are nondimensional parameters which control the shape and the size of the hysteresis loop, while n is a scalar that governs the smoothness of the transition from elastic to plastic response. The Bouc-Wen model is able to capture, in an analytical form, a range of shapes of hysteretic cycles which match the behavior of a wide class of hysteretical systems (Smyth et al, 2002). It has been used experimentally to model piezoelectric elements (Low and Guo, 1995), magnetorheological dampers (Spencer et al, 1997) and wood joints (Foliente, 1995). The experimentally obtained models have been used either to predict the behavior of the physical hysteretic element (Spencer et al, 1997) or for control purposes (Chen et al, 1999).

3. STABILITY PROPERTIES OF THE MODEL

3.1 The bounded input-bounded output stability property

In this section we focus on the so-called bounded input-bounded output (BIBO) stability property of the model. By BIBO stability we mean that, for each bounded input, the output of the system is bounded. Experimental evidence shows that many hysteretic systems of practical relevance in mechanical and structural engineering are BIBO, that is their output is bounded for a bounded input. The Bouc-Wen model should reproduce this property to represent adequately the behavior of the true hysteresis. The physical validation consists in using mathematical analysis to derive the conditions under which this bounded input-bounded output property is verified by the Bouc-Wen model.

We state the problem under study as follows: given the parameters $0 < \alpha < 1$, $k > 0$, $D > 0$, A, β, γ and $n \geq 1$, find the set of initial conditions $z(0)$ for which the Bouc-Wen model (1)-(2) is BIBO. Note that when this set is empty, this means that the Bouc-Wen model is not BIBO. The solution of this problem will lead to classify different sets of parameters and initial conditions and, additionally, to determine explicit bounds for the hysteretic variable $z(t)$.

To this end, let us introduce the following sets:

$$\Omega_{\alpha,k,D,A,\beta,\gamma,n} = \{z(0) \in \mathbb{R} \text{ s.t } \Phi_{BW} \text{ is BIBO}\}$$
$$\Omega_{A,\beta,\gamma,n} = \{z(0) \in \mathbb{R} \text{ s.t } z(t) \text{ is bounded for any } C^1 \text{ bounded input } x(t)\}$$
$$\Omega^{\star}_{A,\beta,\gamma,n} = \{z(0) \in \mathbb{R} \text{ s.t } z(t) \text{ is bounded for any } C^1 \text{ input } x(t)\}$$

The following result derived in (Ikhouane et al, 2005) characterizes the different combinations of initial condition and parameters that lead to a Bouc-Wen model that is BIBO.

Theorem 1. Let $x(t)$, $t \in [0,\infty)$ be a C^1 input signal and

$$z_0 \triangleq \sqrt[n]{\frac{A}{\beta+\gamma}} \text{ and } z_1 \triangleq \sqrt[n]{\frac{A}{\gamma-\beta}}. \quad (3)$$

Then, the solution $z(t)$ of the differential equation (2) exists over $t \in [0,+\infty)$. Moreover, Table 1 holds.

Table 1. Classification of the BIBO Bouc-Wen models

CASE		$\Omega_{A,\beta,\gamma,n}$	$\lvert z(t)\rvert$ bound	Class
$A>0$	$-\beta<\gamma\leq\beta$	$\mathbb{R}$	$\max\left(\lvert z\left(0\right)\rvert, z_0\right)$	I
	$0\leq\beta<\gamma$	$[-z_1,z_1]$	$\max\left(\lvert z\left(0\right)\rvert, z_0\right)$	II
$A<0$	$-\beta\leq\gamma<\beta$	$\mathbb{R}$	$\max\left(\lvert z\left(0\right)\rvert, z_1\right)$	III
	$\gamma<-\beta\leq 0$	$[-z_0,z_0]$	$\max\left(\lvert z\left(0\right)\rvert, z_1\right)$	IV
$A=0$	$-\beta\leq\gamma\leq\beta$	$\mathbb{R}$	$\lvert z\left(0\right)\rvert$	V
ALL OTHER CASES		$\emptyset$		

Furthermore we have $\Omega^{\star}_{A,\beta,\gamma,n}=\Omega_{A,\beta,\gamma,n}=\Omega_{\alpha,k,D,A,\beta,\gamma,n}$

Theorem 1 shows that the Bouc-Wen model is BIBO only for the five classes of Table 1. For any combination of initial condition and parameters that does not belong to the classes I-V, the corresponding Bouc-Wen model delivers unbounded outputs for some bounded inputs, and thus, cannot describe a real physical hysteretic system.

This result shows that, using mathematical analysis, the process of physical validation led to an extra information on the model; namely that there are only five classes of Bouc-Wen model that may represent a physical behavior of the true hysteresis.

3.2 The equilibrium point stability property

In this section, the hysteretic system is assumed to be part of a second-order mechanical/structural system that has an equilibrium point at zero. The existence of an equilibrium point means that, if the initial conditions of the system are zero, then this system remains at zero for all times. In particular, the output of the true hysteresis should be identically zero. The physical validation uses the equations of the movement of the second-order system to show that, for the model to deliver a zero output, its initial condition is to be zero.

We consider a structural isolation scheme, as illustrated in Figure 1, which is modelled as 1 degree-of-freedom system with mass $m>0$ and viscous damping $c>0$ plus a restoring force Φ characterizing a hysteretic behavior of the isolator material. This system is described by the second order differential equation

$$m\ddot{x}+c\dot{x}+\Phi(x,t)=f(t), \tag{4}$$

with initial conditions $x(0)$ and $\dot{x}(0)$ and excited by a force $f(t)$, like the one of the form $-ma(t)$ in the case of an earthquake with ground acceleration $a(t)$. The restoring force is assumed to be described by the Bouc-Wen model:

$$\Phi(x)(t)=\alpha kx(t)+(1-\alpha)Dkz(t), \tag{5}$$

$$\dot{z}=D^{-1}\left[A\dot{x}-\beta|\dot{x}|\,|z|^{n-1}z-\gamma\dot{x}|z|^{n}\right], \tag{6}$$

where $n\geq 1$, $D>0$, $k>0$ and $0<\alpha<1$. Consider the system (4)-(6) with $x(0)=0$,

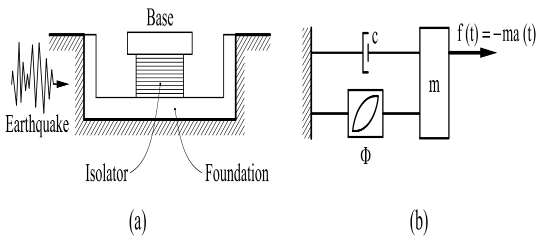

Fig. 1. Hysteretic isolation scheme (a) and its physical model (b).

$\dot{x}(0)=0$, $f(t)\equiv 0$ and assume that $z(0)\neq 0$. By continuity of the solutions of (4)-(6), the signal $z(t)$ will be nonzero at least during some time interval $[0,t_1)$. This implies that in the time interval $(0,t_1)$ the signals $x(t)$ and $\dot{x}(t)$ are not identically zero. In this case, the Bouc-Wen model has delivered non-identically zero signals x and $\dot{x}$ starting from zero initial conditions $x(0)=0$ and $\dot{x}(0)=0$. This will unlikely be the case for the real hysteresis as, in general, the coordinates are chosen in such a way that the point with $x(0)=0$ and $\dot{x}(0)=0$ is an equilibrium position for the real hysteretic system under free motion. With $z(0)$ set to zero, it can be seen from Table 1 that the hysteretic part $z(t)$of the model is always zero for the class V. This means that this class is irrelevant in practice.

4. THE ENERGY DISSIPATION PROPERTY

In this section, we consider that the hysteretic system is part of a second-order mechanical/structural system in free motion as in Figure 1. An in-depth study of the thermodynamic properties of the Bouc-Wen model using the concept of internal energy has been done in reference (Erlicher and Point, 2004). In this section, we are specifically interested in the dissipation of the mechanical energy associated to the movement of the isolated mass. In this case, we cannot expect that the mechanical energy will be decreasing all the time. However, we expect from the true hysteretic system to verify that the mechanical energy at any time is no more than its initial mechanical energy. This property needs to be verified by the Bouc-Wen model in order to be considered as an adequate model candidate.

Write the system (4)-(6) as

$$m\ddot{x}+c\dot{x}+\alpha kx+(1-\alpha)Dkz=0 \tag{7}$$

where z is the solution of the differential equation (6). At each instant t, the total energy $E(t)$ of (7) is the sum of its kinetic energy $\frac{1}{2}m\dot{x}(t)^2$ and its potential elastic energy $\frac{1}{2}\alpha kx(t)^2$. That is

$$E(t) = \frac{1}{2}m\dot{x}(t)^2 + \frac{1}{2}\alpha kx(t)^2. \quad (8)$$

We introduce some definitions.

Definition 1. The Bouc-Wen model defined by its parameters $(A, \beta, \gamma, n, D, \alpha, k)$ is said to be asymptotically dissipative if for every initial conditions $(x(0), \dot{x}(0))$, the final energy of the system (4)-(6) is less than its initial energy. That is we have $E(\infty) < E(0)$ whenever $E(0) \neq 0$.

Definition 2. A class of the Table 1 is said to be asymptotically dissipative if all its elements are asymptotically dissipative.

Definition 3. The Bouc-Wen model defined by its parameters $(A, \beta, \gamma, n, D, \alpha, k)$ is said to generate energy if there exist some initial conditions $(x(0), \dot{x}(0))$ and some finite time t_0 such that $E(t_0) > E(0)$.

With the definitions above we now state the following result (Ikhouane et al, 2004*a*).

Theorem 2. Consider the classes I-V of Table 1. Then, we have the following:

(i) The classes I and II are asymptotically dissipative.
(ii) The classes III and IV contain an infinite number of elements that generate energy.

Theorem 2 shows that the classes I and II are indeed asymptotically dissipative and thus may represent the physical behavior of a true hysteresis. The classes III and IV contain an infinite number of elements which generates energy. This means that both classes are of little practical interest.

The physical validation of the model gave us useful information on its parameters. They should be chosen in such a way that the model is BIBO and dissipative. Due to the fact that the physical validation of the Bouc-Wen has only been done recently (Ikhouane et al, 2004*a*), some previous works used incorrect values for the model parameters. For example, in (Smyth et al, 1999) the following values have been used: $\beta = 0.1$ and $\gamma = -1$, while (Kyprianou et al, 2001) used the values $\beta = 1.5$ and $\gamma = -1.5$. In both cases we do not have $\beta + \gamma > 0$ as should be the case for Classes I and II of Table 1. This means that both references used Bouc-Wen models that are not BIBO and thus do not describe a physical behavior.

5. THE UNIQUENESS OF THE DESCRIPTION

In this section we use the results obtained above to give a new description of the Bouc-Wen model. We show that this model uses more parameters than really needed, and we derive a description of the model that uses exactly the needed number of parameters.

In this section, we consider Bouc-Wen models that belong to classes I or II. As has been shown in the previous sections, they are BIBO and asymptotically dissipative, and thus may represent a physical reality. Consider two Bouc-Wen models (1)-(2) whose parameters are such that $n_2 = n_1 = n$, $A_2 = A_1$, $\beta_2 = \nu^n\beta_1$, $\gamma_2 = \nu^n\gamma_1$, $D_2 = \nu D_1$, $\alpha_2 = \alpha_1$, $k_2 = k_1$ where ν is a positive constant, and with an initial condition $z_2(0) = z_1(0) = 0$. It can be checked that both models belong to the same class according to Table 1, and for any input signal $x(t)$ they deliver exactly the same output $\Phi_{BW}(t)$. This means that the input-output behavior of a Bouc-Wen model is not described by a unique set of parameters $\{\alpha, k, D, A, \beta, \gamma, n\}$. For this reason, it is necessary to elaborate some equivalent "normalized" model whose parameters define in a unique way the input-output behavior of the model. To this end, define $w(t) = \frac{z(t)}{z_0}$ so that the model (1)-(2) can be written as:

$$\Phi_{BW}(x,t) = \kappa_x x(t) + \kappa_w w(t), \quad (9)$$

$$\dot{w}(t) = \rho\left(\dot{x}(t) - \sigma|\dot{x}(t)|\,|w(t)|^{n-1}w(t) + (\sigma - 1)\dot{x}(t)|w(t)|^n\right) \quad (10)$$

where

$$\rho = \frac{A}{Dz_0} > 0, \quad \sigma = \frac{\beta}{\beta+\gamma} \geq 0, \quad \kappa_x = \alpha k > 0,$$
$$\kappa_w = (1-\alpha)Dkz_0 > 0. \quad (11)$$

Note that since the Bouc-Wen model is of classes I or II, it follows from Table 1 that $\beta \geq 0$ and $\beta + \gamma > 0$. We call equations (9)-(10) the normalized form of the Bouc-Wen model. Note that since the initial condition $w(0)$ is such that $w(0) = 0$ then, by Theorem 1, $|w(t)| \leq 1$ for all $t \geq 0$. This means that the variable $z(t)$ has been scaled to unity. It can be checked that the normalized form of the Bouc-Wen model defines a bijective relationship between the input-output behavior of the model and its parameters. Note that the normalized form of the Bouc-Wen model is exactly equivalent to its standard form. Indeed, for any input $x(t)$, both forms deliver exactly the same output $\Phi_{BW}(t)$ taking into account that we have $w(0) = \frac{z(0)}{z_0} = 0$. The normalized model contains only five parameters while the standard one contains seven parameters. This means that two parameters of the standard model are superfluous. The class I of Table 1 corresponds to $\sigma \geq \frac{1}{2}$ while the class II corresponds to $0 \leq \sigma < \frac{1}{2}$. Note that considering that the Bouc-Wen model belongs to classes I or II was instrumental in getting $\rho > 0$ and $\sigma \geq 0$ in equation (11).

6. THE HYSTERETIC PROPERTY

In this section we use the fact that, in a real hysteretic system, the output depends on the sign of the derivative of the input. The Bouc-Wen model is to comply with this property to represent appropriately a hysteretic behavior.

6.1 Amplitude of the input

Consider that the input signal $x(t)$ is bounded and denote $X_{\max}$ its maximal amplitude in absolute value. If $X_{\max}$ is much larger than the ratio κ_w/κ_x, then it follows from equation (9) that the largest value $\kappa_x X_{\max}$ of the linear term $\kappa_x x(t)$ is much larger than the largest value κ_w of the nonlinear term $\kappa_w w(t)$. This means that the behavior of Φ_{BW} versus x becomes almost linear for large values of the input x and that the hysteretic term $\kappa_w w(t)$ will have some influence on $\Phi_{BW}(t)$ only for small values of the input $x(t)$. In particular, for large values of the input signal $x(t)$, the corresponding hysteretic output $\Phi_{BW}(t)$ will be independent of the sign of the derivative $\dot{x}(t)$. This behavior has not been reported experimentally for real hysteretic systems. For this reason, we consider that the Bouc-Wen model does not represent a physical hysteretic behavior if $X_{\max} \gg \frac{\kappa_w}{\kappa_x}$. We thus consider that the Bouc-Wen model may represent a physical behavior only for those signal inputs such that $X_{\max} \leq \frac{\kappa_w}{\kappa_x}$.

6.2 The case $\sigma = 0$

We show in this section that the value $\sigma = 0$ in equation (10) has to be rejected. To this end, the following result is useful (Ikhouane and Rodellar, 2004*b*):

Lemma 1. Take $x(t) = X_{\max}\cos(\omega t)$ for some $\omega > 0$, with $w(0) = 0$. Then we have $|w(t)| < 1$ for all $t \geq 0$.

Assume that we have $\sigma = 0$. Then equation (10) becomes

$$\dot{w}(t) = \rho\left(1 - |w(t)|^n\right)\dot{x}(t) \tag{12}$$

We define the function ξ as

$$\xi(v) = \int_0^v \frac{du}{1 - |u|^n} \tag{13}$$

for all scalars $-1 < v < 1$. We can see that ξ is a bijection from the interval $(-1, 1)$ to $\mathbb{R}$. Denoting ζ its inverse function it follows from equations (12)-(13) and Lemma 1, that we have $w(t) = \zeta\left(\rho x(t) + \eta\right)$ for all $t \geq 0$, where η is an integration constant. This means from equation (9) that the relationship between the hysteretic output Φ_{BW} and the input x does not depend on the sign of $\dot{x}$ which is contrary to the experimental observations for hysteretic systems. Thus the value $\sigma = 0$ does not correspond to the description of a physical hysteretic element.

7. SUMMARY OF THE OBTAINED RESULTS AND CONCLUSIONS

With the results above, we conclude that the Bouc-Wen model needs to be presented under its normalized form (9)-(10) so that its input-output behavior is described uniquely with its parameters. To represent a physical behavior, these parameters need to be such that $\rho > 0$, $\sigma > 0$, $\kappa_x > 0$, $\kappa_w > 0$ and $n \geq 1$. The initial condition should be $w(0) = 0$. This implies that the variable $w(t)$ satisfies $|w(t)| \leq 1$ for all $t \geq 0$. Moreover, the Bouc-Wen model is valid only for those inputs whose maximal value $X_{\max}$ (in absolute value) is such that $X_{\max} \leq \frac{\kappa_w}{\kappa_x}$.

In this paper we have first analyzed the stability of the Bouc-Wen model since most of the hysteretic systems, in practice, are BIBO stable. The result of the mathematical analysis is that only five classes of Bouc-Wen models are stable. Furthermore, the initial condition of the model has to be zero. We have then considered the energy dissipation property inherent to hysteretic systems. The physical validation has shown that only the classes I and II are asymptotically dissipative. A third physical property inherent to hysteretic systems is that the output depends on the sign of the derivative of the input. The mathematical analysis has shown that, for the Bouc-Wen model to comply with this property, the size of the input should be less than a value that depends on the model parameters. Moreover, one of the parameters cannot be zero. These findings have led to a normalized form of the Bouc-Wen model which has a minimal number of parameters that describe uniquely the input-output behavior.

The information that we have derived on the Bouc-Wen model did not necessitate any kind of experimental validation. It only used mathematical analysis in order to express the model in an appropriate form (the normalized one) and establish a range of validity both for the model parameters and the input signals. The next step would be an experimental validation. This means tuning the parameters of the Bouc-Wen model (presented under its normalized form) so that the input-output behavior of the model matches that of the true system for some relevant experimental test inputs.

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REFERENCES

Chen B. M, Lee T. H, Hang C. C, Guo Y and Weerasooriya S (1999). An h_∞ almost disturbance decoupling robust controller design for a piezoelectric bimorph actuator with hysteresis. *IEEE Transactions on Control Systems Technology* **7**(2), 160–174.

Erlicher S, Point N (2004). Thermodynamic admissibility of bouc-wen type hysteresis models. *C. R. Mecanique* **332**, 51–57.

Foliente C (1995). Hysteresis modelling of wood joints and structural systems. *ASCE Journal of Structural Engineering* **121**(6), 1013–1022.

Ikhouane F, Mañosa V, Rodellar J (2004*a*). Input-output and energy dissipation properties of the hysteretic bouc-wen model. *International Journal of Nonlinear Mechanics*, submitted.

Ikhouane F, Mañosa V, Rodellar J (2005). Adaptive control of a hysteretic structural system. *Automatica* **41**, 225–231.

Ikhouane F, Rodellar J (2004*b*). On the hysteretic Bouc-Wen model. Part I: Forced limit cycle characterization. *Nonlinear Dynamics*, to appear.

Wen Y. K (1976). Method of random vibration of hysteretic systems. *Journal of Engineering Mechanics ASCE* **102**(2), 249–263.

Kyprianou A, Worden K and Panet M (2001). Identification of hysteretic systems using the differential evolution algorithm. *Journal of Sound and Vibration* **248**(2), 289–314.

Low T, Guo W (1995). Modelling of a three-layer piezoelectric bimorph beam with hysteresis. *IEEE Journal of Microelectromechanical Systems* **4**(4), 230–237.

Bouc R (1967). Forced vibrations of mechanical systems with hysteresis. In: *Proceedings of the Fourth Conference on Non-Linear Oscillations.*

Smyth A. W, Masri S. F, Chassiakos A. G and Caughey T. K (1999). On-line parametric identification of mdof nonlinear hysteretic systems. *Journal of Engineering Mechanics* pp. 133–142.

Smyth A. W, Masri S. F, Kosmatopoulos E. B, Chassiakos A. G and Caughey T. K (2002). Development of adaptive modeling techniques for non-linear hysteretic systems. *International Journal of Non-Linear Mechanics* **37**, 1435–1451.

Spencer Jr. B. F, Dyke S. J, Sain M. K and Carlson J. D (1997). Phenomenological model for magnetorheological dampers. *J. Engineering Mechanics ASCE* **123**, 230–238.