CONTROLLER FOR A NONLINEAR SYSTEM WITH AN INPUT CONSTRAINT BY USING A CONTROL LYAPUNOV FUNCTION II

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Abstract: Malisoff and Sontag proposed a universal control formula for a nonlinear system such that the k-norm of inputs is less than one, where $1 < k \leq 2$. We have generalized the Malisoff's formula so that it can be applied in any case of $k \geq 1$. However, the generalized controller may become discontinuous if $k = 1$ or $k = \infty$. In this paper, we propose a new control formula that is continuous except the origin in any case of $k \geq 1$. We also confirm the effectiveness of the proposed controller by computer simulation.

Keywords: nonlinear systems, constraints, controllers, Lyapunov methods

1. INTRODUCTION

We consider a nonlinear system such that inputs are restricted to the Minkowski ball U_k. U_k is a subspace of $\mathbb{R}^m$ such that the k-norm of inputs is less than one. Lin and Sontag proposed a universal control formula with respect to U_2 by using a control Lyapunov function (Lin and Sontag, 1991). Malisoff and Sontag provided a universal control formula with respect to U_k, where $1 < k \leq 2$ (Malisoff and Sontag, 2000). We have generalized the Malisoff's controller so that it can be applied in any case of $k \geq 1$ (N. Kidane and Nishitani, 2005). However, the generalized controller may become discontinuous if $k = 1$ or $k = \infty$. Due to discontinuity of the controller, inputs may have chattering.

In this research, we propose a new control formula that is continuous except the origin in any case of $k \geq 1$. We show the design scheme briefly. First, we consider a continuous function $\hat{k}(x)$ and a subspace $\bar{U}'_{\hat{k}} \subset \bar{U}_{\hat{k}}$ such that $\bar{U}_{\hat{k}}$ is the closure of $U_{\hat{k}}$ $\bar{U}'_{\hat{k}}$ is similar to $\bar{U}_{\hat{k}}$ (same shape), $\bar{U}'_{\hat{k}}$ becomes a small ball if $P(x) \leq 0$, and $\bar{U}'_{\hat{k}} \to \bar{U}_k$ as $P(x) \to 1$. Second, we stabilize the system by the input that minimizes $\dot{V}(x, u)$ in $\bar{U}'_{\hat{k}}$. We also confirm the effectiveness of the proposed controller by computer simulation.

2. PRELIMINARY

In this section, we introduce mathematical notation and some definitions. For a vector $x \in \mathbb{R}^n$, k-norm is defined as

$$\|x\|_k = \left(\sum_{i=1}^{n} |x_i|^k \right)^{\frac{1}{k}}. \qquad (1)$$

We obtain the following lemma:

Lemma 1. Assume that $x \in \mathbb{R}^n$ and $1 \le k_1 < k_2$. Then,

$$\|x\|_{k_1} \ge \|x\|_{k_2}. \tag{2}$$

□

Proof 1. Let $e \in \mathbb{R}^n$ be a vector such that $\|e\|_{k_1} = 1$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a bijection defined by

$$f(x) := \left(|x_1|^{\frac{k_1}{k_2}} \operatorname{sgn}(x_1), \dots, |x_n|^{\frac{k_1}{k_2}} \operatorname{sgn}(x_n)\right)^T.$$

Note that $\|f(e)\|_{k_2} = 1$. The norm $\|e\|_{k_2}$ can be written as

$$\|e\|_{k_2} = \left(\sum_{i=1}^{n} |e_i|^{k_2}\right)^{\frac{1}{k_2}} = \left(\sum_{i=1}^{n} |e_i|^{k_1} |e_i|^{k_2-k_1}\right)^{\frac{1}{k_2}}. \tag{3}$$

From $\|e\|_{k_1} = 1$, we get $|e_i| \le 1$ and $|e_i|^{k_2-k_1} \le 1$. From (3), $|e_i|^{k_2-k_1} \le 1$, and $\sum_{i=1}^{n} |e_i|^{k_1} = 1$, we obtain $\|e\|_{k_2} \le 1$. On the other hand, any vector $x \in \mathbb{R}^n$ can be written as

$$x = \|x\|_{k_1} \bar{e} = \|x\|_{k_2} \hat{e}, \tag{4}$$

where $\bar{e} \in \mathbb{R}^n$ and $\hat{e} \in \mathbb{R}^n$ are vectors such that $\|\bar{e}\|_{k_1} = 1$ and $\|\hat{e}\|_{k_2} = 1$. From (4), we get

$$\|x\|_{k_2} = \|x\|_{k_1} \|\bar{e}\|_{k_2}. \tag{5}$$

From $\|\bar{e}\|_{k_2} \le 1$ and (5), we obtain (2). $\|x\|_{k_1}$ and $\|x\|_{k_2}$ are equal if and only if $\|x_i\| = \|x\|$ for some $i \in \{1, \dots, n\}$. □

In this paper, we consider the following affine system:

$$\dot{x} = f(x) + g(x)u, \tag{6}$$

where $x \in \mathbb{R}^n$ is a state vector and $u \in U \subseteq \mathbb{R}^m$ is an input vector. We assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous mappings and $f(0) = 0$. We use the notation $\mathbb{R}_{>0} := (0, \infty)$ and $\mathbb{R}_{\ge 0} := [0, \infty)$.

Definition 1. (control Lyapunov function). A smooth proper positive definite function defined on a neighborhood of the origin $X \in \mathbb{R}^n$, $V : X \to \mathbb{R}_{\ge 0}$ is said to be a local control Lyapunov function for system (6) if the condition

$$\inf_{u\in U} \{L_f V + L_g V \cdot u\} < 0 \tag{7}$$

is satisfied for all $x \in X$, $x \ne 0$. Moreover, $V(x)$ is said to be a control Lyapunov function (clf) for system (6) if $V(x)$ is a function defined on $\mathbb{R}^n$ and condition (7) is satisfied for all $x \in \mathbb{R}^n$, $x \ne 0$. □

Definition 2. (small control property). A (local) control Lyapunov function is said to satisfy the small control property (scp) if for any $\varepsilon > 0$, there is a $\delta > 0$ such that, if $x \ne 0$ satisfies $\|x\| < \delta$, then there is some $u \in U$ with $\|u\| < \varepsilon$ such that $L_f V + L_g V \cdot u < 0$. □

If there exists no input constraint ($U \equiv \mathbb{R}^m$), a smooth radially unbounded positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is a clf if and only if

$$L_g V = 0 \implies L_f V < 0, \quad \forall\, x \ne 0. \tag{8}$$

We define $h(x)$ as the right hand side of system (6) with a state feedback law $u = \beta(x)$;

$$\dot{x} = f(x) + g(x)\beta(x) := h(x). \tag{9}$$

If $\beta(x)$ is continuous except the origin, the closed system has always a Carathéodory solution for each initial state. On the other hand, if $\beta(x)$ is not continuous, Carathéodory solution do not exist. Hence, we associate (9) with a differential inclusion of the form

$$\dot{x} \in F(x). \tag{10}$$

In this paper, we apply the Fillippov's approach

$$F(x) = \bigcap_{\varepsilon>0} \bigcap_{\mu_n(N)} \overline{co}\{h(B_\varepsilon(x)\backslash N)\}, \tag{11}$$

where $B_\varepsilon(x)$ denotes the open ball of center x and radius ε, $\overline{co}$ denotes the convex closure of a set, and μ_n is the Lebesgue measure of $\mathbb{R}^n$.

Definition 3. (Lyapunov function). A smooth and positive definite function defined on a neighborhood of the origin $X \subset \mathbb{R}^n$, $V : X \to \mathbb{R}_{\ge 0}$ is said to be a local Lyapunov function for system (10) if the following condition is satisfied for all $0 \ne x \in X$:

$$\frac{\partial V}{\partial x} \cdot v < 0, \quad \forall\, v \in F(x). \tag{12}$$

Moreover, $V(x)$ is said to be a Lyapunov function for system (10) if $V(x)$ is a radially unbounded function defined on $\mathbb{R}^n$ and condition (12) is satisfied for all $0 \ne x \in \mathbb{R}^n$. □

Theorem 1. (Bacciotti and Rosier, 2001) Let F be a set-valued map such that the local existence of solutions of (10) is insured. If a (local) Lyapunov function exists, then the origin is (locally) asymptotically stabilizable. □

3. PREVIOUS WORK

When there is not any input constraint, Sontag proposed a universal control formula for a nonlinear system (Sontag, 1989). In this paper, we consider a nonlinear system such that inputs are restricted to the Minkowski ball of radius 1;

$$U_k = \left\{ u \in \mathbb{R}^m \;\middle|\; \|u\|_k = \left(\sum_{i=1}^{m} |u_i|^k\right)^{\frac{1}{k}} < 1 \right\}, \tag{13}$$

where $k \ge 1$. Lin and Sontag provided a universal control formula with respect to Minkowski ball U_2 (Lin and Sontag, 1991). Malisoff and Sontag improved the Lin's controller in order to apply for the case of $1 < k \leq 2$ (Malisoff and Sontag, 2000). To construct the controller for the case of $k \geq 1$, we have ganeralized Malisoff's controller. We introduce important results as the followings (N. Kidane and Nishitani, 2005):

Theorem 2. Let $V(x)$ be a local clf for system (6) with input constraint (13), and $a_1 > 0$ be the maximum number such that the condition

$$\inf_{u \in U_k} \{L_f V + L_g V \cdot u\} < 0, \qquad \forall\, x \neq 0 \quad (14)$$

is satisfied for all $x \in W = \{x | V(x) < a_1\}$. Then, W is a domain in which the origin is asymptotically stabilizable. If $V(x)$ is a clf, then $a_1 = \infty$ and $W = \mathbb{R}^n$. □

Proposition 1. We consider system (6) with an input constraint $u \in \bar{U}_k$, where $\bar{U}_k$ is the closure of U_k. Let $V(x)$ be a local clf for the system. Then, the input

$$u_i = \begin{cases} -\dfrac{|L_{g_i} V|^{\frac{1}{k-1}}}{\|L_g V\|_{\frac{k}{k-1}}^{\frac{1}{k-1}}} \operatorname{sgn}(L_{g_i} V) & (L_g V \neq 0) \\ 0 & (L_g V = 0) \end{cases} \quad (i = 1, \ldots, m) \quad (15)$$

minimizes the derivative $\dot{V}(x, u)$. □

Lemma 2. Let $V(x)$ be a local clf for system (6) with input constraint (13), W be a domain in Theorem 2. We define

$$P(x) = \frac{L_f V}{\|L_g V\|_{\frac{k}{k-1}}}. \quad (16)$$

Then,

$$\sup_{x \in \{x \in W | L_g V(x) \neq 0\}} P(x) = 1. \quad (17)$$

□

Theorem 3. Let $V(x)$ be a local clf for system (6) with input constraint (13), W be a domain in Theorem 2, $P(x)$ be a function defined by (16), $c > 0$ and $q \geq 1$ are constants. Then, the input

$$u_i = -\frac{P + |P| + c\|L_g V\|_q}{2 + c\|L_g V\|_q} \cdot \frac{|L_{g_i} V|^{\frac{1}{k-1}}}{\|L_g V\|_{\frac{k}{k-1}}^{\frac{1}{k-1}}} \operatorname{sgn}(L_{g_i} V) \quad (L_g V \neq 0)$$

$$u_i = 0 \quad (L_g V = 0) \quad (i = 1, \ldots, m) \quad (18)$$

asymptotically stabilizes the origin in domain W. If $m = 1$ or $1 < k < \infty$, the input is continuous on $W \backslash \{0\}$. Moreover, if $V(x)$ has the scp, the input is also continuous at the origin. □

If $m = 1$ or $k = 2$, input (18) becomes $u = -b_2(x) L_g V^T / \|L_g V\|_2$ and it causes no chattering. If $m \neq 1$ and $k \neq 2$, however, input (18) may have chattering.

For example, in the case of $m \neq 1$ and $k = \infty$, input (18) becomes $u_i = -b_2(x) \operatorname{sgn}(L_{g_i} V)$. It is discontinuous on $\{x | L_{g_i} V = 0\}$. In the case of $m \neq 1$ and $k = 1$, $u_i = 0$ when $|L_{g_i} V| \neq \max_{j=1,\ldots,m} |L_{g_j} V|$, and $u_i = -b_2(x) \operatorname{sgn}(L_{g_i} V)$ when the other case $|L_{g_i} V| = \max_{j=1,\ldots,m} |L_{g_j} V|$. These controllers may cause chattering in inputs.

Therefore, the closed system may not have Carathéodory solutions in the case of $m \neq 1$ and $k \neq 2$. In this paper, we construct a controller that is continuous except the origin; namely, the controlled system has always a Carathéodory solution for each initial state.

4. CONTROLLER DESIGN

The objective of this paper is to design a stabilizing controller that is continuous except the origin in any case of $k \geq 1$. In our previous work (N. Kidane and Nishitani, 2005), we have proposed controller (18). We show the construction scheme briefly.

First, we consider a subspace $\bar{U}'_k \subset \bar{U}_k$ such that $\bar{U}'_k$ is similar to $\bar{U}_k$, $\bar{U}'_k$ becomes small if $P(x)$ becomes small, and $\bar{U}'_k \to \bar{U}_k$ as $P(x) \to 1$. Second, we design a stabilizing controller by choosing the input that minimizes $\dot{V}(x, u)$ in $\bar{U}'_k$. Consider the (hyper) surface $Q : L_g V \cdot u = a_2$ such that $\bar{U}_k \cap Q \neq \phi$ and a_2 becomes minimum. When input u coincides contact point between Q and $\bar{U}'_k$, $\dot{V}(x, u)$ takes minimum value. Then, the input that minimizes $\dot{V}(x, u)$ in $\bar{U}'_k$ is denoted by the contact point between Q and $\bar{U}'_k$. In the case of $k = 2$, a subspace $\bar{U}'_2 \subset \bar{U}_2$ becomes a ball. Hence, the input that minimizes $\dot{V}(x, u)$ in $\bar{U}'_2$ (namely, the contact point between Q and $\bar{U}'_2$) moves continuously on the boundary of $\bar{U}'_2$. On the other hand, in the case of $k = \infty$, a subspace $\bar{U}'_\infty \subset \bar{U}_\infty$ always becomes a rectangle. Hence, the contact point between Q and $\bar{U}'_\infty$ jumps from a vertex to another vertex at the moment that the sign of $L_{g_i} V$ changes. This causes chattering phenomenon in inputs.

In this section, we propose a stabilizing controller that is continuous except the origin in any case of $k \geq 1$ as the followings: First, we consider a continuous function $\hat{k}$ and a subspace $\bar{U}'_{\hat{k}} \subset \bar{U}_{\hat{k}}$ that satisfies the following conditions: $\bar{U}'_{\hat{k}}$ is similar to $\bar{U}_{\hat{k}}$, $\bar{U}'_{\hat{k}}$ becomes a small ball if $P(x) \leq 0$, and $\bar{U}'_{\hat{k}} \to \bar{U}_k$ as $P(x) \to 1$. Second, we stabilizes the system by the input that minimizes $\dot{V}(x, u)$ in $\bar{U}'_{\hat{k}}$ (See Fig. 1). Note that the subset $\bar{U}'_{\hat{k}}$ has to be large enough to hold $\dot{V}(x, u) < 0$ ($\forall\, 0 \neq x \in W$) under the input constraint $u \in \bar{U}'_{\hat{k}}$.

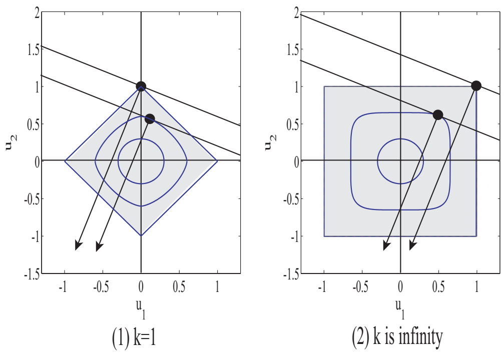

Fig. 1. The input that minimizes $\dot{V}(x,u)$ in $U'_{\hat{k}}$

The input that minimizes $\dot{V}(x,u)$ in $\bar{U}'_{\hat{k}}$ can be written as

$$u_i = -b_4(x)|L_{g_i}V|^{\frac{1}{\hat{k}-1}}\,\mathrm{sgn}(L_{g_i}V) \quad (i = 1,\ldots,m), \tag{19}$$

where $b_4 : \mathbb{R}^n \to \mathbb{R}_{>0}$. We choose a function $b_4(x)$ such that input (19) is continuous on $W\backslash\{0\}$, and it is also continuous at the origin if $V(x)$ has the scp. We define $\hat{k}$ as a monotone increasing or monotone decreasing continuous function such that $\hat{k} = 2$ if $P(x) \le 0$, and $\hat{k} \to k$ as $P(x) \to 1$. In fact, the monotonicity is not necessary. But, we use $\hat{k} \le k$ $(k \ge 2)$ and $\hat{k} > k$ $(1 \le k < 2)$ in the following argument. Note that input constraint (13) and an inequality $\dot{V}(x,u) < 0$ $(\forall\, 0 \ne x \in W)$ have to be satisfied.

In the case of $k \simeq 1$ or $k \simeq \infty$, input (19) may have chattering because $\bar{U}'_{\hat{k}}$ may become a ball too 'slowly'. 'Fast' transformation of $\bar{U}'_{\hat{k}}$ into a ball is necessary for avoiding chettering phenomenon. Namely, $\hat{k}$ have to become 2 'fast' enough. On the other hand, $\bar{U}'_{\hat{k}}$ has to be large enough to hold $\dot{V}(x,u) < 0$ $(\forall\, 0 \ne x \in W)$ under the input constraint $u \in \bar{U}'_{\hat{k}}$. Hence, $\hat{k}$ is limited if $L_fV > 0$. We obtain necessary conditions to hold $\dot{V}(x,u) < 0$ $(\forall\, 0 \ne x \in W)$ as the following:

Remark 1. (Choice of $\hat{k}$). The directional vector of input (19) corresponds to the input that minimizes $\dot{V}(x,u)$ in $\bar{U}_{\hat{k}}$. And input (19) have to satisfy input constraint (13). If $L_gV \ne 0$, the input such that the directional vector corresponds to the input that minimizes $\dot{V}(x,u)$ in $\bar{U}_{\hat{k}}$ and the input exists on the boundary of $\bar{U}_k$ can be writtten as the following:

$$u_i = -\frac{|L_{g_i}V|^{\frac{1}{\hat{k}-1}}}{\|L_gV\|^{\frac{1}{\hat{k}-1}}_{\frac{k}{k-1}}}\,\mathrm{sgn}(L_{g_i}V) \qquad (i = 1,\ldots,m). \tag{20}$$

i) We consider the case of $k \ge 2$. From Lemma 1 and $k \ge \hat{k}$, input (20) achieves

$$\dot{V}(x) \le P\|L_gV\|_{\frac{k}{k-1}} - \|L_gV\|_{\frac{\hat{k}}{\hat{k}-1}}. \tag{21}$$

The right hand side of the equation becomes maximum when $|L_{g_1}V| = \cdots = |L_{g_m}V|$ (See Fig. 2). Assigning the values $|L_{g_1}V| = \cdots = |L_{g_m}V|$ into (21), we can achieve the following necessary condition to satisfy $\dot{V}(x,u) < 0$:

$$\hat{k} \ge \frac{k}{1 - k\log_m P}. \tag{22}$$

Therefore, we have to choose $\hat{k}$ such that inequality (22) is satisfied and $\hat{k}$ becomes to 2 quickly enough to occur no chattering in inputs.

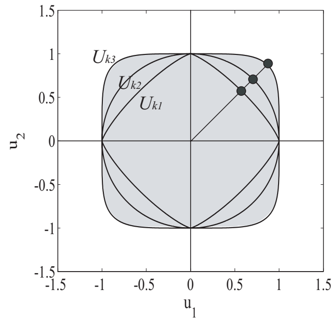

Fig. 2. Comparison of norms $(1 \le k_1 < k_2 < k_3)$

ii) We consider the case of $1 \le k < 2$. From Lemma 1 and $k < \hat{k}$, input (20) achieves

$$\dot{V}(x) \le \frac{\|L_gV\|_{\frac{k}{k-1}}}{\|L_gV\|^{\frac{1}{\hat{k}-1}}_{\frac{k}{k-1}}}\left(P\|L_gV\|^{\frac{1}{\hat{k}-1}}_{\frac{k}{k-1}} - \|L_gV\|^{\frac{1}{\hat{k}-1}}_{\frac{\hat{k}}{\hat{k}-1}}\right). \tag{23}$$

The term in bracket $(\cdot)$ becomes maximum when $|L_{g_1}V| = \cdots = |L_{g_m}V|$. Assigning the values $|L_{g_1}V| = \cdots = |L_{g_m}V|$ into (23), we get a necessary condition to satisfy $\dot{V}(x,u) < 0$ as the following:

$$\hat{k} \le \frac{k}{1 + k\log_m P}. \tag{24}$$

Therefore, we have to choose $\hat{k}$ such that inequality (24) is satisfied and $\hat{k}$ becomes to 2 quickly enough to avoid chattering in inputs.

Although $b_4(x)$ and $\hat{k}$ are not obtained uniquely, we propose the following selection:

Theorem 4. Let $V(x)$ be a local clf for system (6) with input constraint (13), W be a domain in Theorem 2, $P(x)$ be a function defined by (16), $c > 0$ and $q \ge 1$ are constants, and $m \ge 2$. We define $\hat{k}$ and $\bar{k}$ as the following:

$$\hat{k} = \begin{cases} \dfrac{k}{1 - k\log_m\left\{P + (1-P)m^{-\frac{|k-2|}{2k}}\right\}\mathrm{sgn}(k-2)} & (P > 0) \\ 2 & (P \le 0) \end{cases} \tag{25}$$

$$\bar{k} = \begin{cases} \hat{k} & (k \ge 2) \\ k & (1 \le k < 2). \end{cases} \tag{26}$$

Then, the input

$$u_i = \frac{-(P+|P|+c\|L_gV\|_q)}{(P+|P|)\left(1-m^{-\frac{|k-2|}{2k}}\right)+2m^{-\frac{|k-2|}{2k}}+c\|L_gV\|_q} \cdot \frac{|L_{g_i}V|^{\frac{1}{\hat{k}-1}}}{\|L_gV\|^{\frac{1}{\hat{k}-1}}_{\frac{\hat{k}}{\hat{k}-1}}}\,\mathrm{sgn}(L_{g_i}V) \qquad (L_gV \neq 0)$$

$$u_i = 0 \qquad (L_gV = 0) \qquad (i=1,\ldots,m) \tag{27}$$

asymptotically stabilizes the origin in domain W. and it is continuous on $W\backslash\{0\}$. Moreover, if $V(x)$ has the scp, the input is also continuous at the origin.

Proof 2. In the case of $L_gV = 0$, input constraint (13) is satisfied clearly. From Theorem 2, we get $\dot{V}(x) = L_fV < 0$ for all $0 \neq x \in W$.

We consider the case of $L_gV \neq 0$. From Lemma 1, note that $\|\cdot\|_k \le \|\cdot\|_{\hat{k}}$ in the case of $k \ge \hat{k}$. From the fact and $P(x) < 1$, we get

$$\|u\|_k \le \frac{P+|P|+c\|L_gV\|_q}{P+|P|+(2-P-|P|)\,m^{-\frac{|k-2|}{2k}}+c\|L_gV\|_q} < 1.$$

Therefore, input constraint (13) is satisfied. If $\delta < 1$, $\|L_gV\|_q < \delta$, and $L_fV < \delta\|L_gV\|_{k/(k-1)}$, then $\|u\|_k < (2+c)m^{\frac{|k-2|}{2k}}\delta$. Furthermore, $\|u\|_k$ can be made as small as desired when δ is taken to be small enough. In the case of $P(x) \le 0$, the condition $\dot{V}(x) < 0$ is satisfied obviously. We consider the case of $0 < P(x) < 1$.

i) In the case of $k \ge 2$, input (27) achieves

$$\dot{V}(x) < (2P+c\|L_gV\|_q)\,y_1(x) \Big/ \left[2\left\{P+(1-P)m^{\frac{2-k}{2k}}\right\}+\|L_gV\|_{\frac{k}{k-1}}\right],$$

where

$$y_1(x) = \left\{P+(1-P)m^{\frac{2-k}{2k}}\right\}\|L_gV\|_{\frac{k}{k-1}} - \|L_gV\|_{\frac{\hat{k}}{\hat{k}-1}}.$$

$y_1(x)$ becomes maximum when $|L_{g_1}V| = \cdots = |L_{g_m}V|$. From the values $|L_{g_1}V| = \cdots = |L_{g_m}V|$ and (25), we obtain $y_1(x) < 0$ and $\dot{V}(x) < 0$.

ii) In the case of $1 \le k < 2$, input (27) achieves

$$\dot{V}(x) < (2P+c\|L_gV\|_q)\,\|L_gV\|^{-\frac{1}{\hat{k}-1}}_{\frac{k}{\hat{k}-1}}\|L_gV\|_{\frac{\hat{k}}{\hat{k}-1}} \cdot y_2(x) \Big/ \left[2\left\{P+(1-P)m^{\frac{k-2}{2k}}\right\}+\|L_gV\|_{\frac{k}{k-1}}\right],$$

where

$$y_2(x) = \left\{P+(1-P)m^{\frac{k-2}{2k}}\right\}\|L_gV\|^{\frac{1}{\hat{k}-1}}_{\frac{k}{\hat{k}-1}} - \|L_gV\|^{\frac{1}{\hat{k}-1}}_{\frac{\hat{k}}{\hat{k}-1}}.$$

$y_2(x)$ becomes maximum when $|L_{g_1}V| = \cdots = |L_{g_m}V|$. From the values $|L_{g_1}V| = \cdots = |L_{g_m}V|$ and (25), we obtain $y_2(x) < 0$ and $\dot{V}(x) < 0$.

Input (27) asymptotically stabilizes the origin in domain W since $\dot{V}(x) < 0$ ($\forall\, 0 \neq x \in W$) □

5. SIMULATION

In this section, we consider the same example as (N. Kidane and Nishitani, 2005):

$$\begin{aligned}\dot{x}_1 &= x_1 - 4x_2 + u_1\\ \dot{x}_2 &= x_2 + u_2\end{aligned} \tag{28}$$

with an input constraint $\|u\|_\infty < 1$. We choose a local clf as $V(x) = (x_1^2 + x_2^2)/2$. From (16), we get

$$P = \frac{x_1^2 - 4x_1x_2 + x_2^2}{\|x\|_1}. \tag{29}$$

We set $c = 1$ and $q = 1$ in (27). Then, the controller

$$u_i = \begin{cases} -\dfrac{P+|P|+\|x\|_1}{\left(1-\frac{1}{\sqrt{2}}\right)(P+|P|)+\sqrt{2}+\|x\|_1} \cdot \dfrac{|x_i|^{\frac{1}{\hat{k}-1}}}{\|x\|^{\frac{1}{\hat{k}-1}}_{\frac{\hat{k}}{\hat{k}-1}}}\,\mathrm{sgn}(x_i) & (x \neq 0)\\ 0 & (x = 0)\end{cases} \qquad (i=1,2) \tag{30}$$

asymptotically stabilizes the origin in domain $W = \{x | x_1^2 + x_2^2 < 2/9\}$, where

$$\hat{k} = -\frac{1}{\log_2\left\{P+(1-P)\frac{1}{\sqrt{2}}\right\}}.$$

Let $x(0) = (-0.3, 0.3)^T$ be an initial state. Figure 3 and Fig. 4 show the trajectory of the state and the change in the input, respectively. The trajectory converges to zero, and the input constraint $\|u\|_\infty < 1$ is satisfied. In the example of our previous paper (N. Kidane and Nishitani, 2005), we have admited chattering phenomenon in input u_2. On the hand, Fig. 4 demonstrates continuous responce of the input.

6. CONCLUSION

In this paper, we have proposed a stabilizing controller that is continuous except the origin in any case of $k \ge 1$ as the folloeing: First, we considered a continuous function $\hat{k}$ and a subspace $\bar{U}'_{\hat{k}} \subset \bar{U}_{\hat{k}}$ such that $\bar{U}'_{\hat{k}}$ is similar to

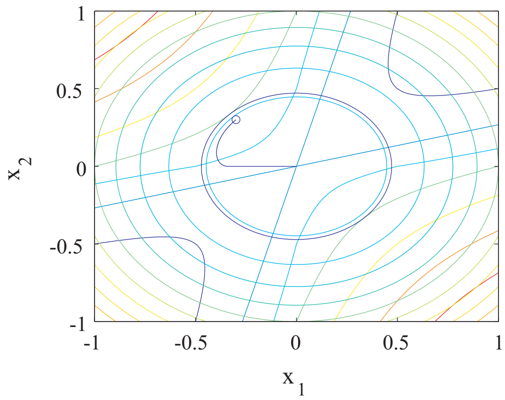

Fig. 3. Trajectory with input (30)

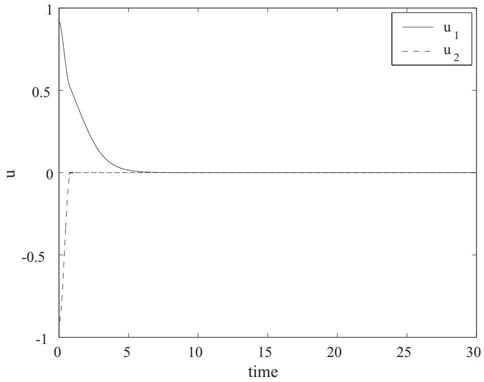

Fig. 4. Change in input (30)

$\bar{U}_{\hat{k}}$ (same shape), $\bar{U}'_{\hat{k}}$ becomes a small ball if $P(x) \leq 0$, and $\bar{U}'_{\hat{k}} \to \bar{U}_k$ as $P(x) \to 1$. Second, we stabilized the system by the input that minimizes $\dot{V}(x,u)$ in $\bar{U}'_{\hat{k}}$. We have obtained necessary conditions to hold $\dot{V}(x,u) < 0$ ($\forall\ 0 \neq x \in W$). Moreover, we have demonstrated the controller's effectiveness by computer simulation.

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