# OUTPUT FEEDBACK STOCHASTIC STABILIZATION OF ACTIVE FAULT TOLERANT CONTROL SYSTEMS: LMI FORMULATION

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Abstract: In this paper, we address the dynamic output feedback control problem of continuous time active fault tolerant control system with Markovian parameters (AFTCSMP). We will first derive a necessary and sufficient condition for the exponential stability in the mean square of the AFTCSMP under a dynamic output feedback control, in terms of coupled matrix inequalities, and then we will give an LMI (Linear Matrix Inequalities) characterization of dynamical compensators that stabilize the closed-loop system in the mean square sense. 

Keywords: Active fault tolerant control - Hybrid systems - Markovian jumping parameters - Linear matrix inequalities - Dynamic output feedback.

## 1. INTRODUCTION

Many physical systems have variable structures subject to random changes, which may result from abrupt phenomena such as component failures, parameters shifting, tracking, and time required to mesure some of the variables at different stages. Systems with this character may be modeled as hybrid ones, *i.e*, the state space of the systems contains both discret and continuous states. Among this kind of systems, fault tolerant control systems (FTCS) have been a subject of great practical importance, which has attracted a lot of interest for the last three decades. FTCS have been developed in order to achieve high levels of reliability and performances in situations where the controlled system can have potentially damaging effects on the environment if failures of its components take place. A bibliographical review on reconfigurable fault tolerant control systems can be found in (Zhang and Jiang, 2003).
The dynamic behaviour of active fault tolerant control systems (AFTCS) is governed by stochastic differential equations (because the failures and failure detection occur randomly) and can be viewed as a general hybrid system(Srichander and Walker, 1993; Mahmoud *et al.*, 2003). A major class of hybrid systems is jump linear systems (JLS). In JLS, a single jump process is used to describe the random variations affecting the system parameters. This process is represented by a finite state Markov chain and is called the plant regime mode. The theory of stability, optimal control and $H_2/H_\infty$ control, as well as important applications of such systems, can be found in several papers in the current literature, for instance in (de Farias *et al.*, 2000; Ji and Chizeck, 1990).
To deal with AFTCS, another class of hybrid systems was defined, denoted as active fault tolerant control systems with Markovian parameters (AFTCSMP). For AFTCSMP, two random processes are defined: the first random process represents system components failures and the second random process represents the FDI (Fault Detection and Isolation) process used to reconfigure the

control law. This model was proposed by Srichander and Walker (Srichander and Walker, 1993). Necessary and sufficient conditions for stochastic stability of AFTCSMP were developed for a single component failure (actuator failures). In (Mahmoud *et al.*, 1999), the authors proposed a dynamical model that takes into account multiple failures occurring at different locations in the system, such as in control actuators and plant components. The authors derived necessary and sufficient conditions for the stochastic stability in the mean square sense. The problem of stochastic stability of AFTCSMP in the presence of noise, parameter uncertainties, detection errors, detection delays and actuators saturation limits has also been investigated in (Mahmoud *et al.*, 1999; Mahmoud *et al.*, 2001; Mahmoud *et al.*, 2003). Another issue related to the synthesis of fault tolerant control laws was also addressed by (Mahmoud *et al.*, 2000; Shi and Boukas, 1997; Shi *et al.*, 2003). In (Mahmoud *et al.*, 2000), the authors designed an optimal control law for AFTCSMP using the matrix minimum principles to minimize an equivalent deterministic cost function. The problem of $H_\infty$ and robust $H_\infty$ control (in the presence of parametric uncertainties) was treated in (Shi and Boukas, 1997; Shi *et al.*, 2003) for both continuous and discret AFTCSMP.

In this paper the problem of dynamic output feedback control of AFTCSMP is addressed under a convex programming approach. We will first derive a testable necessary and sufficient condition for the exponential stability in the mean square of the AFTCSMP, under a dynamic output feedback control, in terms of coupled matrix inequalities and then we will give an LMI characterization of all dynamical compensators that stabilize the closed-loop system in the mean square sense (to the best of our knowledge, this problematic has not been yet fully investigated in the field of AFTCSMP). This problematic was considered by (de Farias *et al.*, 2000) in the field of JLS. Therefore, the JLS model assumes perfect regime knowledge, and does not take into account the location of a fault and the nature of the faulty components. These assumptions are too restrictive to be used in practical AFTCSMP (Mahmoud *et al.*, 2003).

This paper is organized as follows: section 2 describes the dynamical model of the system with appropriately defined random processes. A brief summary of basic stochastic terms, results and definitions are given in section 3. The mathematical formulation of the AFTCSMP is developed in section 4. Section 5 derives the necessary and sufficient conditions for the stochastic stability in the mean square, and the LMI characterization of the dynamic compensators. Finally, a conclusion is given in section 6.

## 2. DYNAMICAL MODEL OF AFTCSMP

Consider an active fault tolerant control system shown in figure 1. The system under normal operation ($\varphi$) can be described by:

$$\varphi : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the system input and $y(t) \in \mathbb{R}^p$ is the system measured output. For the synthesis of the control action $u(t)$, we introduce a dynamical compensator ($\varphi_c$) of the form:

$$\varphi_c : \begin{cases} \dot{v}(t) = A_c v(t) + B_c y(t) \\ u(t) = C_c v(t) \end{cases} \tag{2}$$

where $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times p}$, $C_c \in \mathbb{R}^{m\times n}$. It is

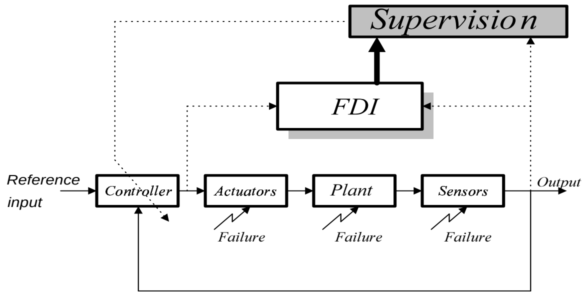


Figure 1. General schematic diagram of AFTCSMP

important to note that a basic point to determine the appropriate dynamical model which describe the faulty system is the location of a fault and the nature of the faulty components. In this paper, we will consider that the system is subject to both actuator and sensor failures. The random changes affecting actuators are represented by a homogeneous Markov process $\eta(t)$ with the finite state space $Z = \{1, 2, ..., z\}$, and the random changes that occur in sensors are represented by another homogeneous Markov process $\xi(t)$ with the finite state space $S = \{1, 2, ..., s\}$. In practice, these random variations are not directly measurable but rather can only be monitored by an FDI scheme. Let $\psi(t)$ denote the state of the FDI process which monitors the states $\eta(t)$ and $\xi(t)$ of the random processes describing the failures. The process $\psi(t)$ is a finite state stochastic process whose random behaviour is conditioned on the failures processes states $\eta(t)$ and $\xi(t)$, therefore, the state space of the FDI process $\psi(t)$ contains the state spaces of the two failure processes (Mahmoud *et al.*, 2003). This state space is also finite and is denoted by $R = \{1, 2, ..., r\}$. In AFTCS, we consider that the control law is only a function of the mesurable FDI process $\psi(t)$. Therefore, the linear AFTCSMP can be modeled as:

$$\varphi : \begin{cases} \dot{x}(t) = Ax(t) + B(\eta(t))u(y(t), \psi(t), t) \\ y(t) = C(\xi(t))x(t) \end{cases} \tag{3}$$

$$\varphi_c : \begin{cases} \dot{v}(t) = A_c(\psi(t))v(t) + B_c(\psi(t))y(t) \\ u(t) = C_c(\psi(t))v(t) \end{cases} \tag{4}$$

where $B(\eta(t))$, $C(\xi(t))$, $A_c(\psi(t))$, $B_c(\psi(t))$ and $C_c(\psi(t))$ are properly dimensioned matrices which depends on random parameters. $\eta(t)$, $\xi(t)$ and $\psi(t)$

are separable measurable Markov processes with finite state spaces $Z = \{1, 2, ..., z\}$, $S = \{1, 2, ..., s\}$ and $R = \{1, 2, ..., r\}$, respectively.

For notational simplicity, we will denote $B(\eta(t)) = B_i$ when $\eta(t) = i \in Z$, $C(\xi(t)) = C_j$ when $\xi(t) = j \in S$, and $A_c(\psi(t)) = A_{ck}$, $B_c(\psi(t)) = B_{ck}$, $C_c(\psi(t)) = C_{ck}$ when $\psi(t) = k \in R$. We also denote $x(t) = x$, $\eta(t) = \eta$, $\xi(t) = \xi$, $\psi(t) = \psi$ and the initial conditions $x(t_0) = x_0$, $\eta(t_0) = \eta_0$, $\xi(t_0) = \xi_0$, and $\psi(t_0) = \psi_0$.

## 3. BASIC DEFINITIONS

In this section, we will summarize some results about exponential stability of AFTCSMP which will be used in the paper. Under the assumption that the system $(\varphi)$ coupled with $(\varphi_c)$ satisfies the global Lipchitz condition, the solution $\chi(t)$ (where $\chi(t) = [x(t), v(t)]^T$) determines a family of unique continuous stochastic processes, one for each choice of the random variable $\chi(t_0)$. The joint process $\{\chi, \eta, \xi, \psi\} = \{\chi(t), \eta(t), \xi(t), \psi(t)\}$ is a Markov process.

### 3.1 *Stochastic Lyapunov Function*

A fondamental tool in the analysis of the stability of stochastic systems is the stochastic Lyapunov function which is used to describe the stability behavior without explicit solution of the differential equation.

**Definition 1** (Srichander and Walker, 1993) The random function $\vartheta(\chi, \eta, \xi, \psi, t)$ of the joint Markov process $\{\chi, \eta, \xi, \psi\}$ qualifies as a stochastic Lyapunov function candidate if the following conditions hold for some fixed $\varepsilon < \infty$:

**a)** The function $\vartheta(\chi, \eta, \xi, \psi, t)$ is positive definite and continuous in $\chi$ and $t$ in the open set $O_\varepsilon = \{\chi(t) : \vartheta(\chi, \eta, \xi, \psi, t) < \varepsilon\}$ $\forall \eta \in Z, \forall \xi \in S, \forall \psi \in R$ and $\forall t \geq t_0$, and $\vartheta(\chi, \eta, \xi, \psi, t) = 0$ only if $\chi = 0$. (The function $\vartheta(\chi, \eta, \xi, \psi, t)$ is said to be positive definite if $\vartheta(\chi, \eta, \xi, \psi, t) \geq W(\chi)$ $\forall \eta \in Z, \forall \xi \in S, \forall \psi \in R$ and $\forall t \geq t_0$, where $W(\chi)$ is positive definite in the sense of Lyapunov).

**b)** The joint Markov process $\{\chi, \eta, \xi, \psi\}$ is defined until $t = \tau_\varepsilon$ where $\tau_\varepsilon = \inf\{t : \chi(t) \notin O_\varepsilon\}$. If $\chi(t) \in O_\varepsilon \forall t < \infty$, then $\tau_\varepsilon = \infty$.

**c)** The function $\vartheta(\chi, \eta, \xi, \psi, t)$ is in the domain of $L$ where $L$ is the *weak infinitesimal operator* of the joint markov process $\{\chi(\tau_t), \eta(\tau_t), \xi(\tau_t), \psi(\tau_t)\}$. where $\tau_t = \min(t, \tau_\varepsilon)$.

The definition of the weak infinitesimal operator is given as follows:

**Definition 2** (Srichander and Walker, 1993) A bounded function $f(\zeta)$ is said to be *in the domain of the weak infinitesimal operator* $L$ *of the random process* $\zeta(t)$ if the limit

$$\lim_{\tau \to 0} \frac{E\{f(\zeta(t+\tau))|\zeta(t)\} - f(\zeta(t))}{\tau} = Lf(\zeta) = h(\zeta) \qquad (5)$$

exists pointwise in $\mathbb{R}$ and satisfies,

$$\lim_{\tau \to 0} E\{h(\zeta(t+\tau))|\zeta(t)\} = h(\zeta(t)) \qquad (6)$$

If we generalize definition 2 to time varying functions $f(\zeta, t)$, then we have

$$Lf(\zeta, t) = \frac{\partial}{\partial t} f(\zeta, t) + h(\zeta, t) \qquad (7)$$

In general, $Lf(\zeta)$ is interpreted as the average time rate of change of the process $f(\zeta)$ at time $t$ given that $\zeta(t) = \zeta$.

### 3.2 *Exponential Stability of AFTCSMP*

**Definition 3** The solution $\chi = 0$ of the system $(\varphi)$ coupled with $(\varphi_c)$ is said to be *exponentially stable in the mean square* if, for any $\eta_0 \in Z$, $\xi_0 \in S$, $\psi_0 \in R$ and some $\gamma(\eta_0, \xi_0, \psi_0) > 0$ there exists two numbers $a > 0$ and $b > 0$ such that when $\|\chi_0\| \leq \gamma(\eta_0, \xi_0, \psi_0)$, the following inequality holds $\forall t \geq t_0$ for all solution of (11) with initial condition $\chi_0$:

$$E\{\|\chi(t)\|^2\} \leq b\|\chi_0\|^2 \exp[-a(t - t_0)] \qquad (8)$$

The following theorem gives a sufficient condition for exponential stability in the mean square sense for the system $(\varphi)$ coupled with $(\varphi_c)$.

**Theorem 1** *The solution* $\chi = 0$ *of the system* $(\varphi)$ *coupled with* $(\varphi_c)$ *is exponentially stable in the mean square for* $t \geq t_0$ *if there exists a function* $\vartheta(\chi, \eta, \xi, \psi, t)$ *satisfying the conditions (a)-(c) in definition 1 such that,*

$$K_1\|\chi(t)\|^2 \leq \vartheta(\chi, \eta, \xi, \psi, t) \leq K_2\|\chi(t)\|^2 \qquad (9)$$

*and*

$$L\vartheta(\chi, \eta, \xi, \psi, t) \leq -K_3\|\chi(t)\|^2 \qquad (10)$$

*for some positive constants* $K_1$, $K_2$ *and* $K_3$.

A necessary condition for exponential stability in the mean square for the system $(\varphi)$ coupled with $(\varphi_c)$ is given by theorem 2.

**Theorem 2** *If the solution* $\chi = 0$ *of the system* $(\varphi)$ *coupled with* $(\varphi_c)$ *is exponentially stable in the mean square, then for any given quadratic positive definite function* $W(\chi, \eta, \xi, \psi, t)$ *in the variables* $\chi$ *which is bounded and continuous* $\forall t \geq t_0$, $\forall \eta \in Z$, $\forall \xi \in S$ *and* $\forall \psi \in R$, *there exists a quadratic positive definite function* $\vartheta(\chi, \eta, \xi, \psi, t)$ *in* $\chi$ *that satisfies the conditions (9)-(10) and is such that* $L\vartheta(\chi, \eta, \xi, \psi, t) = -W(\chi, \eta, \xi, \psi, t)$.

**Remark 1** The proofs of these theorems follow the same arguments as in (Srichander and Walker, 1993) for their proposed stochastic Lyapunov function, so they are not shown in this paper to avoid repetition.

## 4. MATHEMATICAL FORMULATION

The system $(\varphi)$ coupled with $(\varphi_c)$ can be written as follows:

$$\begin{cases} \dot{\chi}(t) = \Lambda(\eta, \xi, \psi)\chi(t) \\ y^*(t) = \Phi(\xi, \psi)\chi(t) \end{cases} \qquad (11)$$

where: $\chi(t) = [x(t), v(t)]^T$, $y^*(t) = [y(t), u(t)]^T$,
$\Lambda(\eta,\xi,\psi) = \begin{bmatrix} A & B(\eta)C_c(\psi) \\ B_c(\psi)C(\xi) & A_c(\psi) \end{bmatrix}$,
$\Phi(\xi,\psi) = \begin{bmatrix} C(\xi) & 0 \\ 0 & C_c(\psi) \end{bmatrix}$.
$\eta(t)$, $\xi(t)$ and $\psi(t)$ being homogeneous Markov processes with finite state spaces, we can define the transition probability of the actuator failure process as (Mahmoud *et al.*, 2003; Srichander and Walker, 1993):

$$\begin{cases} p_{ij}(\Delta t) = \pi_{ij}\Delta t + o(\Delta t) & (i \neq j) \\ p_{ii}(\Delta t) = 1 - \sum\limits_{i\neq j}\pi_{ij}\Delta t + o(\Delta t) & (i = j) \end{cases} \tag{12}$$

The transition probability of the sensor failure process is given by:

$$\begin{cases} p_{kl}(\Delta t) = \nu_{kl}\Delta t + o(\Delta t) & (k \neq l) \\ p_{kk}(\Delta t) = 1 - \sum\limits_{k\neq l}\nu_{kl}\Delta t + o(\Delta t) & (k = l) \end{cases} \tag{13}$$

where $\pi_{ij}$ is the actuator failure rate, and $\nu_{kl}$ is the sensor failure rate. Given that $\eta = k$ and $\xi = l$, the conditional transition probability of the FDI process, $\psi(t)$, is:

$$\begin{cases} p^{kl}_{iv}(\Delta t) = \lambda^{kl}_{iv}\Delta t + o(\Delta t) & (i \neq v) \\ p^{kl}_{ii}(\Delta t) = 1 - \sum\limits_{i\neq v}\lambda^{kl}_{iv}\Delta t + o(\Delta t) & (i = v) \end{cases} \tag{14}$$

Here, $\lambda^{kl}_{iv}$ represents the transition rate from $i$ to $v$ for the Markov process $\psi(t)$ conditioned on $\eta = k \in Z$ and $\xi = l \in S$. Depending on the values of $i$, $v \in R$, $k \in Z$ and $l \in S$, various interpretations, such as rate of false detection and isolation, rate of correct detection and isolation, false alarm recovery rate, etc, can be given to $\lambda^{kl}_{iv}$ (Mahmoud *et al.*, 2003; Srichander and Walker, 1993).

## 5. STABILIZATION OF THE AFTCSMP

In this section, we will first derive a necessary and sufficient condition for the exponential stability in the mean square of the system $(\varphi)$ (subject to both actuator and sensor failures) coupled with $(\varphi_c)$, in terms of coupled matrix inequalities, and then we will give an LMI characterization of dynamical compensators $(\varphi_c)$ that stabilize the closed-loop system in the mean square sense.

**Proposition 1:** a necessary and sufficient condition for exponential stability in the mean square of the system (11) is that there exist symmetric positive-definite matrices $P_{ijk}$, $i \in Z$, $j \in S$ and $k \in R$ such that

$$\tilde{\Lambda}^T_{ijk}P_{ijk} + P_{ijk}\tilde{\Lambda}_{ijk} + \sum_{\substack{h\in Z\\ h\neq i}}\pi_{ih}P_{hjk} + \sum_{\substack{l\in S\\ l\neq j}}\nu_{jl}P_{ilk} + \sum_{\substack{v\in R\\ v\neq k}}\lambda^{ij}_{kv}P_{ijv} < 0 \tag{15}$$

$\forall i \in Z$, $j \in S$ and $k \in R$, where

$$\tilde{\Lambda}_{ijk} = \Lambda_{ijk} - 0.5I\left(\sum_{\substack{h\in Z\\ h\neq i}}\pi_{ih} + \sum_{\substack{l\in S\\ l\neq j}}\nu_{jl} + \sum_{\substack{v\in R\\ v\neq k}}\lambda^{ij}_{kv}\right) \tag{16}$$

**Proof**

**a)** Sufficiency

Assume that there exist $P_{ijk} > 0$, $i \in Z$, $j \in S$ and $k \in R$ such that (15) is verified. Then $\vartheta(\chi,\eta,\xi,\psi,t) = \chi^T P(\eta,\xi,\psi)\chi$ is a stochastic Lyapunov function which satisfies conditions (a)-(c) of definition 1 and also the condition (9) in theorem 1. Evaluating $L\vartheta(\chi,\eta,\xi,\psi,t)$ for the system (11) when the quantities $\eta = i \in Z$, $\xi = j \in S$ and $\psi = k \in R$ have occurred at some $t \in [0,\infty)$, we get:

$$L\vartheta = \chi^T\left\{\tilde{\Lambda}^T_{ijk}P_{ijk} + P_{ijk}\tilde{\Lambda}_{ijk} + \sum_{\substack{h\in Z\\ h\neq i}}\pi_{ih}P_{hjk} + \sum_{\substack{l\in S\\ l\neq j}}\nu_{jl}P_{ilk} + \sum_{\substack{v\in R\\ v\neq k}}\lambda^{ij}_{kv}P_{ijv}\right\}\chi \tag{17}$$

$i \in Z$, $j \in S$, $k \in R$, where $\tilde{\Lambda}_{ijk}$ is given by (16). Since by hypothesis $\{P_{ijk}, i \in Z, j \in S, k \in R\}$ satisfies (15), then $\exists\ Q(\eta,\xi,\psi) > 0$, such that $L\vartheta(\chi,\eta,\xi,\psi) = -\chi^T Q(\eta,\xi,\psi)\chi < 0$, and by theorem 1 the dynamical system (11) is exponentially stable in the mean square $\forall t > t_0$.

**b)** Necessity

Assume that the system (11) is exponentially stable in the mean square. Let $W(\chi,\eta,\xi,\psi,t) = \chi^T Q(\eta,\xi,\psi)\chi$, where $Q(\eta,\xi,\psi)$ are symmetric positive-definite matrices $\forall\eta \in Z$, $\forall\xi \in S$ and $\forall\psi \in R$. Then by theorem 2, there exists a quadratic positive definite function $\vartheta(\chi,\eta,\xi,\psi,t)$ $\forall\eta \in Z$, $\forall\xi \in S$ and $\forall\psi \in R$ that satisfies the condition (9) in theorem 1 and is in the domain of the weak infinitesimal operator $L$ such that $L\vartheta(\chi,\eta,\xi,\psi,t) = -W(\chi,\eta,\xi,\psi,t)$. Let us denote the quadratic function that satisfies these conditions by $\vartheta(\chi,\eta,\xi,\psi,t) = \chi^T P(\eta,\xi,\psi)\chi$, $P(\eta,\xi,\psi)$ being symmetric positive-definite matrices $\forall\eta \in Z, \forall\xi \in S$ and $\forall\psi \in R$. Evaluating $L\vartheta(\chi,\eta,\xi,\psi,t)$ for the system (11), when the quantities $\eta = i \in Z$, $\xi = j \in S$ and $\psi = k \in R$ have occurred at some $t \in [0,\infty)$, we have:

$$L\vartheta = \chi^T\left\{\tilde{\Lambda}^T_{ijk}P_{ijk} + P_{ijk}\tilde{\Lambda}_{ijk} + \sum_{\substack{h\in Z\\ h\neq i}}\pi_{ih}P_{hjk} + \sum_{\substack{l\in S\\ l\neq j}}\nu_{jl}P_{ilk} + \sum_{\substack{v\in R\\ v\neq k}}\lambda^{ij}_{kv}P_{ijv}\right\}\chi = -\chi^T Q_{ijk}\chi < 0 \tag{18}$$

From (18), we conclude that there exist $P_{ijk} > 0$, $i \in Z$, $j \in S$ and $k \in R$ such that (15) is verified. Hence the proof is complete. ■

**Remark 2** For the LMI characterization of $(\varphi_c)$, we make the assumption, as in (Shi and Boukas, 1997; Shi *et al.*, 2003; Mahmoud *et al.*, 2003), that all jump states $\eta$, $\xi$ and $\psi$ are available for feedback.

**Proposition 2:** a necessary and sufficient condition for exponential stability in the mean square of the system (11) is that the following matrix inequalities

$$\begin{pmatrix} \tilde{A}_{ijk}Y_{ijk} + Y_{ijk}\tilde{A}_{ijk}^T + F_{ijk}^T B_i^T + B_i F_{ijk} & R_{ijk}(Y) \\ R_{ijk}(Y)^T & S_{ijk}(Y) \end{pmatrix} < 0 \tag{19}$$

$$\begin{aligned} &\tilde{A}_{ijk}^T X_{ijk} + X_{ijk}\tilde{A}_{ijk} + C_j^T H_{ijk}^T + H_{ijk}C_j \\ &+ \sum_{\substack{h\in Z\\ h\neq i}} \pi_{ih} X_{hjk} + \sum_{\substack{l\in S\\ l\neq j}} \nu_{jl} X_{ilk} + \sum_{\substack{v\in R\\ v\neq k}} \lambda_{kv}^{ij} X_{ijv} < 0 \end{aligned} \tag{20}$$

$$\begin{pmatrix} Y_{ijk} & I \\ I & X_{ijk} \end{pmatrix} > 0 \tag{21}$$

where

$$\begin{cases} R_{ijk} = [R1_{ijk}, R2_{ijk}, R3_{ijk}] \\ R1_{ijk} = \left[\alpha_{i1}Y_{ijk}, ...\alpha_{i(i-1)}Y_{ijk}, \alpha_{i(i+1)}Y_{ijk}, ..., \alpha_{iz}Y_{ijk}\right] \\ R2_{ijk} = \left[\beta_{j1}Y_{ijk}, ...\beta_{j(j-1)}Y_{ijk}, \beta_{j(j+1)}Y_{ijk}, ..., \beta_{js}Y_{ijk}\right] \\ R3_{ijk} = \left[\gamma_{k1}Y_{ijk}, ...\gamma_{k(k-1)}Y_{ijk}, \gamma_{k(k+1)}Y_{ijk}, ..., \gamma_{kr}Y_{ijk}\right] \\ \alpha_{il} = \sqrt{\pi_{il}}; \beta_{jl} = \sqrt{\nu_{jl}}; \gamma_{kl} = \sqrt{\lambda_{kl}^{ij}} \\ S_{ijk} = -\mathrm{diag}[S1_{ijk}, S2_{ijk}, S3ijk] \\ S1_{ijk} = [Y_{1jk}, ..., Y_{(i-1)jk}, Y_{(i+1)jk}, ..., Y_{zjk}] \\ S2_{ijk} = [Y_{i1k}, ..., Y_{i(j-1)k}, Y_{i(j+1)k}, ..., Y_{isk}] \\ S3_{ijk} = [Y_{ij1}, ..., Y_{ij(k-1)}, Y_{ij(k+1)}, ..., Y_{ijr}] \\ \tilde{A}_{ijk} = A - 0.5I\sum_{\substack{h\in Z\\ h\neq i}}\pi_{ih} - 0.5I\sum_{\substack{l\in S\\ l\neq j}}\nu_{jl} - 0.5I\sum_{\substack{v\in R\\ v\neq k}}\lambda_{kv}^{ij} \end{cases}$$

have feasible solutions $X_{ijk} = X_{ijk}^T$, $Y_{ijk} = Y_{ijk}^T$, $H_{ijk}$, and $F_{ijk}$. The corresponding compensator $(\varphi_c)$ is given by

$$\begin{aligned} A_{cijk} &= (X_{ijk} - Y_{ijk}^{-1})^{-1}\Big[\tilde{A}_{ijk}^T + X_{ijk}\tilde{A}_{ijk}Y_{ijk} \\ &+ X_{ijk}B_iF_{ijk} + H_{ijk}C_jY_{ijk} + \Big(\sum_{\substack{h\in Z\\ h\neq i}}\pi_{ih}Y_{hjk}^{-1} \\ &+ \sum_{\substack{l\in S\\ l\neq j}}\nu_{jl}Y_{ilk}^{-1} + \sum_{\substack{v\in R\\ v\neq k}}\lambda_{kv}^{ij}Y_{ijv}^{-1}\Big)Y_{ijk}\Big]Y_{ijk}^{-1} \\ &+ 0.5I\sum_{\substack{h\in Z\\ h\neq i}}\pi_{ih} + 0.5I\sum_{\substack{l\in S\\ l\neq j}}\nu_{jl} + 0.5I\sum_{\substack{v\in R\\ v\neq k}}\lambda_{kv}^{ij} \end{aligned} \tag{22}$$

$$B_{cijk} = (Y_{ijk}^{-1} - X_{ijk})^{-1}H_{ijk} \tag{23}$$

$$C_{cijk} = F_{ijk}Y_{ijk}^{-1} \tag{24}$$

**Proof** The proof essentially follows a similar line to the proof of a result in the work of (de Farias *et al.*, 2000), except here we take three Markov processes $\eta(t)$, $\xi(t)$ and $\psi(t)$ into account.

**a)** Sufficiency

Assume that $X_{ijk} = X_{ijk}^T$, $Y_{ijk} = Y_{ijk}^T$, $H_{ijk}$, and $F_{ijk}$, $\forall i \in Z$, $j \in S$ and $k \in R$ are feasible solutions of (19)-(21). Then, taken for each, $i$, $j$, $k$

$$P_{ijk} = \begin{pmatrix} X_{ijk} & Y_{ijk}^{-1} - X_{ijk} \\ Y_{ijk}^{-1} - X_{ijk} & X_{ijk} - Y_{ijk}^{-1} \end{pmatrix} > 0 \tag{25}$$

$$T_{ijk} = \begin{pmatrix} Y_{ijk} & I \\ Y_{ijk} & 0 \end{pmatrix} \tag{26}$$

and $A_{cijk}$, $B_{cijk}$, $C_{cijk}$ as defined in (22)-(24). It follows (by using the Schur complement) that (27) holds, and hence, (15) is verified. Then from proposition 1, the system (11) is exponentially stable in the mean square sense.

$$\begin{aligned} &T_{ijk}^T\Big(\tilde{\Lambda}_{ijk}^T P_{ijk} + P_{ijk}\tilde{\Lambda}_{ijk} + \sum_{\substack{h\in Z\\ h\neq i}}\pi_{ih}P_{hjk} + \sum_{\substack{l\in S\\ l\neq j}}\nu_{jl}P_{ilk} \\ &+ \sum_{\substack{v\in R\\ v\neq k}}\lambda_{kv}^{ij}P_{ijv}\Big)T_{ijk} = \begin{pmatrix} Z1_{ijk} & 0 \\ 0 & Z2_{ijk} \end{pmatrix} < 0 \end{aligned} \tag{27}$$

where

$$\begin{aligned} Z1_{ijk} &= \tilde{A}_{ijk}Y_{ijk} + Y_{ijk}\tilde{A}_{ijk}^T + F_{ijk}^T B_i^T + B_iF_{ijk} \\ &+ Y_{ijk}\Big(\sum_{\substack{h\in Z\\ h\neq i}}\pi_{ih}Y_{hjk}^{-1} + \sum_{\substack{l\in S\\ l\neq j}}\nu_{jl}Y_{ilk}^{-1} \\ &+ \sum_{\substack{v\in R\\ v\neq k}}\lambda_{kv}^{ij}Y_{ijv}^{-1}\Big)Y_{ijk} \end{aligned} \tag{28}$$

$$\begin{aligned} Z2_{ijk} &= \tilde{A}_{ijk}^T X_{ijk} + X_{ijk}\tilde{A}_{ijk} + C_j^T H_{ijk}^T + H_{ijk}C_j \\ &+ \sum_{\substack{h\in Z\\ h\neq i}}\pi_{ih}X_{hjk} + \sum_{\substack{l\in S\\ l\neq j}}\nu_{jl}X_{ilk} + \sum_{\substack{v\in R\\ v\neq k}}\lambda_{kv}^{ij}X_{ijv} \end{aligned} \tag{29}$$

**b)** Necessity

Assume that (11) is exponentially stable in the mean square, then from proposition 1, (15) is verified. Consider the following partition of $P_{ijk}$:

$$P_{ijk} = \begin{pmatrix} P_{1ijk} & P_{2ijk} \\ P_{2ijk}^T & P_{3ijk} \end{pmatrix} \tag{30}$$

Let us define the matrices

$$Y_{ijk} = (P_{1ijk} - P_{2ijk}P_{3ijk}^{-1}P_{2ijk}^T)^{-1} > 0 \tag{31}$$

$$T_{ijk} = \begin{pmatrix} Y_{ijk} & I \\ Y_{ijk} & 0 \end{pmatrix}; J_{ijk} = \begin{pmatrix} I & 0 \\ 0 & -P_{3ijk}^{-1}P_{2ijk}^T \end{pmatrix} \tag{32}$$

by multiplying (15) to the left by $T_{ijk}^T J_{ijk}^T$, and to the right by $J_{ijk}T_{ijk}$ we get:

$$\begin{pmatrix} N1_{ijk} & M_{ijk}^T \\ M_{ijk} & N2_{ijk} \end{pmatrix} + \begin{pmatrix} N3_{ijk} & 0 \\ 0 & N4_{ijk} \end{pmatrix} < 0 \tag{33}$$

where

$$N1_{ijk} = \tilde{A}_{ijk}Y_{ijk} + Y_{ijk}\tilde{A}_{ijk}^T + F_{ijk}^T B_i^T + B_iF_{ijk} \tag{34}$$

$$N2_{ijk} = \tilde{A}_{ijk}^T P_{1ijk} + P_{1ijk}\tilde{A}_{ijk} + C_j^T H_{ijk}^T + H_{ijk}C_j \tag{35}$$

$$\begin{aligned} M_{ijk} &= \tilde{A}_{ijk}^T + P_{1ijk}\tilde{A}_{ijk}Y_{ijk} + P_{1ijk}B_iF_{ijk} \\ &+ H_{ijk}C_jY_{ijk} - P_{2ijk}\tilde{A}_{cijk}P_{3ijk}^{-1}P_{2ijk}^T Y_{ijk} \\ &+ \Big(\sum_{\substack{h\in Z\\ h\neq i}}\pi_{ih}(P_{1hjk} - P_{2hjk}P_{3ijk}^{-1}P_{2ijk}^T) \\ &+ \sum_{\substack{l\in S\\ l\neq j}}\nu_{jl}(P_{1ilk} - P_{2ilk}P_{3ijk}^{-1}P_{2ijk}^T) \\ &+ \sum_{\substack{v\in R\\ v\neq k}}\lambda_{kv}^{ij}(P_{1ijv} - P_{2ijv}P_{3ijk}^{-1}P_{2ijk}^T)\Big)Y_{ijk} \end{aligned} \tag{36}$$

$$\tilde{A}_{cijk} = A_{cijk} - 0.5I \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} - 0.5I \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} - 0.5I \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} \quad (37)$$

$$F_{ijk} = -C_{cijk} P_{3ijk}^{-1} P_{2ijk}^T Y_{ijk}; H_{ijk} = P_{2ijk} B_{cijk} \quad (38)$$

$$N3_{ijk} = \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} Y_{ijk} \Big[ Y_{ijv}^{-1} + (P_{2ijk} P_{3ijk}^{-1} P_{3ijv} - P_{2ijv}) P_{3ijv}^{-1} (P_{2ijk} P_{3ijk}^{-1} P_{3ijv} - P_{2ijv})^T \Big] Y_{ijk} + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} Y_{ijk} \Big[ Y_{ilk}^{-1} + (P_{2ijk} P_{3ijk}^{-1} P_{3ilk} - P_{2ilk}) P_{3ilk}^{-1} (P_{2ijk} P_{3ijk}^{-1} P_{3ilk} - P_{2ilk})^T \Big] Y_{ijk} + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} Y_{ijk} \Big[ Y_{hjk}^{-1} + (P_{2ijk} P_{3ijk}^{-1} P_{3hjk} - P_{2hjk}) P_{3hjk}^{-1} (P_{2ijk} P_{3ijk}^{-1} P_{3hjk} - P_{2hjk})^T \Big] Y_{ijk} \quad (39)$$

$$N4_{ijk} = \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} P_{1hjk} + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} P_{1ilk} + \sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} P_{1ijv} \quad (40)$$

The conditions $P_{ijk} > 0$, $\forall i \in Z$, $j \in S$ and $k \in R$ are equivalent to

$$T_{ijk}^T J_{ijk}^T P_{ijk} J_{ijk} T_{ijk} = \begin{pmatrix} Y_{ijk} & I \\ I & P_{1ijk} \end{pmatrix} > 0 \quad (41)$$

Since

$$\sum_{\substack{v \in R \\ v \neq k}} \lambda_{kv}^{ij} (P_{2ijk} P_{3ijk}^{-1} P_{3ijv} - P_{2ijv}) P_{3ijv}^{-1} (P_{2ijk} P_{3ijk}^{-1} P_{3ijv} - P_{2ijv})^T + \sum_{\substack{l \in S \\ l \neq j}} \nu_{jl} (P_{2ijk} P_{3ijk}^{-1} P_{3ilk} - P_{2ilk}) P_{3ilk}^{-1} (P_{2ijk} P_{3ijk}^{-1} P_{3ilk} - P_{2ilk})^T + \sum_{\substack{h \in Z \\ h \neq i}} \pi_{ih} (P_{2ijk} P_{3ijk}^{-1} P_{3hjk} - P_{2hjk}) P_{3hjk}^{-1} (P_{2ijk} P_{3ijk}^{-1} P_{3hjk} - P_{2hjk})^T \geq 0 \quad (42)$$

and using the Schur complement (Boyd *et al.*, 1994), it follows that (19)-(21) are verified for, $X_{ijk} = P_{1ijk}$, $F_{ijk}$ and $H_{ijk}$ as defined in (38), $\forall i \in Z$, $j \in S$ and $k \in R$. Hence the proof is complete. ■

## 6. CONCLUSION

In this paper, the problem of dynamic output feedback control of AFTCSMP has been considered. This last one being subject to both actuator and sensor failures. We have shown that the problem addressed can be recast as a convex optimization problem characterized by a linear matrix inequalities (LMI); therefore, an LMI approach was developed to derive the necessary and sufficient conditions for the existence of all desired dynamic output feedback controllers that achieve the stochastic stabilization of the AFTCSMP. An effective design procedure for the expected controllers was also presented. Our forthcoming works will treat about the ability of the AFTCSMP to cope with unknown-but-bounded transition probability rates.

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