EIGENSTRUCTURE ASSIGNMENT FOR SEMI-PROPER SYSTEMS: PSEUDO-STATE FEEDBACK [1]

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Abstract: The general problem of assigning the eigenstructure of semi-proper systems, using state feedback, is first considered. Standard eigenstructure assignment algorithms invariably assume the direct transmission matrix to be null. Consequently, they are suitable only for strictly proper systems. Algorithms do exist for assigning the eigenstructure of semi-proper systems, but they suffer from a lack of visibility and attention has not been paid to certain important aspects of the problem. A new exposition of the problem is presented here and then used directly to develop a novel algorithm which overcomes these issues. An extension to state feedback for semi-proper systems, the pseudo-state feedback case, is then considered.

Keywords: Eigenstructure assignment, Linear systems, Matrix algebra

1. INTRODUCTION AND EXISTING WORK

Standard eigenstructure assignment (EA) algorithms can be divided roughly into two groups: state-feedback, where the output ($\mathbf{C}$) matrix is assumed to be an identity and output-feedback, where assumptions are made only about the number of inputs and outputs (White, 1995). However, both groups of algorithms generally rely on the direct transmission ($\mathbf{D}$) matrix being null. This is valid only if the system is strictly proper. Given the multivariable transfer function matrix $\mathbf{G}(s)$, a system is strictly proper if

$$\lim_{s\to\infty} \mathbf{G}(s) = 0 \tag{1}$$

and semi-proper if

$$\lim_{s\to\infty} \mathbf{G}(s) = \mathbf{D} \quad (\neq \mathbf{0}) \tag{2}$$

Although semi-proper systems are mathematically feasible, all physical systems are strictly proper (Skogestad and Postlethwaite, 1996). However, semi-proper systems are often useful approximations.

For example, an aircraft mathematical model will often contain several velocity states. Velocities are almost impossible to measure in the absence of a fixed reference frame and, therefore, accelerometers are used to obtain state information. Incorporating the measured accelerations into the model results in the addition of nonzero entries in the $\mathbf{D}$ matrix. Additionally, many control structures familiar to designers of classical control systems, including PID controllers and phase-advance networks, involve a differentiation and will have the same effect. These are all approximations since no accelerometer or controller has an infinite bandwidth. However, the realisation of these differ-

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entiations in the form of a semi-proper system formulation is convenient and, provided that the bandwidths of the approximated components is sufficiently high, is fit for practical purposes.

To the authors' knowledge, the only published work involving the assignment of eigenstructure to semi-proper systems is that of Fletcher (Fletcher, 1981*a*; Fletcher, 1981*b*). These papers, published during the formative years of EA, are concerned with output-feedback pole placement rather then EA specifically. Eigenvectors are selected but no mention is made of their importance to the design or solution, or how they should be chosen. Moreover, Fletcher's technique is essentially a protection method (White, 1995) and consequently does not serve our purposes due to the lack of design visibility offered by these approaches. Finally the development of the method does not include a formulation of the complete closed-loop system, and therefore fails to show that the input and output matrices change when the loop is closed, a fact which can be important in the design process.

This paper proceeds as follows. Firstly, in Section 2, a closed-loop system description is produced that forms the basis for assignment algorithms presented later on. It is noted that all four system matrices are subject to change when the loop is closed, and the implications of this are discussed in Section 3. A pre-condition is introduced in Section 2 which, if not fulfilled, would prevent the calculation of the closed-loop system matrices. The physical meaning of this pre-condition is discussed in Section 4, along with a necessary and sufficient condition for its satisfaction.

The pseudo-state feedback algorithm, itself, is introduced in Section 5, together with necessary and sufficient conditions for the construction of a gain matrix. In Section 6, the possibility that there are more outputs than states is considered, and the algorithm of Section 5 is adapted to account for this. Finally, conclusions may be found in Section 7.

2. PROBLEM FORMULATION

Consider a semi-proper state-space system under the influence of feedback such that

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v} \tag{3}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{v} \tag{4}$$
$$\mathbf{v} = \mathbf{K}\mathbf{y} + \mathbf{u} \tag{5}$$

where $\mathbf{u}$ is an exogenous input, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times r}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$, $\mathbf{D} \in \mathbb{R}^{m\times r}$ and $\mathbf{K} \in \mathbb{R}^{r\times m}$.

By substitution we may readily obtain

$$\mathbf{v} = (\mathbf{I} - \mathbf{KD})^{-1}\mathbf{u} + (\mathbf{I} - \mathbf{KD})^{-1}\mathbf{KCx} \tag{6}$$

under the assumption that the term $\mathbf{I} - \mathbf{KD}$ is nonsingular (the implications of this restriction are discussed later).

To simplify subsequent analysis, we define

$$\mathbf{N} \triangleq (\mathbf{I} - \mathbf{KD})^{-1}\mathbf{K} \tag{7}$$

giving, after substitution:

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{BNC})\,\mathbf{x} + \mathbf{B}\,(\mathbf{I} - \mathbf{KD})^{-1}\,\mathbf{u} \tag{8}$$
$$\mathbf{y} = (\mathbf{C} + \mathbf{DNC})\,\mathbf{x} + \mathbf{D}\,(\mathbf{I} - \mathbf{KD})^{-1}\,\mathbf{u} \tag{9}$$

We may therefore define

$$\mathbf{A}_{cl} \triangleq \mathbf{A} + \mathbf{BNC} \tag{10}$$
$$\mathbf{B}_{cl} \triangleq \mathbf{B}\,(\mathbf{I} - \mathbf{KD})^{-1} \tag{11}$$
$$\mathbf{C}_{cl} \triangleq \mathbf{C} + \mathbf{DNC} \tag{12}$$
$$\mathbf{D}_{cl} \triangleq \mathbf{D}\,(\mathbf{I} - \mathbf{KD})^{-1} \tag{13}$$

Note that the closed-loop $\mathbf{A}_{cl}$, $\mathbf{B}_{cl}$, $\mathbf{C}_{cl}$, $\mathbf{D}_{cl}$ all differ from the open-loop $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ if $\mathbf{D} \neq \mathbf{0}$. Hence, if the open-loop system is semi-proper, not only do the system dynamics change when the loop is closed, but also do the input-state, state-output and input-output couplings.

3. IMPLICATIONS OF THE CLOSED LOOP SYSTEM STRUCTURE

In Fletcher's paper (Fletcher, 1981*a*), Equation 10 is stated (in expanded form) but it is not derived. The other three closed-loop matrices are not presented. However, the effect of loop closure on $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ is important.

Consider the case of right eigenvector assignment in order to control the coupling of modes into system outputs. The closed-loop output matrix $\mathbf{C}_{cl}$ (Equation 12) depends upon an inverse involving the gain matrix which, at the time of assignment, is unknown. Consequently the change from $\mathbf{C}$ to $\mathbf{C}_{cl}$, when the loop is closed, cannot be predicted. Therefore, the assignment of eigenvectors to determine mode-output coupling is not appropriate.

Whether or not the change in coupling between the states and outputs is of concern depends upon the nature of the assignment taking place. If it is necessary to ensure a specific coupling of modes into states, then assignment of eigenvectors is apporopriate. If, instead, it is desired to control the appearance of the modes in the outputs, then techniques leading to the direct assignment of the output-coupling vectors are required.

A secondary benefit of assigning output coupling vectors *directly* is that the algorithm is immediately suitable for those systems in which the states themselves have no direct physical interpretation. Models derived using identification techniques are likely to fall into this category if the identification process can only approximate input-output relationships.

4. SINGULARITIES IN THE CLOSED LOOP SYSTEM

Section 2 introduced the pre-condition on the gain matrix that $\mathbf{I}-\mathbf{KD}$ must be nonsingular. Here, we consider the reasons for this, and its implications for control system design.

The constraint is not a curiosity of the exposition presented here. Rather, it represents a system singularity. The feedforward $\mathbf{D}_{cl}$ matrix and the feedback $\mathbf{K}$ matrix form direct forward and backward transmission paths, coupling the input and output through a pair of simultaneous equations. When the constraint is not satisfied, no instantaneous solution exists to these equations for $\mathbf{y}$ given $\mathbf{u}$ and $\mathbf{x}$. It is reasonable to assume that a control system design approach based on meeting performance goals would never give rise to such a situation. Nevertheless, ensuring that this is the case is a simple matter.

Consider rearranging the defined structure of $\mathbf{N}$, given in Equation 7:

$$\mathbf{N} = (\mathbf{I} - \mathbf{KD})^{-1}\mathbf{K} \tag{14}$$

$$\mathbf{N} - \mathbf{KDN} = \mathbf{K} \tag{15}$$

$$\mathbf{N} = \mathbf{K} + \mathbf{KDN} \tag{16}$$

$$\mathbf{N}(\mathbf{I} + \mathbf{DN})^{-1} = \mathbf{K} \tag{17}$$

Note the bijective transformation between $\mathbf{N}$ and $\mathbf{K}$. This implies that if $\mathbf{I} - \mathbf{KD}$ is nonsingular, then the term $\mathbf{I} + \mathbf{DN}$ must also be nonsingular. The term $\mathbf{I} + \mathbf{DN}$, implied in Equation 12, is a factor that links the open-loop $\mathbf{C}$ and closed-loop $\mathbf{C}_{cl}$ matrices. Consequently, for this term to be nonsingular, $\mathbf{C}_{cl}$ must be of lower rank than $\mathbf{C}$.

If the design is specified using only output modal-coupling vectors (for example, using the technique to be presented in Section 5), it is clear that assigning a linearly independent set of coupling vectors is sufficient to ensure that the outputs are not co-linear and, hence, that $\mathbf{C}_{cl}$ is not rank-deficient.

5. PSEUDO-STATE FEEDBACK

We now present a novel algorithm that forms a simple extension to standard state-feedback EA. It appears elsewhere in a simplified, less generic form (Pomfret and Clarke, 2003). The term 'pseudo-state feedback' is coined here to describe the application of output feedback to a controllable, observable system with the same number of independent outputs as states. It is not a misnomer, since *state* feedback implies that the states are measurable directly (ie. that $\mathbf{C} = \mathbf{I}$ and $\mathbf{D} = \mathbf{0}$). *Pseudo*-state feedback simply requires that $\operatorname{rank}(\mathbf{C}) = m = n$ and otherwise carries no constraints beyond those of output feedback. The condition, $\operatorname{rank}(\mathbf{C}) = m = n$, allows for the placement of all the system poles by assigning only right-eigenvectors. This is the common characteristic of all state-feedback EA algorithms. Therefore, the term 'pseudo-state feedback' is deemed appropriate here.

By definition, for any closed-loop eigenvalue-eigenvector pair $\{\mathbf{v}_i, \lambda_i\}$,

$$\mathbf{A}_{cl}\mathbf{v}_i = \mathbf{v}_i\lambda_i \tag{18}$$

and consequently

$$(\mathbf{A} + \mathbf{BNC})\,\mathbf{v}_i = \mathbf{v}_i\lambda_i \tag{19}$$

$$(\mathbf{A} - \lambda_i\mathbf{I})\,\mathbf{v}_i = \mathbf{BNC}\mathbf{v}_i \tag{20}$$

$$\left[\mathbf{A} - \lambda_i\mathbf{I} \,\vdots\, \mathbf{B}\right]\begin{bmatrix}\mathbf{v}_i\\ \mathbf{NC}\mathbf{v}_i\end{bmatrix} = \mathbf{0} \tag{21}$$

Equation 21 may be parameterised by setting

$$\begin{bmatrix}\mathbf{v}_i\\ \mathbf{NC}\mathbf{v}_i\end{bmatrix} = \begin{bmatrix}\mathbf{P}_i\\ \mathbf{Q}_i\end{bmatrix}\cdot\mathbf{f}_i \tag{22}$$

where

$$\operatorname{range}\left(\begin{bmatrix}\mathbf{P}_i\\ \mathbf{Q}_i\end{bmatrix}\right) = \ker\left(\left[\mathbf{A} - \lambda_i\mathbf{I} \,\vdots\, \mathbf{B}\right]\right) \tag{23}$$

The output-coupling vector $\mathbf{o}_i$ describes the distribution of a given mode into the outputs:

$$\mathbf{o}_i = \mathbf{C}_{cl}\mathbf{v}_i \tag{24}$$

$$= (\mathbf{C} + \mathbf{DNC})\,\mathbf{v}_i \tag{25}$$

$$= \left[\mathbf{C} \,\vdots\, \mathbf{D}\right]\begin{bmatrix}\mathbf{v}_i\\ \mathbf{NC}\mathbf{v}_i\end{bmatrix} \tag{26}$$

$$\mathbf{o}_i = \left[\mathbf{C} \,\vdots\, \mathbf{D}\right]\begin{bmatrix}\mathbf{P}_i\\ \mathbf{Q}_i\end{bmatrix}\cdot\mathbf{f}_i \tag{27}$$

Consequently, the design vector, $\mathbf{f}_i$, for a given mode may used to select either an output-coupling vector $\mathbf{o}_i$, or an eigenvector $\mathbf{v}_i$ using, for example, a least-squares projection of a desired vector into the allowable subspace. Note that, since $\mathbf{C}_{cl}$ is square and full-rank (see Section 4), the condition that the closed-loop eigenvectors must be linearly independent can be satisfied by ensuring instead that the selected output-coupling vectors are linearly independent.

Having selected the design vectors $\{\mathbf{f}_i\}$, the matrix $\mathbf{N}$ may be recovered:

$$\mathbf{V} = \left[\, \mathbf{v}_1 \;\, \mathbf{v}_2 \; \ldots \; \mathbf{v}_n \,\right] \tag{28}$$

$$= \left[\, \mathbf{P}_1\mathbf{f}_1 \;\, \mathbf{P}_2\mathbf{f}_2 \; \ldots \; \mathbf{P}_n\mathbf{f}_n \,\right] \tag{29}$$

$$\mathbf{S} = \mathbf{NCV} \tag{30}$$

$$= \left[\, \mathbf{Q}_1\mathbf{f}_1 \;\, \mathbf{Q}_2\mathbf{f}_2 \; \ldots \; \mathbf{Q}_n\mathbf{f}_n \,\right] \tag{31}$$

$$\mathbf{N} = \mathbf{SV}^{-1}\mathbf{C}^{-1} \tag{32}$$

It only remains to rearrange $\mathbf{N}$ (defined at Equation 7) to find $\mathbf{K}$ (using Equation 17). In Section 4, it was shown that a solution is guaranteed for a linearly independent set of output-coupling vectors. Consequently, assigning a linearly independent set of output vectors guarantees both a solution to Equation 32 *and* a solution to Equation 7.

6. EXCESS FREEDOM

If accelerometers, or other forms of derivative feedback, are used in order to increase the number of system outputs to the point where pseudo-state feedback is practical, it is also feasible that the number of outputs may be made to exceed the number of states, ie. $m > n$. In this case, the gain matrix solution is not unique.

If the condition that $m = n$ is replaced by the new condition $m \geq n$, the procedure of Section 5 may be followed until Equation 31. At this point, the algorithm relies on the inversion of $\mathbf{C}$ and must therefore be modified.

From Equations 29 and 31, it is clear that

$$\mathbf{NC} = \mathbf{SV^{-1}} \tag{33}$$

where $\mathbf{C}$, $\mathbf{S}$ and $\mathbf{V}$ are known.

From Equation 17,

$$\mathbf{N}\left(\mathbf{I} + \mathbf{DN}\right)^{-1} = \mathbf{K} \tag{34}$$

$$\mathbf{NC} = \mathbf{K}\left(\mathbf{I} + \mathbf{DN}\right)\mathbf{C} \tag{35}$$

Substituting Equation 33 gives

$$\mathbf{SV^{-1}} = \mathbf{K}\left(\mathbf{I} + \mathbf{DN}\right)\mathbf{C} \tag{36}$$

and solving for $\mathbf{K}$ (Ben-Israel and Greville, 1974, 39)

$$\mathbf{K}\left(\mathbf{I} + \mathbf{DN}\right) = \mathbf{SV^{-1}C^{\dagger}} + \mathbf{Y}\left(\mathbf{I} - \mathbf{CC^{\dagger}}\right) \tag{37}$$

$$\mathbf{K} = \mathbf{SV^{-1}C^{\dagger}}\left(\mathbf{I} + \mathbf{DN}\right)^{-1} + \mathbf{Y}\left(\mathbf{I} - \mathbf{CC^{\dagger}}\right)\left(\mathbf{I} + \mathbf{DN}\right)^{-1} \tag{38}$$

where $\mathbf{C}^{\dagger}$ is the Moore-Penrose pseudo-inverse of $\mathbf{C}$ and $\mathbf{Y}$ is a matrix of free parameters. This parameter matrix is expressed in a similar way to the matrix of free parameters existing at the end of the output-feedback EA algorithm of Clarke *et al.* (2003). Hence, for a system with $m > n$, it is possible, not only to assign n eigenvalues and right-eigenvectors, but also to recover the unused design freedom and employ it for another purpose.

7. CONCLUSIONS

A novel state-feedback eigenstructure assignment algorithm has been presented which is capable of operating upon semi-proper systems. Attention has been paid to the changes in mode-output couplings that occur when the loop is closed around a semi-proper system. A simple output-coupling vector assignment technique has been developed to overcome these changes. Straightforward necessary and sufficient conditions have been developed for the construction of a gain matrix. Finally, the algorithm has been modified to allow for the situation where there are more outputs than states and to encapsulate the excess design freedom in a usable form.

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