TRACKING CONTROL BASED ON NUMERICAL METHODS

Gustavo Scaglia, José F. Postigo, Vicente Mut, Zulma Millan, Carlos Calvo

Instituto de Automática (INAUT). Universidad Nacional de San Juan, Argentina.
Av. Libertador San Martín (oeste) 1109. J5400ARL. San Juan, Argentina.
e-mail:{gscaglia, jpostigo, vmut}@inaut.unsj.edu.ar

Abstract: This work proposes a methodology to find a control structure such that a system may follow a pre-established trajectory. To accomplish this, a system of linear equations should be solved for each sampling period. This approach can be applied to systems that are either linear or non-linear, time-varying or non-varying. For the case of non-minimal phase, a trajectory tracking is made after a finite-time interval.
The proposed control structure is simple and acceptably robust under uncertainties about model knowledge. Three cases of study, applying the proposed control structure, are presented: a pendulum with friction, a system of non-minimun phase and a trajectory control of a mobile robot. Some conclusions about the performance of the control structure are also sketched.

Keywords: Control system design, Discrete systems, Models, Tracking, Trajectory control, Numerical methods.

1. INTRODUCTION

Several works dealing with the problem of trajectory tracking have been published in the literature, which nevertheless, are focused onto specific system types. For example, (Kim,2003) proposes a receding horizon tracking control for time-varying linear systems with constraints both on the control signal and on the tracking error, on which it is based the minimization of a functional for finite-time costs. Besides, Linear Matrix Inequalities (LMI) are used in order to synthesize the controller. In (Chem et al.,1995) a controller is proposed only for linear, non-varying systems which are exponentially stable and of non-minimum phase, which needs a set of input/output data.

In (Fujimoto et al.,2001), a perfect-tracking control is presented that is based on multirate feedforward control for linear systems. Specifically, the design is made for a SISO system which, nevertheless, can be extended for a MIMO system. In this latter case, it is also necessary that the system be completely controllable, which could thus pose a limitation.

The present work proposes a methodology which does not require that the system be controllable but, instead, that the desired state can be reach for the system. If this were not the case, the control actions will lead the system to the reachable state closest to the desired state.

The main idea of this paper is to propose an approximation of the system using numerical methods, and then to find the best control actions that lead the system from actual state to the desired one.

Here, the proposed methodology is applied to non-linear SISO systems (through a case of a pendulum with friction), to non-minimum phase linear systems (considering a general problem) and to multivariable non-linear systems (considering the problem of tracking trajectory of a mobile robot). Experimental and simulation results for this last case, show the advantages of the proposed methodology.

2. STATEMENT OF THE PROBLEM

Let's consider the following differential equation:

$$y' = f(y,t) \quad y(0) = y_0, \tag{1}$$

where the aim is to know the value of $y(t)$ at discrete time instants $t = nTo$, where To is the sampling period and $n \in \{0,1,2,3.....\}$, the value for variable $y(t)$ at $t = nTo$ will be denoted as y_n. If, for example, one wishes to compute the value for y_{n+1} by knowing the value y_n, one should integrate Eq. (1) over the time interval $nTo \le t \le (n+1)To$, as shown in Eq. (2),

$$y_{n+1} = y_n + \int_{nTo}^{(n+1)To} f(y,t)dt \tag{2}$$

There are several numerical integration methods, with their corresponding algorithms to calculate the value of y_{n+1}. For instance, an approach could be through Eqs. (3) and (4),

$$y_{n+1} \cong y_n + To\, f(y_n, t_n) \tag{3}$$

$$y_{n+1} \cong y_n + \frac{To}{2}\{f(y_n,t_n) + f(y_{n+1},t_{n+1})\} \tag{4}$$

where y_{n+1} on the rigth-side member of Eq.(4) is not known and, therefore, can be estimated by Eq. (3). In numerical methods, these approximations, given by Eqs (3) and (4), are called the Euler and 2nd order Runge-Kutta methods, respectively.

3. APLICATIONS TO HIGHER-ORDER SYSTEMS

3.1 Non-linear system of second order: Pendulum with friction.

The pendulum with friction is described through a 2nd order differential equation (see Eq. (5)). The procedure to solve the system is by expressing the model as a set of coupled 1st order differential equations, as shown by Eqs.(6) and (7),

$$l^2 m\ddot{\theta} + b\dot{\theta} + mglsin(\theta) = u(t) \tag{5}$$

$$x_1 = \theta \quad ; \quad x_2 = \dot{\theta} \tag{6}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{l^2 m}\left(-mglsin(x_1) - bx_2 + u\right) \end{cases} \tag{7}$$

where l is the pendulum´s length, m is its mass, b is the friction coefficient at the hinge, and g is the gravity constant. By applying the procedure used for Eqs. (3) and (4) to Eq. (7), yields in,

$$\begin{cases} x_{1\,n+1} = x_{1n} + \dfrac{To}{2}[x_{2n} + x_{2n+1}] \\ x_{2\,n+1} = x_{2n} + \dfrac{To}{2l^2 m}(-mglsin(x_{1n}) - bx_{2n} + u_n + \\ \qquad -mglsin(x_{1n+1}) - bx_{2n+1} + u_{n+1}) \end{cases} \tag{8}$$

Now, by approaching $x_{1\,n+1}$ and $x_{2\,n+1}$ through the Euler method on the rigth-side member of (8), and by re-arranging the new equation,

$$\begin{aligned} x_{1n+1} &= x_{1n} + x_{2n}\frac{To}{2}\left(2 - \frac{Tob}{l^2 m}\right) - \frac{To^2 g}{2l} sin(x_{1n}) + \frac{To^2}{2l^2 m}u_n \\ x_{2n+1}\left(1 + \frac{Tob}{2l^2 m}\right) &= x_{2n} + \frac{To}{2l^2 m}[-mgl\sin(x_{1n}) - bx_{2n} + u_n - mgl\sin(x_{1n} + To\ x_{2n}) + u_{n+1}] \end{aligned} \tag{9}$$

Eq. (9) expresses the influence of u_n, u_{n+1} and of $[x_{1n} \quad x_{2n}]^T$ (the state vector at the current time instant) on the state vector on a later time instant. Therefore, by knowing the desired trajectory to be followed by the system states, which henceforth will be called x_{1d} and x_{2d}, the control actions can consequently be computed in order to reach the desired state $[x_{1d\ n+1} \quad x_{2d\ n+1}]^T$. Then, by substituting $[x_{1n+1} \quad x_{2n+1}]^T$ by $[x_{1d\ n+1} \quad x_{2d\ n+1}]^T$ in Eq.(9) and defining,

$$\begin{bmatrix} e_{1n} \\ e_{2n} \end{bmatrix} = \begin{bmatrix} x_{1dn+1} \\ x_{2dn+1}\left(1 + \dfrac{Tob}{2l^2 m}\right) \end{bmatrix} - \begin{bmatrix} x_{1n} + x_{2n}\dfrac{To}{2}\left(2 - \dfrac{Tob}{l^2 m}\right) - \dfrac{To^2 g}{2l}sin(x_{1n}) \\ x_{2n} + \dfrac{To}{2l^2 m}[-mglsin(x_{1n}) - bx_{2n} - mglsin(x_{1n} + Tox_{2n})] \end{bmatrix} \tag{10}$$

$$B = \frac{To}{2l^2 m}\begin{bmatrix} To & 0 \\ 1 & 1 \end{bmatrix} \tag{11}$$

Then, operating with Eqs. (9), (10) and (11), it yields,

$$B\begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} e_{1n} \\ e_{2n} \end{bmatrix}, \tag{12}$$

which is a system of two equations with two unknowns that can be solved before implementing the controller. This set given by Eq. (12), can also be computed on line by using an iterative method, for example, the Gauss-Seidel method (Strang, 1980), because the value for u_{n+1} is already available and it can be used at a future time as the first value for u_n to begin the iteration.

Then, the values for u_n and u_{n+1}, attained through Eq. (12), are:

$$u_n = e_{1n}\frac{2l^2 m}{To^2}$$
$$u_{n+1} = (e_{2n} - \frac{e_{1n}}{To})\frac{2l^2 m}{To} \quad (13)$$

where u_n is the control signal at the instant nTo and u_{n+1} is the control signal at the instant $(n+1)To$, which is computed at instant nTo.

On account of the type of approximation used (a trapezoidal approximation, see Eq. (4)), the system output may present small oscillations about the desired value and, consequently, the control action will oscillate as well.

Figures 1 and 2 show the system response and the control action, respectively, when the following parameters are considered in Eq. (5): $l = 1[m]$, $m = 1[kg]$, $b = 1[Nm/(rad/\sec.)]$, $g = 9.8[m/\sec.^2]$ and $To = 0.1\sec$. (Eq. (5)). Now, it is proposed to define $x_{2d\ n+1}$ as in Eq. (14), yielding in,

$$x_{2d\ n+1} = \frac{x_{1d\ n+1} - x_{1n}}{To} \quad (14)$$

The initial conditions of system (5) are given by $[\theta(0) \quad \dot{\theta}(0)]^T = [-0.5rad \quad -0.5rad/s]^T$.

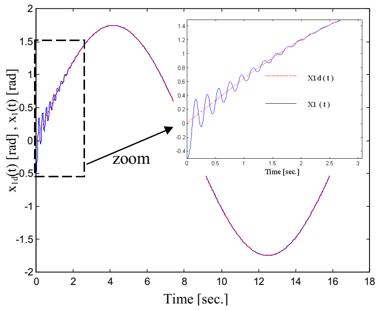

Fig. 1: Desired trajectory $x_{1d}(t)$ and real $x_1(t)$ of the pendulum with friction

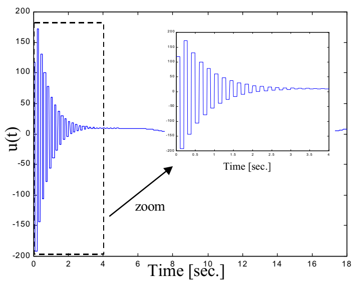

Fig. 2: Control action $u(t)$

Figure 1 shows that the system response presents small oscillations about the desired value, whereas Fig. 2 shows that $u(t) = u(nTo) = u_n$ oscillate about a certain value such that a smoother control action can be attained when using a linear combination of u_n and u_{n+1} in Eq. (13). In such a case, the new control signal can be obtained through Eq. (15), as,

$$u_{new}(t) = u_{new}(nTo) = k_1 * u_n + k_2 * u_{n+1}$$
$$\text{where} \quad k_1, k_2 \geq 0 \ \ and \ \ k_1 + k_2 = 1 \quad (15)$$

Fig. 3 shows the system response when $k_1 = 0.8$ and $k_2 = 0.2$ in Eq. (15), whereas Fig. 4 shows the resulting control action, $u_{new}(t)$.

When comparing Figs 1 and 3, it should be noticed that the system response is less oscillatory in the last one. Fig. 4 indicates that a smoother control action than that of Fig. 2 has been obtained. An additional advantage is that $u_{new}(t)$ depends now on u_{n+1} which is a function of x_{2d}, defined by Eq. (14), and also establishes the response speed of the system. It is then proposed to re-define Eq. (14) as,

$$x_{2d\ n+1} = k\frac{x_{1d\ n+1} - x_{1n}}{To}, \ \ where \ \ 0 < k \leq 1 \quad (16)$$

Figures 5 and 6 depict the system response and the control action signal for $k = 0.2$, $k_1 = 0.7$ and $k_2 = 0.3$, respectively. From Fig. 5, it should be noticed that the system response doesn´t show any overshoot.

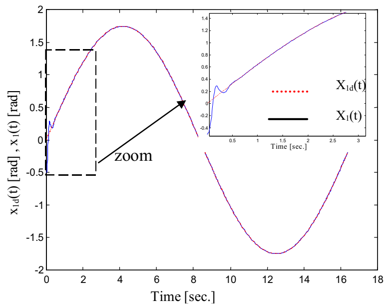

Fig. 3: Desired and real trajectories, $x_{1d}(t)$ and $x_1(t)$, respectively.

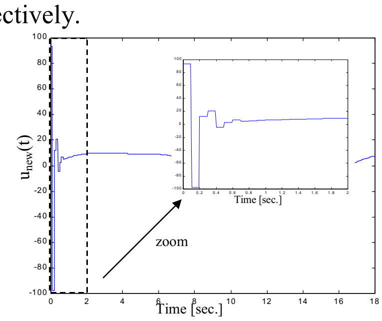

Fig. 4: Control action $u_{new}(t)$

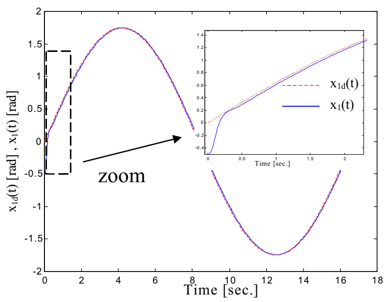

Fig. 5: Desired and real trajectories $x_{1d}(t)$ and $x_1(t)$, respectively.

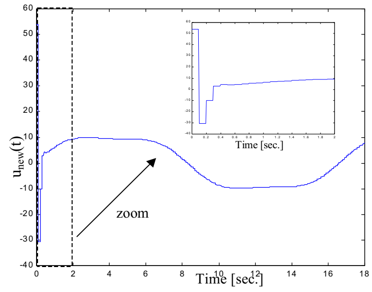

Fig. 6: Control action $u_{new}(t)$

The fact that the control action depends on x_{2d} is advantageous, because this allows to decrease the overshoot of the system response, as shown in Fig. 7, when $x_{1d} = 1.75 \; \forall \; t \geq 0$. Fig. 7 also shows the system response for different values of *k, k_1* and *k_2*, and *for* two different instances: one case with errors in 15 % of system parameters, and another when there is no error.

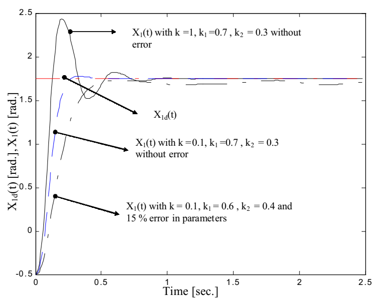

Fig. 7: Desired and real trajectories $x_{1d}(t)$ and $x_1(t)$, respectively.

The system response with $k = 0.1$ is smoother and with less overshoot than what it was for $k = 1$. It can also be noted that the system response against errors in the parameters is acceptable. This demonstrates as well the robustness of the proposed controller against parametric uncertainty.

3.2 Linear System of Non-Minimum phase

The non-minimum phase linear system to be considered is described by Eq. (17), as,

$$G(s) = \frac{-2s+1}{s^2+s+1} \qquad (17)$$

If the procedure used in Section 3.1 is applied to system (17), a precise tracking of the trajectory will be attained, though the control signal is not bounded. On this account, the controller will be computed with the following model (Morari ,1989):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+s+1} \qquad (18)$$

A model of state variables of Eq. (18) is given by the system of Eqs. (19), as,

$$\begin{cases} \dot{x}_1 &= -x_1 - x_2 + 2u \\ \dot{x}_2 &= x_1 \end{cases} \\ y = x_1 + 0.5x_2 \qquad (19)$$

If the Euler's approximation is applied to the system (19), it yields,

$$\begin{cases} x_{1n+1} &= x_{1n} + To*(-x_{1n} - x_{2n} + 2u_n) \\ x_{2n+1} &= x_{2n} + To*x_{1n} \end{cases} \\ y_n = x_{1n} + 0.5x_{2n} \qquad (20)$$

where $To = 0.1\,\text{sec}$.

Since $x_1(t) \neq y(t)$, it is proposed to find $x_{1d}(t)$ as follows: y_n from Eq. (20) should comply with $y_{dn} = x_{1dn} + 0.5x_{2dn}$, with,

y_{dn}: Desired output value at time instant nTo.

x_{1dn}: Desired value of x_1 at time instant nTo.

x_{2dn}: Desired value of x_2 at time instant nTo.

Then:

$$x_{2d\;n+1} = \frac{y_{d\;n+1} - x_{1d\;n+1}}{0.5} \qquad (21)$$

By repeating the procedure of the previous example (given by Eq.14), it yields,

$$x_{1d\;n+1} = \frac{x_{2d\;n+1} - x_{2n}}{To} \qquad (22)$$

By substituting Eq. (22) into Eq. (21):

$$x_{1d} = \frac{1}{(1+0.5*To)}\left(\frac{yd}{0.5*To} - \frac{x_2}{To}\right) \qquad (23)$$

where $n+1$ is not denoted, for simplicity reasons.

Now, by substituting Eq. (23), instead of x_{n+1}, into Eq. (20), and arranging the result to obtain u_n, it yields,

$$e = x_{1d\;n+1} - x_{1n} - To*(-x_{1n} - x_{2n}) \qquad (24)$$

$$u(t) = u(nTo) = u_n = \frac{e}{2*To} \qquad (25)$$

where Eq. (25) represents the *control action.*

Figure 8 shows both the real and desired trajectories of the system given by Eq. (17), and Figure 9 depicts the control action of the system (17).

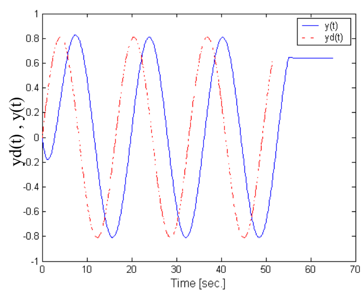

Fig. 8: Desired and real trajectories $yd(t)$, and y(t) of system (17).

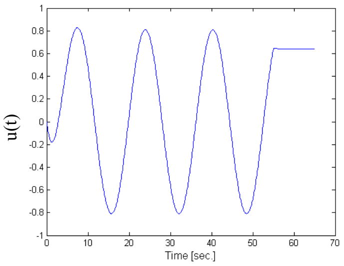

Fig. 9: Control action $u(t)$ of the system (17).

It should be noticed from Fig. 8 that the system output follows the reference trajectory after a finite interval (Δt), which is a function of the desired trajectory. Figure 9 shows that the control action remains bounded.

Therefore, if the trajectory is known in advance, $y_d(t+\Delta t)$ can be used in Eq. (23) instead of $y_d(t+To)$. Figures 10 and 11 respectively show the system response and the control action, when $\Delta t = 35 * To$. In order that $u(t)$ be *"smooth"*, Δt was increased progressively at each sampling time until reaching the desired value ($\Delta t = 35 * To$).

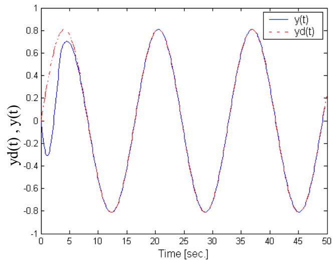

Fig. 10: Desired and real trajectories $yd(t)$ and y(t)

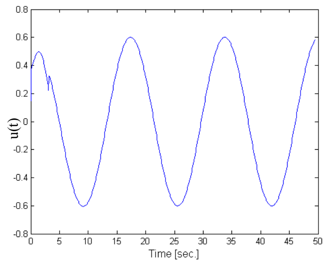

Fig. 11: Control action $u(t)$

Figure 10 shows that the system output follows precisely the reference trajectory after a finite time, and it should be noticed that this is achieved with bounded control energy (Fig. 11).

3.3 Non-linear multivariable system: Trajectory control of a mobile robot.

In this part of the work, a non-linear kinematic model for a mobile robot, represented by Eqs. (25), will be used (Campion et al.,1996), as shown in Fig. 12.

$$\begin{cases} \dot{x} &= V\cos\theta \\ \dot{y} &= V\sin\theta \\ \dot{\theta} &= W \end{cases} \qquad (25)$$

where V = linear velocity of the mobile robot, W = angular velocity of the mobile robot, (x, y) = Cartesian position , θ = orientation of the mobile rob ot.

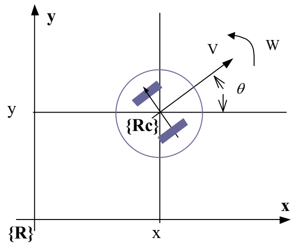

Fig. 12: Geometric description of the mobile robot

In this case, the aim is at finding the values for V and W at each sample time, such that the mobile robot may follow a pre-established trajectory.

Through Euler's approximation for the kinematic model of the mobile robot, the following set of equations is obtained,

$$\begin{cases} x_{n+1} &= x_n + ToV_n\cos\theta_n \\ y_{n+1} &= y_n + ToV_n\sin\theta_n \\ \theta_{n+1} &= \theta_n + ToW_n \end{cases} \qquad (26)$$

which can be expressed in vector form as,

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ \theta_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \\ \theta_n \end{bmatrix} + To \begin{bmatrix} \cos\theta_n & 0 \\ \sin\theta_n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_n \\ W_n \end{bmatrix} \qquad (27).$$

where $To = 0.1\,\text{sec}$.

If the desired trajectory $[x_{dn+1}\ y_{dn+1}\ \theta_{dn+1}]^T$ is known, then it can be replaced into Eq. (27) instead of $[x_{n+1}\ y_{n+1}\ \theta_{n+1}]^T$ and, thus, be able to calculate the control actions V_n, W_n necessary to make the mobile robot go from the current state, $[x_n\, y_n\, \theta_n]^T$ to the desired one $[x_{dn+1}\, y_{dn+1}\, \theta_{dn+1}]^T$ at the next sampling instant. By defining,

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta\theta \end{bmatrix} = \begin{bmatrix} x_{dn+1} - x_n \\ y_{dn+1} - y_n \\ \theta_{dn+1} - \theta_n \end{bmatrix}, \quad B = \begin{bmatrix} \cos\theta_n & 0 \\ \sin\theta_n & 0 \\ 0 & 1 \end{bmatrix} \qquad (28)$$

from Eqs. (27) and (28), it yields,

$$B \begin{bmatrix} V_n \\ W_n \end{bmatrix} = \frac{1}{To} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta\theta \end{bmatrix} \qquad (29)$$

Equation (29) is a set of three equations of two unknowns each, whose optimal solution is given by (Strang ,1980),

$$B^T B \begin{bmatrix} V_n \\ W_n \end{bmatrix} = \frac{1}{To} B^T \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta\theta \end{bmatrix} \qquad (30)$$

$$\begin{bmatrix} V_n \\ W_n \end{bmatrix} = \begin{bmatrix} \frac{\Delta x}{To}\cos\theta_n + \frac{\Delta y}{To}\cos\theta_n \\ \frac{\Delta\theta}{To} \end{bmatrix} \qquad (31)$$

where Eq. (31) represents the *proposed controller*.

Experimental studies were carried out with a mobile robot PIONNER 2DX available at INAUT laboratories to test the proposed controller performance. Fig. 13 depicts the Pioneer 2DX structure and the laboratory facilities where the experiences were carried out.

Figure 13: Pioneer 2DX mobile robot and its laboratory environment.

The reference trajectory for the experiments was a circle of 600mm radius. The robot was placed initially at the centre of the circle. Figs. 14 and 15 show the time-evolution of coordinates x and y of the mobile robot. Figure 16 shows the path followed by the mobile robot on the $x-y$ plane. The maximum robot speed was $200mm/s$.

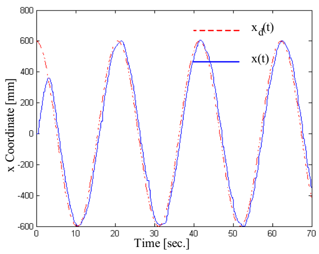

Fig 14:Evolution of $xd(t)$ and $x(t)$ of the mobile robot.

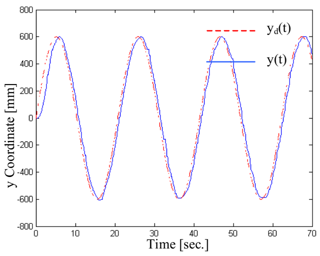

Fig 15:Evolution of $yd(t)$ and $y(t)$ of the mobile robot

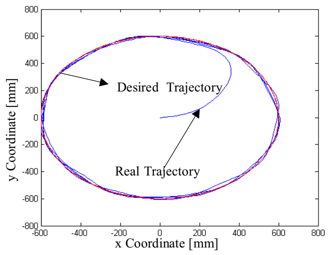

Fig. 16: Trajectory followed by the mobile robot in the x-y plane

From Fig. 16, it can also be noticed that the difference between the trajectory followed by the mobile robot and the desired one is only 20 mm, maximum, which compared with the distance between wheels (330 mm), it turns to be very small, showing the precision of the proposed control structure.

CONCLUSIONS

This paper has proposed a methodology to find the control action in such a way that the system may follow a desired trajectory, provided the desired states are reachable by the system. If this were not the case, the system will find the control actions that may lead it to the closest state to the desired one within the set of reachable states.

This methodology is applicable to systems that can be approached through numerical methods, by intervening into the model straightforwardly for the generation of control actions. This is advantageous because any decrease in system uncertainty will mean an improvement on controller performance.

The necessary precision of the numerical method used here is smaller than that required to simulate a system. This is so because, when using state feedback at each sampling time, any shift between the approximation and the real system can then be corrected. The approximation is used only to find the best way to go from the current state to the following one, and not to duplicate the system's evolution.

Future work will entail the generalization of this methodology to the cases where the states cannot be measured and, consequently, observers are needed. Simulation tests were performed, on systems where the state variable is the derivative of another one, by using derivative filters. These devices have given good results when using the Runge-Kutta approximation. An additional future work is to find a synthesis procedure that explicitly address questions of modelling errors.

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