CONTROLLER FOR A NONLINEAR SYSTEM WITH AN INPUT CONSTRAINT BY USING A CONTROL LYAPUNOV FUNCTION I

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Abstract: In this paper, we generalize the Malisoff's controller for a nonlinear system with an input constraint. Malisoff and Sontag proposed a universal control formula for a nonlinear system such that the k-norm of inputs is less than one. However, k is limited to $1 < k \leq 2$. We improve the Malisoff's formula so that it can be applied in any case of $k \geq 1$. We also confirm the effectiveness of the improved controller by computer simulation.

Keywords: nonlinear systems, constraints, controllers, Lyapunov methods

1. INTRODUCTION

Actual systems have input constraints due to the limits on performance and for protection of the systems. Lin and Sontag proposed a universal control formula for a nonlinear system such that the 2-norm of inputs is less than one by using a control Lyapunov function (Lin and Sontag, 1991). Malisoff and Sontag provided a universal control formula for a nonlinear system such that the k-norm of inputs is less than one (Malisoff and Sontag, 2000). However, k is limited to $1 < k \leq 2$.

In this research, we propose a new control formula that can be applied in any case of $k \geq 1$. First, we look for a domain in which the origin is asymptotically stabilizable. Secondly, we derive the input that minimizes the derivative of a local control Lyapunov function under the input constraint. Thirdly, we design a stabilizing controller that can be applied in any case of $k \geq 1$. Finally, we confirm the effectiveness of the proposed controller by computer simulation.

2. PRELIMINARY

In this section, we introduce mathematical notation and some definitions. We consider the following affine system:

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is a state vector and $u \in U \subseteq \mathbb{R}^m$ is an input vector. We assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous mappings and $f(0) = 0$. We use the notation $\mathbb{R}_{>0} := (0, \infty)$ and $\mathbb{R}_{\geq 0} := [0, \infty)$.

Definition 1. (control Lyapunov function). A smooth proper positive definite function defined on a neighborhood of the origin $X \in \mathbb{R}^n$, $V : X \to \mathbb{R}_{\geq 0}$ is said to be a local control Lyapunov function for system (1) if the condition

$$\inf_{u \in U} \{L_f V + L_g V \cdot u\} < 0 \tag{2}$$

is satisfied for all $x \in X$, $x \neq 0$. Moreover, $V(x)$ is said to be a control Lyapunov function (clf) for system (1) if $V(x)$ is a function defined on $\mathbb{R}^n$ and condition (2) is satisfied for all $x \in \mathbb{R}^n$, $x \neq 0$. □

Definition 2. (small control property). A (local) control Lyapunov function is said to satisfy the small control property (scp) if for any $\varepsilon > 0$, there is a $\delta > 0$ such that, if $x \neq 0$ satisfies $\|x\| < \delta$, then there is some $u \in U$ with $\|u\| < \varepsilon$ such that $L_f V + L_g V \cdot u < 0$. □

If there exists no input constraint ($U \equiv \mathbb{R}^m$), a smooth radially unbounded positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a clf if and only if

$$L_g V = 0 \quad \Longrightarrow \quad L_f V < 0, \quad \forall\, x \neq 0. \tag{3}$$

We define $h(x)$ as the right hand side of system (1) with a state feedback law $u = \beta(x)$;

$$\dot{x} = f(x) + g(x)\beta(x) := h(x). \tag{4}$$

If $\beta(x)$ is continuous except the origin, the closed system has always a Carathéodory solution for each initial state. On the other hand, if $\beta(x)$ is not continuous, Carathéodory solution do not exist. Hence, we associate (4) with a differential inclusion of the form

$$\dot{x} \in F(x). \tag{5}$$

In this paper, we apply the Fillippov's approach

$$F(x) = \bigcap_{\varepsilon > 0} \bigcap_{\mu_n(N)} \overline{co}\{h(B_\varepsilon(x) \backslash N)\}, \tag{6}$$

where $B_\varepsilon(x)$ denotes the open ball of center x and radius ε, $\overline{co}$ denotes the convex closure of a set, and μ_n is the Lebesgue measure of $\mathbb{R}^n$.

Definition 3. (Lyapunov function). A smooth and positive definite function defined on a neighborhood of the origin $X \subset \mathbb{R}^n$, $V : X \to \mathbb{R}_{\geq 0}$ is said to be a local Lyapunov function for system (5) if the following condition is satisfied for all $0 \neq x \in X$:

$$\frac{\partial V}{\partial x} \cdot v < 0, \quad \forall\, v \in F(x). \tag{7}$$

Moreover, $V(x)$ is said to be a Lyapunov function for system (5) if $V(x)$ is a radially unbounded function defined on $\mathbb{R}^n$ and condition (7) is satisfied for all $0 \neq x \in \mathbb{R}^n$. □

Theorem 1. (Bacciotti and Rosier, 2001) Let F be a set-valued map such that the local existence of solutions of (5) is insured. If a (local) Lyapunov function exists, then the origin is (locally) asymptotically stable. □

3. MALISOFF'S UNIVERSAL FORMULA

When there is not any input constraint, Sontag proposed a universal control formula for a nonlinear system (Sontag, 1989). In this paper, we consider a nonlinear system such that inputs are restricted to the Minkowski ball of radius 1;

$$U_k = \left\{ u \in \mathbb{R}^m \;\middle|\; \|u\|_k = \left(\sum_{i=1}^{m} |u_i|^k \right)^{\frac{1}{k}} < 1 \right\}, \tag{8}$$

where $k \geq 1$. Lin and Sontag provided a universal control formula with respect to Minkowski ball U_2 (Lin and Sontag, 1991). Malisoff and Sontag improved the Lin's controller so that it can be applied in the case of $1 < k \leq 2$ (Malisoff and Sontag, 2000).

Theorem 2. (Malisoff's universal control formula). We consider system (1) with input constraint (8). We assume that $f(x)$ and $g(x)$ are smooth. Let $V(x)$ be a clf for the system and $k = 2r/(2r-1)$, where $r > 0$ is an integer. Then, the input

$$u_i = -\frac{L_f V + \left(L_f V^{2r} + \|L_g V\|_{2r}^{4r^2} \right)^{\frac{1}{2r}}}{1 + \left(1 + \|L_g V\|_{2r}^{2r(2r-1)} \right)^{\frac{1}{2r}}} \cdot \frac{L_{g_i} V^{2r-1}}{\|L_g V\|_{2r}^{2r}} \quad (L_g V \neq 0)$$

$$u_i = 0 \quad (L_g V = 0)$$

$$(i = 1, \ldots, m) \tag{9}$$

is smooth on $\mathbb{R}^n \backslash \{0\}$, and globally asymptotically stabilizes the origin. Moreover, if the right hand side of (1) is real analytic in x and $V(x)$ is real analytic, then the input is real analytic on $\mathbb{R}^n \backslash \{0\}$. Furthermore, if $V(x)$ has the scp, then the input is continuous at the origin. □

In the case of $L_g V \neq 0$, Malisoff's controller (9) can be rewritten as

$$u_i = -b_1(x) \frac{L_{g_i} V^{2r-1}}{\|L_g V\|_{2r}^{2r-1}} = -b_1(x) \frac{L_{g_i} V^{\frac{1}{k-1}}}{\|L_g V\|_{\frac{k}{k-1}}^{\frac{1}{k-1}}} \quad (i = 1, \ldots, m), \tag{10}$$

where $b_1 : \mathbb{R}^n \to \mathbb{R}_{>0}$. Note that k is limited to $1 < k \leq 2$.

4. DOMAIN IN WHICH THE ORIGIN IS ASYMPTOTICALLY STABILIZABLE

The objective of this paper is to propose a stabilizing controller that can be applied in any case of $k \geq 1$ by using a local control Lyapunov function. For a simplicity, we assume that a local clf satisfies condition (3) for all $x \in \mathbb{R}^n$. We define the derivative of a local clf $\dot{V}(x,u)$ by the following equation:

$$\dot{V}(x,u) = L_f V + L_g V \cdot u, \tag{11}$$

where we permit discontinuous inputs. We can guarantee a domain in which the origin is asymptotically stabilizable as the following:

Theorem 3. Let $V(x)$ be a local clf for system (1) with input constraint (8), and $a_1 > 0$ be the maximum number such that the condition

$$\inf_{u \in U_k} \{L_f V + L_g V \cdot u\} < 0, \qquad \forall\, x \neq 0 \tag{12}$$

is satisfied for all $x \in W = \{x | V(x) < a_1\}$. Then, W is a domain in which the origin is asymptotically stabilizable. If $V(x)$ is a clf, then $a_1 = \infty$ and $W = \mathbb{R}^n$. □

Since condition (12) is satisfied for all $x \in W$, we can design a controller such that $\dot{V}(x,u) < 0$ ($\forall\, 0 \neq x \in W$). Such a controller takes the state into the level set of the local clf, and the trajectory does not go out of W. Hence, W is a domain in which the origin is asymptotically stabilizable. We prove Theorem 3 after we design a stabilizing controller.

5. INPUT THAT MINIMIZES $\dot{V}(X,U)$ IN $\bar{U}_K$

In this section, we derive the input that minimizes $\dot{V}(x,u)$ under an input constraint $u \in \bar{U}_k$, where $\bar{U}_k$ is the closure of U_k. We also reach an important condition that is satisfied in domain W.

From (11), we find that the input that minimizes $\dot{V}(x,u)$ in $\bar{U}_k$ is equivalent to the input that minimizes $L_g V \cdot u$ in $\bar{U}_k$. Noticing the fact, we obtain the following proposition:

Proposition 1. We consider system (1) with an input constraint $u \in \bar{U}_k$. Let $V(x)$ be a local clf for the system. Then, the input

$$u_i = \begin{cases} -\dfrac{|L_{g_i} V|^{\frac{1}{k-1}}}{\|L_g V\|_{\frac{k}{k-1}}^{\frac{1}{k-1}}} \operatorname{sgn}(L_{g_i} V) & (L_g V \neq 0) \\ 0 & (L_g V = 0) \end{cases} \quad (i = 1, \ldots, m) \tag{13}$$

minimizes the derivative $\dot{V}(x,u)$ for each x. □

Proof 1. An input that minimizes $\dot{V}(x,u)$ in $\bar{U}_k$ is equivalent to an input that minimizes $L_g V \cdot u$ in $\bar{U}_k$. For each x such that $L_g V \neq 0$, we consider (hyper) surface $Q : L_g V \cdot u = a_2$ such that $\bar{U}_k \cap Q \neq \phi$. Consider the problem of minimizing $a_2 \in \mathbb{R}$. If Q is tangent to $\bar{U}_k$ at a point $\bar{u}$ and $L_g V \cdot \bar{u} < 0$, the contact point $\bar{u}$ denotes the input that minimizes $\dot{V}(x,u)$ in $\bar{U}_k$ (See Fig. 1).

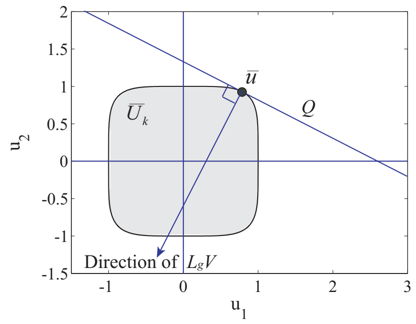

Fig. 1. Input that minimizes $\dot{V}(x,u)$ in $\bar{U}_k$

Let $\partial \bar{U}_k$ be the boundary of $\bar{U}_k$. Then, the following equation denotes $\partial \bar{U}_k$:

$$S := \sum_{i=1}^{m} |u_i|^k - 1 = 0. \tag{14}$$

i) In the case of $1 < k < \infty$, S is differentiable. From (14), the plane that is tangent to $\bar{U}_k$ at $\bar{u}$ can be described as the following:

$$\sum_{i=1}^{m} |\bar{u}_i|^{k-1} \operatorname{sgn}(\bar{u}_i) u_i = 1. \tag{15}$$

By (15), we can obtain the following equation such that plane $Q : L_g V \cdot u = a_2$ is tangent to $\bar{U}_k$ at $\bar{u}$, and $L_g V \cdot \bar{u} < 0$:

$$\begin{aligned} &\left(|\bar{u}_1|^{k-1} \operatorname{sgn}(\bar{u}_1), \ldots, |\bar{u}_m|^{k-1} \operatorname{sgn}(\bar{u}_m)\right) \\ &\qquad = -a_3 \left(L_{g_1} V, \ldots, L_{g_m} V\right), \end{aligned} \tag{16}$$

where $a_3 > 0$. From (14) and (16), we obtain input (13).

ii) In the case of $k = 1$, $\bar{U}_1$ becomes

$$\bar{U}_1 = \left\{ u \in \mathbb{R}^m \,\middle|\, \|u\|_1 = \sum_{i=1}^{m} |u_i| \leq 1 \right\}, \tag{17}$$

and we get

$$L_g V \cdot u \geq -\sum_{i=1}^{m} |L_{g_i} V| \cdot |u_i| \geq -\|L_g V\|_\infty. \tag{18}$$

Set $u_i = 0$ ($|L_{g_i} V| \neq \max_{i=1,\ldots,m} |L_{g_i} V|$) and $u_i = -|L_{g_i} V|^\infty \operatorname{sgn}(L_{g_i} V) / \sum_{i=1,\ldots,m} |L_{g_i} V|^\infty$ ($|L_{g_i} V| = \max_{i=1,\ldots,m} |L_{g_i} V|$). Then, the input achieves $L_g V \cdot u = -\max_{i=1,\ldots,m} |L_{g_i} V|$. From (18), we find that the input minimizes $L_g V \cdot u$ in $\bar{U}_1$. The input corresponds to input (13) in the case of $k = 1$.

iii) In the case of $k = \infty$, $\bar{U}_\infty$ becomes

$$\bar{U}_\infty = \left\{ u \in \mathbb{R}^m \,\middle|\, \|u\|_\infty = \max_{i=1,\ldots,m} |u_i| \leq 1 \right\}, \tag{19}$$

and we obtain

$$L_g V \cdot u \geq -\sum_{i=1}^{m} |L_{g_i} V| \cdot |u_i| \geq -\sum_{i=1}^{m} |L_{g_i} V|. \tag{20}$$

Set $u_i = -\operatorname{sgn}(L_{g_i} V)$. Then, the input achieves $L_g V \cdot u = -\sum_{i=1}^{m} |L_{g_i} V|$. From (20), we find that the input minimizes $L_gV \cdot u$ in $\bar{U}_\infty$. The input corresponds to input (13) in the case of $k = \infty$. □

The k-norm of (13) does not become small even if the state is near the origin. So, it is not appropriate to stabilize the system by input (13). Note that the directional vector of Malisoff's controller (9) corresponds to input (13).

Input (13) gives the minimum value of $\dot{V}(x,u)$ as the following:

$$\dot{V}(x) = L_fV - \|L_gV\|_{\frac{k}{k-1}}. \tag{21}$$

If $L_gV \neq 0$, we define

$$P(x) = \frac{L_fV}{\|L_gV\|_{\frac{k}{k-1}}}. \tag{22}$$

If $P(x) < 1$, condition (12) is satisfied. If $P(x) \geq 1$, however, condition (12) is not satisfied. Namely, the domain in which $P(x) < 1$ contains domain W. Hence, we obtain the following lemma:

Lemma 1. Let $V(x)$ be a local clf for system (1) with input constraint (8), W be a domain in Theorem 3, $P(x)$ be a function defined by (22). Then,

$$\sup_{x \in \{x \in W | L_gV(x) \neq 0\}} P(x) = 1. \tag{23}$$

□

6. CONTROLLER DESIGN

In this section, we propose a new controller such that the directional vector corresponds to input (13). The controller is continuous on $W \backslash \{0\}$ if $m = 1$ or $1 < k < \infty$, and it is also continuous at the origin if $V(x)$ has the scp. For this purpose, we consider a subset $\bar{U}'_k \subset \bar{U}_k$ such that $\bar{U}'_k$ is similar to $\bar{U}_k$ (same shape), $\bar{U}'_k$ becomes small if $P(x)$ becomes small, and $\bar{U}'_k \to \bar{U}_k$ as $P(x) \to 1$. To stabilize the system, we choose the input that minimizes $\dot{V}(x,u)$ in $\bar{U}'_k$ (See Fig. 2). Note that subset $\bar{U}'_k$ has to be large enough to hold $\dot{V}(x,u) < 0$ ($\forall\ 0 \neq x \in W$) under the input constraint $u \in \bar{U}'_k$.

If $L_gV \neq 0$, the input that minimizes $\dot{V}(x,u)$ in $\bar{U}'_k$ can be written as

$$u_i = -b_2(x) \frac{|L_{g_i}V|^{\frac{1}{k-1}}}{\|L_gV\|^{\frac{1}{k-1}}_{\frac{k}{k-1}}} \operatorname{sgn}(L_{g_i}V) \quad (i = 1, \ldots, m), \tag{24}$$

where $b_2 : \mathbb{R}^n \to \mathbb{R}_{>0}$. Input (24) achieves

$$\dot{V}(x) = L_fV - b_2(x)\|L_gV\|_{\frac{k}{k-1}}. \tag{25}$$

In order to hold $\dot{V}(x) < 0$, we choose $b_2(x)$ as the following:

$$b_2(x) = \frac{P(x) + |P(x)|}{2} + b_3(x), \tag{26}$$

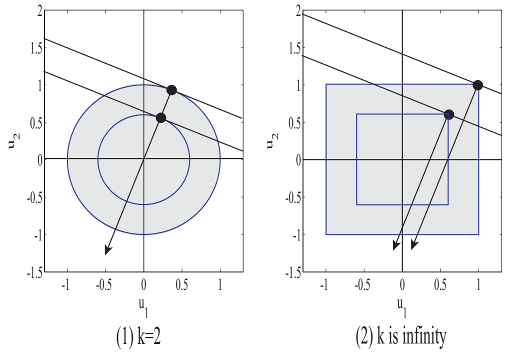

Fig. 2. The input that minimizes $\dot{V}(x,u)$ in $\bar{U}'_k$

where $b_3 : \mathbb{R}^n \to \mathbb{R}_{>0}$. Then, $\dot{V}(x) = L_fV - b_3(x)\|L_gV\|_{k/(k-1)} < 0$ if $L_fV \leq 0$, and $\dot{V}(x) = -b_3(x)\|L_gV\|_{k/(k-1)} < 0$ if $L_fV > 0$. We choose $b_3(x)$ such that input constraint (8) is satisfied, input (24) is continuous on $\mathbb{R}^n \backslash \{0\}$ if $m = 1$ or $1 < k < \infty$, and it is also continuous at the origin if $V(x)$ has the scp. Although $b_3(x)$ is not obtained uniquely, we propose the following selection:

Theorem 4. Let $V(x)$ be a local clf for system (1) with input constraint (8), W be a domain in Theorem 3, $P(x)$ be a function defined by (22), $c > 0$ and $q \geq 1$ are constants. Then, the input

$$\begin{aligned} u_i &= -\frac{P + |P| + c\|L_gV\|_q}{2 + c\|L_gV\|_q} \cdot \frac{|L_{g_i}V|^{\frac{1}{k-1}}}{\|L_gV\|^{\frac{1}{k-1}}_{\frac{k}{k-1}}} \operatorname{sgn}(L_{g_i}V) \\ &\qquad\qquad (L_gV \neq 0) \\ u_i &= 0 \qquad\qquad (L_gV = 0) \\ &\qquad\qquad (i = 1, \ldots, m) \end{aligned} \tag{27}$$

asymptotically stabilizes the origin in domain W. If $m = 1$ or $1 < k < \infty$, the input is continuous on $W \backslash \{0\}$. Moreover, if $V(x)$ has the scp, the input is also continuous at the origin. □

Proof 2. i) In the case of $L_gV = 0$, input constraint (8) is satisfied clearly. From Theorem 3, we get $\dot{V}(x) = L_fV < 0$ for all $0 \neq x \in W$.

ii) We consider the case of $L_gV \neq 0$. From $P(x) < 1$, we obtain

$$\|u\|_k = \frac{P + |P| + c\|L_gV\|_q}{2 + c\|L_gV\|_q} < 1.$$

Therefore, input constraint (8) is satisfied. If $\delta < 1$, $\|L_gV\|_q < \delta$, and $L_fV < \delta\|L_gV\|_{k/(k-1)}$, then $\|u\|_k < (2 + c)\delta$. Furthermore, $\|u\|_k$ can be made as small as desired when δ is taken to be small enough. If $P(x) \leq 0$, we get $\dot{V}(x) < 0$ obviously. If $0 < P(x) < 1$, input (27) becomes

$$u_i = -\left\{P + \frac{c(1-P)\|L_gV\|_q}{2 + c\|L_gV\|_q}\right\} \cdot \frac{|L_{g_i}V|^{\frac{1}{k-1}}}{\|L_gV\|^{\frac{1}{k-1}}_{\frac{k}{k-1}}} \operatorname{sgn}(L_{g_i}V),$$

and it achieves

$$\dot{V}(x) = \frac{c(P-1)\|L_gV\|_q\|L_gV\|_{\frac{k}{k-1}}}{2+c\|L_gV\|_q} < 0.$$

Input (27) asymptotically stabilizes the origin in domain W since $\dot{V}(x) < 0$ $(\forall\, 0 \neq x \in W)$ □

The q-norm $\|L_gV\|_q$ in (27) can be replaced by other appropriate functions. Note that the directional vector of input (27) corresponds to the directional vector of Malisoff's controller (9). This implies that (27) is a generalized controller with respect to U_k. From Malisoff's formula (9) and $2r = k/(k-1)$, we can obtain another controller that can be applied in any case of $k \geq 1$ as the following:

$$u_i = \begin{cases} -\dfrac{L_fV + \left(|L_fV|^{\frac{k}{k-1}} + \|L_gV\|_{\frac{k}{k-1}}^{\frac{k}{(k-1)^2}}\right)^{\frac{k-1}{k}}}{1+\left(1+\|L_gV\|_{\frac{k}{k-1}}^{\frac{k}{(k-1)^2}}\right)^{\frac{k-1}{k}}} \\ \cdot \dfrac{|L_{g_i}V|^{\frac{1}{k-1}}}{\|L_gV\|_{\frac{k}{k-1}}^{\frac{k}{k-1}}}\,\mathrm{sgn}(L_{g_i}V) & (L_gV \neq 0) \\ 0 & (L_gV = 0) \end{cases}$$
$$(i = 1, \ldots, m). \quad (28)$$

If $f(x)$ and $g(x)$ are smooth, Malisoff's controller (9) is smooth except the origin. It is natural to ask whether input (24) becomes smooth as well. We consider the case of $k > 2$. From $1/(k-1) < 1$, $|L_{g_i}V|^{1/(k-1)}\,\mathrm{sgn}(L_{g_i}V)$ is not differentiable at points such that $L_{g_i}V = 0$. Therefore, input (24) is not smooth for any $b_2(x)$. In the case of $1 < k \leq 2$, we obtain $1/(k-1) \geq 1$. Hence, if we choose smooth $b_2(x)$, input (24) becomes at least a C^1 function. The $k/(k-1)$th root in Malisoff's controller (9) is employed in order to make (9) real analytic when $k = 2r/(2r-1)$. In this paper, we use the absolute value instead of the $k/(k-1)$th root because we need only continuity.

Proof 3. We prove Theorem 3. Input (27) ahieves $\dot{V}(x) < 0$ for all $0 \neq x \in W$. The trajectory of the state goes into the level set of the local clf and does not go out of domain W. So, W is a domain in which the origin is asymptotically stabilizable. □

7. SIMULATION

We consider a system

$$\begin{aligned} \dot{x}_1 &= x_1 - 4x_2 + u_1 \\ \dot{x}_2 &= x_2 + u_2 \end{aligned} \quad (29)$$

with an input constraint $\|u\|_k < 1$. We choose a local clf as $V(x) = (x_1^2 + x_2^2)/2$.

i) We consider the case of $k = 3$. From (22), we get

$$P = \frac{x_1^2 - 4x_1x_2 + x_2^2}{\|x\|_{\frac{3}{2}}}. \quad (30)$$

We set $c = 1$ and $q = 1$ in (27). Then, the controller

$$u_i = \begin{cases} -\dfrac{P+|P|+\|x\|_1}{2+\|x\|_1} \cdot \dfrac{|x_i|^{\frac{1}{2}}}{\|x\|_{\frac{3}{2}}^{\frac{1}{2}}}\,\mathrm{sgn}(x_i) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$
$$(i = 1, 2) \quad (31)$$

asymptotically stabilizes the origin in domain $W = \{x | x_1^2 + x_2^2 < \sqrt[3]{2}/9\}$, where we get W from the definition of $V(x)$ and $P(x) = 1$. Let $x(0) = (-0.26, 0.26)^T$ be an initial state. Figure 3, Fig. 4, and Fig. 5 show the trajectory of the state, the change in the input, and the norm $\|u\|_3$, respectively. The trajectory converges to zero, and the input constraint $\|u\|_3 < 1$ is satisfied. Moreover, we can confirm that the input is continuous.

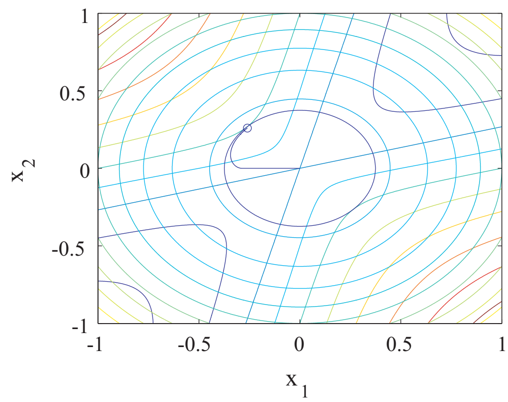

Fig. 3. Trajectory with input (31)

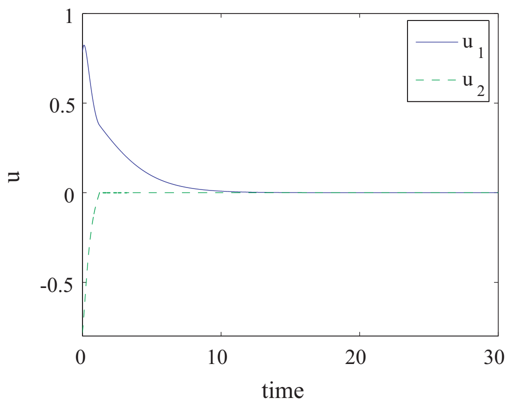

Fig. 4. Change in input (31)

ii) We consider the case of $k = \infty$. From (22), we get

$$P = \frac{x_1^2 - 4x_1x_2 + x_2^2}{\|x\|_1}. \quad (32)$$

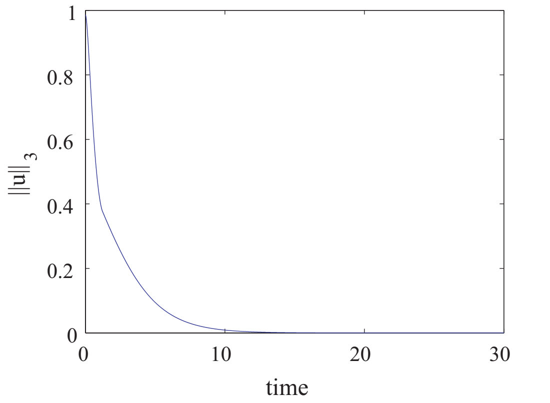

Fig. 5. Change in input (31)

We set $c = 1$ and $q = 1$ in (27). Then, the controller

$$u_i = \begin{cases} -\dfrac{P + |P| + \|x\|_1}{2 + \|x\|_1} \operatorname{sgn}(x_i) & (x \neq 0) \\ 0 & (x = 0) \end{cases} \quad (33)$$
$$(i = 1, 2)$$

asymptotically stabilizes the origin in domain $W = \{x | x_1^2 + x_2^2 < 2/9\}$, where we get W from the definition of $V(x)$ and $P(x) = 1$. Let $x(0) = (-0.3, 0.3)^T$ be an initial state. Figure 6 and Fig. 7 show the trajectory of the state and the change in the input, respectively. The trajectory converges to zero, and the input constraint $\|u\|_\infty < 1$ is satisfied. However, we find that input u_2 has chattering. Fig. 7 shows that the Carathéodory solution to the closed system do not exist.

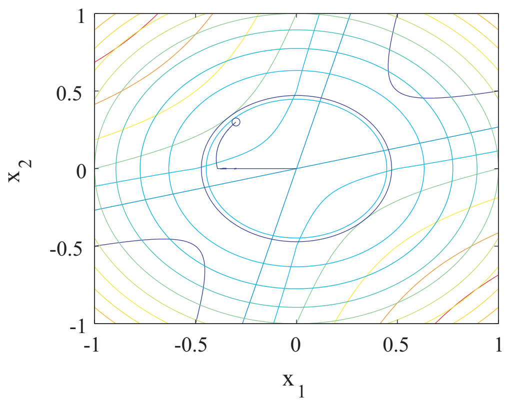

Fig. 6. Trajectory with input (33)

8. CONCLUSION

In this paper, we have generalized the Malisoff's controller so that it can be applied in any case of $k \geq 1$. In Section 4, we have obtained a domain in which the origin is asymptotically stabilizable, W. In Section 5, we have derived the input that minimizes $\dot{V}(x, u)$ in $\bar{U}_k$, and have shown that $P(x) < 1$ is satisfied for all $x \in W$ such that

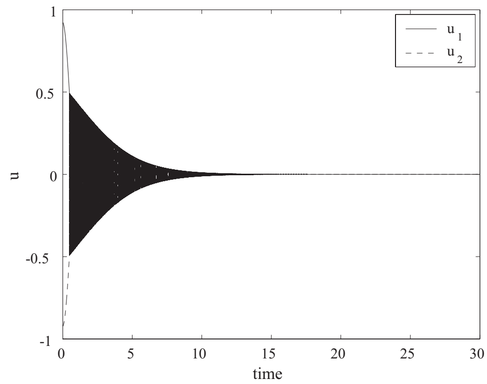

Fig. 7. Change in input (33)

$L_g V \neq 0$. In Section 6, we have proposed a stabilizing controller that can be applied in any case of $k \geq 1$ as the followings: First, we considered a subspace $\bar{U}'_k \subset \bar{U}_k$ such that $\bar{U}'_k$ is similar to $\bar{U}_k$, $\bar{U}'_k$ becomes small if $P(x)$ becomes small, and $\bar{U}'_k \to \bar{U}_k$ as $P(x) \to 1$. Second, we stabilized the origin by the input that minimizes $\dot{V}(x, u)$ in $\bar{U}'_k$. In Section 7, we have demonstrated the controller's effectiveness by computer simulation.

If $m = 1$ or $1 < k < \infty$, the proposed controller is continuous on $W \backslash \{0\}$. If $k = 1$ or $k = \infty$, however, the proposed controller may become discontinuous. This causes a chattering phenomenon. We propose a new stabilizing controller that is continuous on $W \backslash \{0\}$ in any case of $k \geq 1$ in the following paper.

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