# CONTROLLER WITH MINIMAL INTERACTION

**A.A. Julius J.W. Polderman A.J. van der Schaft**

*Dept. Applied Mathematics*
*University of Twente*
*PO Box 217, Enschede 7500AE*
*The Netherlands*
*Email:*`a.a.julius@math.utwente.nl`
`j.w.polderman@math.utwente.nl`
`a.j.vanderschaft@math.utwente.nl`

Abstract: We study controller design from the behavioral point of view. Given a plant and a regularly implementable specification, our goal is to design a regular controller that uses as few control variables as possible. It turns out that the solution basically consists of two main steps. The first step is to design a regular controller that is equivalent to the canonical controller. The second step is searching for the desired controller in a class parameterized by the controller designed in the first step. 

Keywords: behavior, controller design, canonical controller, minimal interaction

## 1. PRELIMINARY MATERIAL

Standard control problems in the behavioral approach to systems theory can be formulated as follows (Willems 1997, Polderman and Willems 1998, Belur 2003). Given is a plant to be controlled, which has two kinds of variables: *to-be-controlled variables* and *control variables*. A *controller* is a device that is attached to (or an algorithm that acts on) the control variables and restricts their behavior. This restriction is imposed on the plant, such that it eventually affects the behavior of the to-be-controlled variables (see Figure 1). The resulting behavior is called the *controlled system*.

As part of the control problem, one is given a *specification*, which is expressed in terms of the to-be-controlled variables. The objective of the control problem is to make the controlled system satisfy the specification. If there exists a controller such that this objective is satisfied, we say that the specification is *implementable*.

Throughout this paper, we denote the control variables as $\mathbf{c}$ and the to-be-controlled variables as $\mathbf{w}$. The cardinality of $\mathbf{c}$ and $\mathbf{w}$ are denoted as

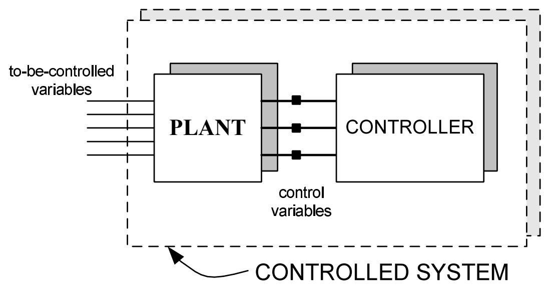


Fig. 1. Control in the behavioral approach.

$\mathtt{c}$ and $\mathtt{w}$ respectively. A behavioral model of the plant system that captures the relevant relation between $\mathbf{w}$ and $\mathbf{c}$ is called the *full plant behavior*, and is denoted by $\mathcal{P}_{\text{full}}$. The full plant behavior is represented by a minimal kernel representation

$$R\left(\frac{d}{dt}\right) w + M\left(\frac{d}{dt}\right) c = 0. \qquad (1)$$

In this paper, we restrict our attention to infinitely differentiable functions. Thus, the full plant behavior consists of all signal pairs $(w, c)$ that are strong solutions to the kernel representation (1) (Polderman and Willems 1998).

$$\mathcal{P}_{\text{full}} := \{(w,c) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}+\mathtt{c}}) \mid R\left(\frac{d}{dt}\right) w + M\left(\frac{d}{dt}\right) c = 0\}. \quad (2)$$

If we eliminate the control variables from the full behavior, we obtain the so called *manifest behavior*, which is denoted by $\mathcal{P}$. Thus,

$$\mathcal{P} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \text{ such that } (w,c) \in \mathcal{P}_{\text{full}}\}. \quad (3)$$

If we rewrite the kernel representation (1) as

$$\tilde{R}_1\left(\frac{d}{dt}\right) w + \tilde{M}_1\left(\frac{d}{dt}\right) c = 0, \qquad \tilde{R}_2\left(\frac{d}{dt}\right) w = 0, \quad (4)$$

where $\tilde{M}_1$ and $\tilde{R}_2$ are full row rank matrices, then the manifest behavior $\mathcal{P}$ is the kernel of $\tilde{R}_2\left(\frac{d}{dt}\right)$ (cf. (Polderman and Willems 1998) Chapter 6).

A controller $\mathcal{C}$ is a behavior containing all signals $c$ allowed by the controller:

$$\mathcal{C} := \left\{c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \mid C\left(\frac{d}{dt}\right) c = 0\right\}. \quad (5)$$

The *controlled behavior* is then defined as

$$\mathcal{K} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \text{ such that } (w,c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}. \quad (6)$$

The controlled behavior $\mathcal{K}$ is obtained by eliminating the control variables from the following kernel representation.

$$R\left(\frac{d}{dt}\right) w + M\left(\frac{d}{dt}\right) c = 0, \qquad C\left(\frac{d}{dt}\right) c = 0. \quad (7)$$

The specification $\mathcal{S}$ is given by the following kernel representation

$$S\left(\frac{d}{dt}\right) w = 0. \quad (8)$$

The objective of the control problem is to find a controller $\mathcal{C}$ such that $\mathcal{K} = \mathcal{S}$. If such controller exists, then $\mathcal{S}$ is said to be implementable and the controller $\mathcal{C}$ is said to implement $\mathcal{S}$.

Clearly, the implementability of a specification $\mathcal{S}$ is a property that depends on the specification itself as well as the plant. The following result is proven in (Willems 1999).

*Theorem 1.* (Willems' lemma). Given $\mathcal{P}_{\text{full}}$ as a kernel representation of (1). A specification $\mathcal{S}$ is implementable if and only if

$$\mathcal{N} \subseteq \mathcal{S} \subseteq \mathcal{P}, \quad (9)$$

where $\mathcal{N} \in \mathfrak{L}^{\mathtt{w}}$ is the *hidden behavior* defined by

$$\mathcal{N} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid (w,0) \in \mathcal{P}_{\text{full}}\}.$$

Quite often, in addition to requiring that the controller implements the desired specification, we also require that the controller possesses a certain property with respect to the plant. A property that has been quite extensively studied is the so called *regularity* (Polderman 2000, Belur and Trentelman 2002, Julius and van der Schaft 2003, Willems *et al.* 2003). Without giving any behavioral interpretation to the concept of regularity (the reader is referred to the above mentioned references), we define a controller

$$\mathcal{C} = \left\{c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \mid C\left(\frac{d}{dt}\right) c = 0\right\}, \quad (10)$$

where $C$ is full row rank, to be regular if

$$\operatorname{rank}\begin{bmatrix} R & M \\ 0 & C \end{bmatrix} = \operatorname{rank}\,[\, R \;\; M \,] + \operatorname{rank} C. \quad (11)$$

It can be proven that nonregular interconnections affect the autonomous part of the systems, which, in many cases would be undesirable or unrealistic.

*Remark 2.* Although the characterization of the regular controller suggests that regularity is a representation dependent property, it is actually not. The interested readers are referred to (Belur and Trentelman 2002, Julius and van der Schaft 2003, Willems *et al.* 2003) for some behavioral interpretation of regularity.

If the specification $\mathcal{S}$ is such that there exists a regular controller $\mathcal{C}$ that implements it, then $\mathcal{S}$ is said to be regularly implementable. The necessary and sufficient condition for regular implementability was derived in (Belur and Trentelman 2002):

*Theorem 3.* Given the full plant behavior $\mathcal{P}_{\text{full}}$. A specification $\mathcal{S}$ is implementable if and only if
1) it is implementable, i.e., $\mathcal{N} \subseteq \mathcal{S} \subseteq \mathcal{P}$ and
2) $\mathcal{S} + \mathcal{P}^{\text{ctr}} = \mathcal{P}$.
The symbol $\mathcal{P}^{\text{ctr}}$ denotes the controllable part of the manifest behavior $\mathcal{P}$.

## 2. PROBLEM FORMULATION

Consider the following definition of irrelevant variables.

*Definition 4.* Let a behavior $\mathfrak{B}$ be given by the kernel representation

$$R_1\left(\frac{d}{dt}\right) w_1 + R_2\left(\frac{d}{dt}\right) w_2 = 0. \quad (12)$$

If $R_1$ is the zero matrix, then the variables in $\mathbf{w}_1$ are said to be *irrelevant* to $\mathfrak{B}$.

Notice that whether or not some variables are irrelevant to a behavior is not a matter of representation. Rather, it is a property of the behavior. It means for every $(w_1, w_2) \in \mathfrak{B}$ we can always replace $w_1$ by any infinitely differentiable trajectory $w_1'$ and have that $(w_1', w_2)$ is still an element of $\mathfrak{B}$.

The problem that we are addressing in this paper can be formulated as follows. Given the control problem as discussed in the previous section. We assume that the specification $\mathcal{S}$ is regularly implementable. Construct a regular controller $\mathcal{C}$ that has as many irrelevant variables as possible. This controller is called the *controller with minimal interaction.*

Since the number of variables is finite, the maximal number of irrelevant variables that can be attained exists. However, generally there is no unique selection of variables to make up this maximal number.

The motivation behind this problem is as follows. Consider a situation where the plant and the controller are separated by a large physical distance. We need a communication link between the plant and the controller to establish the interconnection. It is therefore favorable to have as few control variables as possible, so that the amount of communication links/channels can be minimized.

## 3. THE CONSTRUCTION OF THE CONTROLLER

**Notation.** We denote the class of regular controllers that implement $\mathcal{S}$ as $\mathfrak{C}_{\mathcal{S}}^{\text{reg}}$.

Obviously, the problem can be formulated as finding an element in $\mathfrak{C}_{\mathcal{S}}^{\text{reg}}$ that has as many irrelevant variables as possible. To find such a controller, we use the following result.

*Lemma 5.* Let $X$ be subset of $\mathfrak{C}_{\mathcal{S}}^{\text{reg}}$ such that for any $\mathcal{C} \in \mathfrak{C}_{\mathcal{S}}^{\text{reg}}$ there exists a $\mathcal{C}' \in X$ such that $\mathcal{C} \subseteq \mathcal{C}'$. If $\mathcal{C} \in \mathfrak{C}_{\mathcal{S}}^{\text{reg}}$ is a controller that has the maximal number of irrelevant variables, then there exists a $\mathcal{C}' \in X$ that has at least as many irrelevant variables as $\mathcal{C}$.

**Proof.** If a variable is irrelevant in $\mathcal{C}$, it is also irrelevant in any $\mathcal{C}' \supseteq \mathcal{C}$. Therefore if a $\mathcal{C} \in \mathfrak{C}_{\mathcal{S}}^{\text{reg}}$ has $n$ irrelevant variables, there is a $\mathcal{C}' \in X$ that has at least $n$ irrelevant variables. ■

Lemma 5 tells us that if we can construct a subset $X$ of $\mathfrak{C}_{\mathcal{S}}^{\text{reg}}$ with the property as in the premise of the lemma, then it is sufficient to search for the controller with minimal interaction in $X$ rather than in the whole $\mathfrak{C}_{\mathcal{S}}^{\text{reg}}$. To construct such a subset $X$, we need the following result.

First, we write the kernel representation of the full plant behavior as

$$\begin{aligned} R_1\left(\frac{d}{dt}\right) w + M_1\left(\frac{d}{dt}\right) c = 0, \\ M_2\left(\frac{d}{dt}\right) c = 0, \end{aligned} \tag{13}$$

with $R_1$ and $M_2$ both full row rank. We define

$$\mathcal{P}_c := \left\{ c \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \mid M_2\left(\frac{d}{dt}\right) c = 0 \right\}. \tag{14}$$

This is the manifest plant behavior with respect to the control variables $\mathbf{c}$, which is obtained by eliminating $\mathbf{w}$ from $\mathcal{P}_{\text{full}}$.

**Notation.** The interconnection of behaviors is denoted with the operator $\|$. The behavior $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is defined as $\mathfrak{B}_1 \cap \mathfrak{B}_2$. In terms of the kernel representation, if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ can be expressed as the kernels of $R_1\left(\frac{d}{dt}\right)$ and $R_2\left(\frac{d}{dt}\right)$ respectively, then $\mathfrak{B}_1 \parallel \mathfrak{B}_2$ is the kernel of $[R_1\ R_2]^{\mathrm{T}}$.

*Theorem 6.* There exists a controller

$$\mathcal{C}_{\text{can}} := \left\{ c \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \mid C_{\text{can}}\left(\frac{d}{dt}\right) c = 0 \right\}, \tag{15}$$

such that
(i) $\mathcal{C}_{\text{can}}$ implements the specification $\mathcal{S}$,
(ii) $\mathcal{C}_{\text{can}} \subseteq \mathcal{P}_c$,
(iii) for any other controller $\mathcal{C}'$ that implements $\mathcal{S}$, we have that $(\mathcal{C}' \parallel \mathcal{P}_c) \subset (\mathcal{C}_{\text{can}} \parallel \mathcal{P}_c)$.

**Proof.** The canonical controller introduced in (Van der Schaft and Julius 2002, Van der Schaft 2003, Willems *et al.* 2003) satisfies all the three properties above. A kernel representation of the canonical controller can be obtained by eliminating $\mathbf{w}$ from the following kernel representation.

$$\begin{aligned} R\left(\frac{d}{dt}\right) w + M\left(\frac{d}{dt}\right) c = 0, \\ S\left(\frac{d}{dt}\right) w = 0. \end{aligned} \tag{16}$$

■

We use the canonical controller in Theorem 6 to construct the set $X$ as:

$$X := \{\mathcal{C} \in \mathfrak{C}_{\mathcal{S}}^{\text{reg}} \mid (\mathcal{C} \parallel \mathcal{P}_c) = \mathcal{C}_{\text{can}}\}. \tag{17}$$

*Lemma 7.* Define $X$ as in (17). The following statement holds. For all $\mathcal{C} \in \mathfrak{C}_{\mathcal{S}}^{\text{reg}}$, there exists a $\mathcal{C}' \in X$ such that $\mathcal{C} \subseteq \mathcal{C}'$.

**Proof.** Take any $\mathcal{C} \in \mathfrak{C}_{\mathcal{S}}^{\text{reg}}$. By definition of $\mathfrak{C}_{\mathcal{S}}^{\text{reg}}$ we know that
(a) $\mathcal{C}$ is a regular controller.
(b) For all $w \in \mathcal{S}$, there exists a $c \in \mathcal{C}$ such that $(w, c) \in \mathcal{P}_{\text{full}}$.
(c) For all $c \in \mathcal{C}$, $(w, c) \in \mathcal{P}_{\text{full}}$ implies $w \in \mathcal{S}$.

We construct $\mathcal{C}' := \mathcal{C} + \mathcal{C}_{\text{can}}$. Clearly $\mathcal{C} \subseteq \mathcal{C}'$. We have to prove that $\mathcal{C}' \in X$. That is, we have to prove that
(a') $\mathcal{C}'$ is regular.
(b') $\mathcal{C}' \parallel \mathcal{P}_c = \mathcal{C}_{\text{can}}$.
The statement (a') follows from the fact that $\mathcal{C} \subseteq \mathcal{C}'$ and the regularity of $\mathcal{C}$. To prove (b'), first we show that $\mathcal{C}'$ implements $\mathcal{S}$. From here, (b') follows from the fact that $\mathcal{C}_{\text{can}} \subseteq \mathcal{C}'$ and the property of $\mathcal{C}_{\text{can}}$ being the least restrictive controller. Showing that $\mathcal{C}'$ implements $\mathcal{S}$ means showing that
(a") For all $w \in \mathcal{S}$, there exists a $c' \in \mathcal{C}'$ such that $(w, c') \in \mathcal{P}_{\text{full}}$.
(b") For all $c' \in \mathcal{C}'$, $(w, c') \in \mathcal{P}_{\text{full}}$ implies $w \in \mathcal{S}$.
Statement (a") follows immediately from (b). To show that (b") holds, notice that any $c' \in \mathcal{C}'$ can be written as $c + c_{\text{can}}$ with $c \in \mathcal{C}$ and $c_{\text{can}} \in \mathcal{C}_{\text{can}}$.

Also notice that for all $c_{\mathrm{can}} \in \mathcal{C}_{\mathrm{can}}$, there exists a $w_{\mathrm{can}} \in \mathcal{S}$ such that $(w_{\mathrm{can}}, c_{\mathrm{can}}) \in \mathcal{P}_{\mathrm{full}}$. Thus,

$$\begin{aligned}(w, c') \in \mathcal{P}_{\mathrm{full}} &\Rightarrow (w - w_{\mathrm{can}} + w_{\mathrm{can}}, c + c_{\mathrm{can}}) \in \mathcal{P}_{\mathrm{full}} \\ &\overset{\text{linearity}}{\Rightarrow} ((w - w_{\mathrm{can}}), c) \in \mathcal{P}_{\mathrm{full}} \\ &\overset{(c)}{\Rightarrow} (w - w_{\mathrm{can}}) \in \mathcal{S} \overset{\text{linearity}}{\Rightarrow} w \in \mathcal{S}.\end{aligned}$$

■

The reason we construct $X$ as in (17) is because we can parameterize all elements of $X$. We shall now find a parametrization of the kernel representation of the elements of $X$. For that purpose, we use the following results.

It is easily seen that the controller $\mathcal{C}$ (as in (10)) is regular if and only if $\mathcal{P}_c \parallel \mathcal{C}$ is a regular full interconnection,

$$\operatorname{rank} \begin{bmatrix} M_2 \\ C \end{bmatrix} = \operatorname{rank} M_2 + \operatorname{rank} C. \tag{18}$$

*Lemma 8.* Let a plant $\mathcal{P}$ be given as the kernel of a full row rank $R(\frac{d}{dt})$ and a regular controller $\mathcal{C}$ be given as the kernel of a full row rank $C(\frac{d}{dt})$. Denote the controlled behavior by

$$\mathcal{K} := \mathcal{P} \parallel \mathcal{C}.$$

Let $\mathcal{C}'$ be another regular controller such that $\mathcal{P} \parallel \mathcal{C}' = \mathcal{K}$. A minimal kernel representation of $\mathcal{C}'$ has exactly as many rows as $C(\frac{d}{dt})$.

**Proof.** Straightforward from the definition of regularity. ■

*Lemma 9.* Let a plant $\mathcal{P}$ be given as the kernel of a full row rank $R(\frac{d}{dt})$ and a regular controller $\mathcal{C}$ be given as the kernel of a full row rank $C(\frac{d}{dt})$. Denote the full interconnection

$$\mathcal{K} := \mathcal{P} \parallel \mathcal{C}.$$

Let $\mathfrak{C}_{\mathcal{K}}$ denote the set of all controllers (not necessarily regular ones) that
(i) have at most as many rows in the minimal kernel representation as $\mathcal{C}$ and
(ii) also implement $\mathcal{K}$ when interconnected with $\mathcal{P}$.
A controller $\mathcal{C}' \in \mathfrak{C}_{\mathcal{K}}$ if and only if its kernel representation can be written as $VR+C$ for some matrix $V$. Moreover, every controller in $\mathcal{C}' \in \mathfrak{C}_{\mathcal{K}}$ has the following properties.
(a) $\mathcal{C}'$ is regular.
(b) Its minimal kernel representation has exactly as many rows as that of $\mathcal{C}$.

**Proof.** (if) Suppose that a controller $\mathcal{C}'$ has $(VR+C)$ as its kernel representation, then $\mathcal{P} \parallel \mathcal{C}'$ is given by the kernel of

$$\begin{bmatrix} R \\ VR+C \end{bmatrix} = \begin{bmatrix} I & 0 \\ V & I \end{bmatrix} \begin{bmatrix} R \\ C \end{bmatrix}. \tag{19}$$

This shows that $\mathcal{P} \parallel \mathcal{C}' = \mathcal{K}$. Moreover, since $\mathcal{C}$ is a regular controller, it follows that $(VR+C)$ is a minimal kernel representation of $\mathcal{C}'$. Thus, properties (a) and (b) are verified.

(only if) Suppose that a controller $\mathcal{C}'$ satisfies (i) and (ii) above. This controller can be written as the kernel of a matrix (not necessarily minimal) $C'(\frac{d}{dt})$ with as many rows as $C(\frac{d}{dt})$. We know that there is a unimodular matrix $U$ such that

$$U \begin{bmatrix} R \\ C \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} R \\ C \end{bmatrix} = \begin{bmatrix} R \\ C' \end{bmatrix}. \tag{20}$$

We shall prove that we can assume $U$ to be of the form

$$U = \begin{bmatrix} I & 0 \\ V & I \end{bmatrix}. \tag{21}$$

First, we find a unimodular matrix $W$ such that

$$RW = [\, D \;\; 0 \,], \tag{22}$$

where $D$ is a square nonsingular matrix. We then use the following notation

$$\begin{bmatrix} R \\ C \end{bmatrix} W =: \begin{bmatrix} D & 0 \\ C_1 & C_2 \end{bmatrix}, \tag{23}$$

$$\begin{bmatrix} R \\ C' \end{bmatrix} W =: \begin{bmatrix} D & 0 \\ C_1' & C_2' \end{bmatrix}. \tag{24}$$

It follows that (20) can be rewritten as

$$U \begin{bmatrix} D & 0 \\ C_1 & C_2 \end{bmatrix} W^{-1} = \begin{bmatrix} D & 0 \\ C_1' & C_2' \end{bmatrix} W^{-1}, \tag{25}$$

and since $W$ is unimodular,

$$U \begin{bmatrix} D & 0 \\ C_1 & C_2 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} D & 0 \\ C_1 & C_2 \end{bmatrix} = \begin{bmatrix} D & 0 \\ C_1' & C_2' \end{bmatrix}. \tag{26}$$

Consequently, we have the following equations

$$U_{11}D + U_{12}C_1 = D, \tag{27a}$$
$$U_{12}C_2 = 0, \tag{27b}$$
$$U_{21}D + U_{22}C_1 = C_1', \tag{27c}$$
$$U_{22}C_2 = C_2'. \tag{27d}$$

Since the controller $\mathcal{C}$ is regular, $C_2$ must be full row rank. Now, (27b) implies that $U_{12}$ is a left annihilator of $C_2$. Consequently

$$U_{12} = 0. \tag{28}$$

Substituting this to (27a) yields

$$U_{11} = I. \tag{29}$$

Since $U$ is unimodular, this implies that $U_{22}$ is unimodular. Thus, we can conclude that

$$U = \begin{bmatrix} I & 0 \\ U_{21} & U_{22} \end{bmatrix}, \tag{30}$$

with $U_{22}$ unimodular. Furthermore, $C'' := U_{22}C'$ is also a kernel representation of $\mathcal{C}'$ so we can assume $U_{22}$ to be the identity matrix without any loss of generality. ■

If in the statement of Lemma 9 we replace $\mathcal{K}$ by $\mathcal{C}_{\mathrm{can}}$ and $\mathcal{P}$ by $\mathcal{P}_c$, we can conclude that the kernel representation of the elements of $X$ can be parameterized if we follow these steps. First, we construct an element of $X$. Denote this element as $\mathcal{C}$ and its kernel representation as $C\left(\frac{d}{dt}\right)$.

Then, a controller $\mathcal{C}'$ is an element of $X$ if and only if its kernel representation can be written as $VM_2 + C$ for some matrix $V$. The quest to make the controller has as many irrelevant variables as possible becomes a matter of finding $V$, such that $VM_2 + C$ has as many zero columns as possible. The procedure to compute a regular controller that implements $\mathcal{S}$ and has as many irrelevant variables as possible can be summarized as follows.

**Step 1.** Construct the canonical controller $C_{\text{can}}$ for the problem. Since $\mathcal{S}$ is regularly implementable, we know that the canonical controller implements $\mathcal{S}$.

**Step 2.** Construct a controller $\mathcal{C} \in \mathfrak{C}_{\mathcal{S}}^{\text{reg}}$ such that $(\mathcal{C} \parallel \mathcal{P}_c) = \mathcal{C}_{\text{can}}$. Lemma 7 guarantees that this can be done. Denote the kernel representation of $\mathcal{C}$ and $\mathcal{P}_c$ by $C(\frac{d}{dt})$ and $M_2(\frac{d}{dt})$ respectively.

**Step 3.** The kernel representation of the controller with minimal interaction can be found by finding a matrix $V$ such that $C + VM_2$ has as many zero columns as possible.

The algebraic problem related to the third step has a combinatorial aspect in it, as we generally need to search for the answer by trying all possible subsets of the columns. This situation gives rise to a computational challenge, namely to design a clever algorithm that can handle this combinatorial problem efficiently. We shall not venture into this direction in this paper. Rather, we shall only continue with a special case, in which we can solve this problem relatively easily. Thus, although we believe that it is an important and interesting task to develop such an algorithm, we shall proceed primarily by investigating the systems theoretic properties of this problem.

For a special case, where $M_2$ is in Smith form, we can get a straightforward answer. If $M_2$ is in the Smith form, then we generally can write it as

$$M_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & D & 0 \end{bmatrix}, \tag{31}$$

where $D$ is a diagonal matrix whose diagonal entries are polynomials with degree at least one. If $\mathcal{P}_c$ is controllable then the size of $D$ is actually zero. Similarly, $C$ is partitioned as

$$C = [\, C_1 \;\; C_2 \;\; C_3 \,]. \tag{32}$$

The problem becomes to find $V = [V_1 \;\; V_2]$ such that

$$\begin{aligned} [\, V_1 \;\; V_2 \,] \begin{bmatrix} I & 0 & 0 \\ 0 & D & 0 \end{bmatrix} + [\, C_1 \;\; C_2 \;\; C_3 \,] \\ = [\, V_1 + C_1 \;\; V_2 D + C_2 \;\; C_3 \,] \end{aligned}$$

has as many zero columns as possible. The solution is obvious. The columns in the left most partition can be nullified by choosing $V_1 = -C_1$. The columns in the right most partition cannot be nullified, except for those that happen to be zero columns. The i-th column of the middle partition can be nullified if and only if it is a multiple of the polynomial $D_{ii}$. Here $D_{ii}$ denotes the i-th entry of the diagonal of $D$.

Although we cannot present a complete procedure to compute the controller with minimal interaction in the general case, we can present an upper bound for the number of irrelevant variables in the controller.

*Lemma 10.* The controller with minimal interaction can have at most $\mathtt{c} - \mathtt{p}(\mathcal{C})$ irrelevant variables. Here $\mathtt{c}$ denotes the number of all control variables (the cardinality of $\mathbf{c}$) and $\mathtt{p}(\mathcal{C})$ denotes the number of output variables in $\mathcal{C}$ (Willems 1997), which is any element of $\mathfrak{C}_{\mathcal{S}}^{\text{reg}}$.

**Proof.** From Lemma 8 we know that all elements of $\mathfrak{C}_{\mathcal{S}}^{\text{reg}}$ have the same number of output, i.e., $\mathtt{p}(\mathcal{C})$. This is the number of rows in a minimal kernel representation of any controller in $\mathfrak{C}_{\mathcal{S}}^{\text{reg}}$. It is easily seen that the number of columns is $\mathtt{c}$. If a controller $\mathcal{C} \in \mathfrak{C}_{\mathcal{S}}^{\text{reg}}$ has more than $\mathtt{c} - \mathtt{p}(\mathcal{C})$ irrelevant variables, then the nonzero entries of its kernel representation form a tall matrix, and thus cannot be minimal. A tall matrix is a matrix, in which there are more nonzero rows than there are columns. This contradicts Lemma 8. ■

We shall see in the numerical example presented in the last section of this paper that the upper bound given by Lemma 10 is tight. That is, there are some situations where this upper bound is actually reached.

## 4. MINIMAL INTERACTION WITH CONTROL VARIABLES TRANSFORMATION

An alternative problem of interest can be formulated as follows. Recall that in the problem description, our goal is to use as few control variables as possible. These control variables are a part of the initial control variables. Now, suppose that instead of picking a part of the original control variables as the new control variables directly, we first allow for an isomorphic transformation of variables to take place. This means, we construct a new set of control variables $\mathbf{c}_{\text{new}}$ from the old ones $\mathbf{c}$ by

$$c_{\text{new}} = T\left(\frac{d}{dt}\right) c. \tag{33}$$

The matrix $T$ is a *unimodular* matrix to be designed. Our goal is to design the transformation $T$ such that we can use as few variables in the new control variables $\mathbf{c}_{\text{new}}$ as possible. Of course, with this new selection of control variables, we have to maintain regular implementability of the specification.

This alternative description of the problem departs from the practical motivation given at the end of Section 2. However, from the systems theoretic point of view, this alternative problem can also be interpreted as control with minimal information. This is because $\mathbf{c}_{\text{new}}$ and $\mathbf{c}$ contain the same 'amount' of information, as they are related through an isomorphic transformation. It turns out that this problem has a simple solution.

It can be verified that with this new problem formulation, the problem changes from 'finding a $V$ such that $VM_2 + C$ has as many zero columns as possible' to 'finding a $V$ and a unimodular $T$ such that $(VM_2 + C)T^{-1}$ has as many zero columns as possible'. Obviously the new problem formulation is equivalent to 'finding a $V$ such that $VM_2 + C$ has as small column rank as possible'. From Lemma 9 we know that $VM_2 + C$ always has the same row rank as $C$, thus it always has the same column rank as $C$. We can then take any matrix to be $V$. For simplicity, we take $V = 0$. We then compute a unimodular matrix $U$ such that

$$CU = \left[ \tilde{C} \ 0 \right], \tag{34}$$

where $\tilde{C}$ has full column rank. The transformation is then

$$c_{\text{new}} = U^{-1} \left( \frac{d}{dt} \right) c, \tag{35}$$

and the new control variables that are relevant to the controller are the first $\mathtt{p}(\mathcal{C})$ components of $\mathbf{c}_{\text{new}}$. Here the symbol $\mathtt{p}(\mathcal{C})$ indicates the number of output variables of $\mathcal{C}$ (Willems 1997). Therefore, if we are allowed to transform the control variables, we can obtain exactly $\mathtt{c} - \mathtt{p}(\mathcal{C})$ irrelevant variables, which is the upper bound stipulated by Lemma 10.

## 5. EXAMPLE

Consider the control problem, where

$$R(\xi) = \begin{bmatrix} \xi & -1 & 0 & 0 \\ 0 & \xi & -1 & 0 \\ 0 & 0 & \xi & -1 \\ 1 & 1 & 1 & \xi+1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{36}$$

$$M(\xi) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{37}$$

$$S(\xi) = \begin{bmatrix} \xi+1 & -1 & 0 & 0 \\ 1 & 1 & 1 & \xi+2 \\ 0 & \xi & -1 & 0 \\ 0 & 0 & \xi & -1 \end{bmatrix}. \tag{38}$$

We can compute that

$$M_2(\xi) = \begin{bmatrix} \xi^2 & 0 & -\xi^3 & 1 \\ \xi^3 + \xi^2 + \xi + 1 & 1 & -\xi^4 - \xi^3 - \xi^2 - \xi - 1 & 0 \end{bmatrix}. \tag{39}$$

We can also verify that the specification $\mathcal{S}$ is regularly implementable. In fact, it is implemented by a regular controller $\mathcal{C}'$, which is the kernel of $C'(\frac{d}{dt})$, where

$$C'(\xi) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \tag{40}$$

We see that the controller $\mathcal{C}'$ does not have any irrelevant variables.

The canonical controller for this problem is given by the kernel of $C_{\text{can}}(\frac{d}{dt})$, where

$$C_{\text{can}}(\xi) = \begin{bmatrix} \xi^2 & 0 & -\xi^3 & 1 \\ 1 & 0 & 1 & 0 \\ \left(\xi^4 + 2\xi^3 + 2\xi^2 + 2\xi + 2\right) & 1 & 0 & 0 \\ \left(\xi^4 + 3\xi^3 + 3\xi^2 + 2\xi + 2\right) & 0 & 0 & 0 \end{bmatrix}. \tag{41}$$

A regular controller $\mathcal{C} \in X$ can be constructed as the kernel of $C(\frac{d}{dt})$, where

$$C(\xi) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ \left(\xi^4 + 3\xi^3 + 3\xi^2 + 2\xi + 2\right) & 0 & 0 & 0 \end{bmatrix}. \tag{42}$$

We can see that $C(\xi)$ already has two zero columns. This means the second and fourth control variables are irrelevant in $\mathcal{C}$. Following Lemma 10, we know that there cannot be more than two irrelevant variables in a regular controller that implements $\mathcal{S}$. Thus $\mathcal{C}$ is a controller with minimal interaction.

As a final remark, we would like to point out the fact that the controller with minimal interaction $\mathcal{C}$ has a McMillan degree of 4, while another regular controller $\mathcal{C}'$ has a McMillan degree of 0. This fact shows that while the controller with minimal interaction uses fewer control variables to interact with the controller, it can be more complex than a controller that uses more control variables.

## REFERENCES


Belur, M. N. (2003). Control in a behavioral context. PhD thesis. Rijksuniversiteit Groningen.

Belur, M. N. and H. L. Trentelman (2002). Stabilization, pole placement and regular implementability. *IEEE Trans. Automatic Control* **47(5)**, 735–744.

Julius, A. A. and A. J. van der Schaft (2003). Compatibility of behavior interconnections. In: *Proc. European Control Conference*. IEE. Cambridge.

Polderman, J. W. (2000). Sequential continuous time adaptive control: a behavioral approach. In: *Proc. 39th IEEE Conf. Decision and Control*. IEEE. Sydney. pp. 2484–2487.

Polderman, J. W. and J. C. Willems (1998). *Introduction to Mathematical Systems Theory : A Behavioral Approach*. Springer. New York.

Van der Schaft, A. J. (2003). Achievable behavior of general systems. *Systems and Control Letters* **49**, 141–149.

Van der Schaft, A. J. and A. A. Julius (2002). Achievable behavior by composition. In: *Proceedings 41st IEEE Conf. Decision and Control*. IEEE. Las Vegas. pp. 7–12.

Willems, J. C. (1997). On interconnections, control and feedback. *IEEE Trans. Automatic Control* **42(3)**, 326–339.

Willems, J. C. (1999). Behaviors, latent variables, and interconnections. *Systems, Control and Information (Japan)* **43**, 453–464.

Willems, J. C., M. N. Belur, A. A. Julius and H. L. Trentelman (2003). The canonical controller and its regularity. In: *Proc. IEEE Conference on Decision and Control*. Hawaii. pp. 1639–1644.