USING POST-EIGENSTRUCTURE ASSIGNMENT DESIGN FREEDOM FOR THE IMPOSITION OF CONTROLLER STRUCTURE [1]

Andrew Pomfret* Tim Clarke*

* *Intelligent Systems Group, University of York,*
Heslington, York, YO10 5DD, UK
http://www.elec.york.ac.uk/intsys/

Abstract: Recent algorithm developments in the field of output-feedback eigenstructure assignment make use of the available design freedom in a multi-stage assignment process. Depending on the number of degrees of freedom available and the manner in which they are distributed between the stages, it is possible that not all will be used. This paper develops an algorithm by which these excess degrees of freedom may be put to use by nulling individual elements of the gain matrix, thereby imposing structure upon the resulting controller. Consideration is given to the effect of this process on the remaining gain matrix elements.

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Keywords: Eigenstructure assignment, Linear systems, Structural constraints, Matrix algebra

1. INTRODUCTION

The recent work of Clarke *et al.* (2003) detailed new algorithms for output-feedback eigenstructure assignment control for linear systems. The algorithms employ a multi-stage approach, in which the available design freedom is reduced stepwise by the assignment of eigenvectors and associated eigenvalues. Depending on the number of degrees of freedom available and the manner in which they are distributed between the design stages, it is possible that some may remain unused after the assignment is complete. In other published work, Clarke and Griffin (2004) introduce an algorithm (the 'retro-assignment stage') that makes use of this post-assignment freedom to assign complementary eigenvectors to those assigned using the original algorithm.

It is likely, however, that further eigenvector assignment is not the most appropriate use for this design freedom. Typically, only a few right eigenvectors (corresponding to dominant modes) are crucial to the system specification, but the formation of the general nonlinear eigenstructure assignment problem into a problem with a linear solution requires that one eigenvector is assigned for every eigenvalue. The control of modal coupling is therefore likely to have been satisfied by the primary assignment algorithm and the design freedom could, instead, be employed to achieve some other objective.

One such objective is a defined controller structure. Eigenstructure assignment, in common with most multi-input multi-output control system design techniques, will, in general, generate a fully-populated matrix of feedback gains. The resulting complex, fully-interconnected controller bears little resemblance to the sparse, modular control systems achieved using classical approaches. In

[1] Corresponding author A. Pomfret. Tel. +44 (0) 1904 432419, Fax +44 (0) 1904 432335.

order to impose structure upon a controller, it is necessary to reduce a subset of the gains to zero, thereby reducing the complexity of the connections from plant outputs to plant inputs. This paper presents a method by which design freedom, remaining after eigenstructure assignment is completed, may be used for this purpose.

Section 2 defines the Kronecker product and vec operator, both of which are at the heart of the development of the problem in Section 3. A solution is then derived in Section 5. Nulling individual gain matrix entries must affect the remaining entries. The nature and magnitude of this effect is investigated in Section 6. Conclusions are presented in Section 7.

2. DEFINITIONS

The following definitions will be employed throughout this paper.

Kronecker Product: The Kronecker product (direct product, tensor product) of $\mathbf{A}^{m\times n} = [a_{ij}]$ and $\mathbf{B}^{r\times s} = [b_{ij}]$ is defined as the partitioned matrix

$$\mathbf{A}\otimes\mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix} \quad (1)$$

which can be seen to be of order $(mr \times ns)$.

Vec Operator: The vec operator converts a matrix of order $(m \times n)$ into a vector of length mn and is defined as

$$\mathrm{vec}\mathbf{A} = \begin{bmatrix} \mathbf{A}_{\cdot 1} \\ \mathbf{A}_{\cdot 2} \\ \vdots \\ \mathbf{A}_{\cdot n} \end{bmatrix} \quad (2)$$

3. PROBLEM DEFINITION

The algorithms described in Clarke *et al.* (2003) yield a gain matrix $\mathbf{K}$ which is dependant upon a matrix of free parameters $\mathbf{Z}$. The matrix $\mathbf{Z}$ may be chosen arbitrarily, and any changes exhibited by the gain matrix $\mathbf{K}$ as a result will not affect the eigenvalues or assigned eigenvectors of the closed loop system. The gain matrix equation takes the following form:

$$\mathbf{K} = \mathbf{K}_0 + \left(\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}\right)\mathbf{Z}\left(\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger\right) \quad (3)$$

where $\mathbf{X} \in \mathbb{C}^{w\times r}$, $\mathbf{Y} \in \mathbb{C}^{m\times v}$, and $\mathbf{K}, \mathbf{K}_0, \mathbf{Z} \in \mathbb{R}^{r\times m}$. $\mathbf{A}^\dagger$ is the Moore-Penrose inverse of $\mathbf{A}$ and satisfies the following four equations:

$$\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A} \quad (4)$$

$$\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger \quad (5)$$

$$\left(\mathbf{A}\mathbf{A}^\dagger\right)^\star = \mathbf{A}\mathbf{A}^\dagger \quad (6)$$

$$\left(\mathbf{A}^\dagger\mathbf{A}\right)^\star = \mathbf{A}^\dagger\mathbf{A} \quad (7)$$

where $\mathbf{A}^\star$ represents the complex conjugate transpose of $\mathbf{A}$.

The mapping of $\mathbf{Z}$ to $\mathbf{K}$ through Equation 3 is not bijective, so a multiplicity of values for $\mathbf{Z}$ can yield the same $\mathbf{K}$. Clarke and Griffin (2004) show that the number of degrees of freedom available at this stage is given by

$$f = (m - v)\,(r - w) \quad (8)$$

and that, if $f = 0$, the term involving $\mathbf{Z}$ in Equation 3 will evaluate to zero.

In order to reduce an arbitrary set of gain matrix entries to zero, it is necessary to find a solution to the equation

$$\mathbf{U}\mathrm{vec}\mathbf{K} = \mathbf{0} \quad (9)$$

where the permutation matrix $\mathbf{U}_{\delta\times mr}$ posesses exactly one unity element per row and is zero elsewhere. The parameter δ is therefore equal to the number of gain matrix elements that are to be nulled. Substituting Equation 3,

$$\mathbf{U}\mathrm{vec}\left(\mathbf{K}_0 + \left(\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}\right)\mathbf{Z}\left(\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger\right)\right) = \mathbf{0} \quad (10)$$

$$\mathbf{U}\mathrm{vec}\left(\left(\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}\right)\mathbf{Z}\left(\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger\right)\right) = -\mathbf{U}\mathrm{vec}\mathbf{K}_0 \quad (11)$$

The identity (Graham, 1981, p25)

$$\mathrm{vec}\left(\mathbf{ABC}\right) = \left(\mathbf{C}^T \otimes \mathbf{A}\right)\mathrm{vec}\mathbf{B} \quad (12)$$

can now be applied, yielding

$$\mathbf{U}\left(\left(\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger\right) \otimes \left(\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}\right)\right)\mathrm{vec}\mathbf{Z} = -\mathbf{U}\mathrm{vec}\mathbf{K}_0 \quad (13)$$

Note the lack of a transpose operator since the matrices forming the Kronecker product are symmetric. We may now define

$$\mathbf{\Xi} \triangleq \left(\mathbf{I} - \mathbf{Y}\mathbf{Y}^\dagger\right) \otimes \left(\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}\right) \quad (14)$$

so that

$$\mathbf{U\Xi}\mathrm{vec}\mathbf{Z} = -\mathbf{U}\mathrm{vec}\mathbf{K}_0 \quad (15)$$

Section 5 will concentrate upon finding a solution to Equation 15. But, beforehand, we must consider essential properties of $\mathbf{\Xi}$.

4. PROPERTIES OF $\mathbf{\Xi}$

The term $\mathbf{\Xi}$, defined in Equation 14, has two properties that will assist in the analysis presented below.

Firstly, $\mathbf{\Xi}$ is idempotent. The idempotence of its component terms is easily shown, and the product of two Kronecker products (Graham, 1981, p24) is given by

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD} \tag{16}$$

provided that the dimensions are such that the various matrices exist. Consequently, if both $\mathbf{E}$ and $\mathbf{F}$ below are idempotent, then

$$\begin{aligned}(\mathbf{E} \otimes \mathbf{F})^2 &= \mathbf{E}^2 \otimes \mathbf{F}^2 \\ &= \mathbf{E} \otimes \mathbf{F} \end{aligned} \tag{17}$$

and, therefore, $\boldsymbol{\Xi}$ is idempotent.

Secondly, $\boldsymbol{\Xi}$ is symmetric. From Equations 6 and 7, the expressions $\mathbf{X}^\dagger\mathbf{X}$ and $\mathbf{YY}^\dagger$ can be seen to be symmetric and consequently so are $\left(\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}\right)$ and $\left(\mathbf{I} - \mathbf{YY}^\dagger\right)$. The transpose of a Kronecker product is given by Graham (1981, p24) as

$$(\mathbf{A} \otimes \mathbf{B})^\star = \mathbf{A}^\star \otimes \mathbf{B}^\star \tag{18}$$

and so it is clear that the term $\boldsymbol{\Xi}$ is symmetric.

5. SOLUTION FOR Z AND K

We now consider the solution for Equation 15, which is of the form

$$\mathbf{Ax} = \mathbf{b} \tag{19}$$

and therefore (Ben-Israel and Greville, 1974, p40) has a solution if and only if

$$\mathbf{AA}^\dagger\mathbf{b} = \mathbf{b} \tag{20}$$

The solution is therefore given by

$$\mathbf{x} = \mathbf{A}^\dagger\mathbf{b} + \left(\mathbf{I} - \mathbf{A}^\dagger\mathbf{A}\right)\mathbf{y} \tag{21}$$

Consequently, a solution to Equation 15 exists if and only if

$$\mathbf{U}\boldsymbol{\Xi}\left(\mathbf{U}\boldsymbol{\Xi}\right)^\dagger \mathbf{U}\text{vec}\mathbf{K}_0 = \mathbf{U}\text{vec}\mathbf{K}_0 \tag{22}$$

A matrix $\mathbf{E}$ is idempotent if $\mathbf{E}^2 = \mathbf{E}$; the term $\mathbf{U}\boldsymbol{\Xi}\left(\mathbf{U}\boldsymbol{\Xi}\right)^\dagger$ is idempotent (via Equations 4 and 5), and it holds for an idempotent matrix $\mathbf{E}$ (Ben-Israel and Greville, 1974, p49) that

$$\mathbf{Ex} = \mathbf{x} \tag{23}$$

if and only if

$$\mathbf{x} \in \text{range}\left(\mathbf{E}\right) \tag{24}$$

Therefore a solution exists for Equation 15 if and only if

$$\begin{aligned}\mathbf{U}\text{vec}\mathbf{K}_0 &\in \text{range}\left(\mathbf{U}\boldsymbol{\Xi}\left(\mathbf{U}\boldsymbol{\Xi}\right)^\dagger\right) &(25)\\ &\in \text{range}\left(\mathbf{U}\boldsymbol{\Xi}\right) &(26)\end{aligned}$$

A simple sufficient condition is easily seen to be

$$\text{rank}\left(\mathbf{U}\boldsymbol{\Xi}\right) = \delta \tag{27}$$

This condition, although not strictly necessary, is necessary for *general* $\mathbf{K}_0$ and $\boldsymbol{\Xi}$ since, otherwise, there is no guarantee of the existence of a $\mathbf{U}$ which satisfies Equation 26.

A necessary (but not sufficient) condition for the fulfillment of Equation 27 is that

$$\begin{aligned}\delta &\le \text{rank}\left(\boldsymbol{\Xi}\right) &(28)\\ &\le \text{rank}\left(\left(\mathbf{I} - \mathbf{YY}^\dagger\right) \otimes \left(\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}\right)\right) &(29)\\ &\le \text{rank}\left(\mathbf{I} - \mathbf{YY}^\dagger\right) \cdot \text{rank}\left(\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}\right) &(30)\\ &\le \left(m - \text{rank}\left(\mathbf{YY}^\dagger\right)\right)\left(r - \text{rank}\left(\mathbf{X}^\dagger\mathbf{X}\right)\right) &(31)\\ &\le (m - v)(r - w) &(32)\end{aligned}$$

Comparison with Equation 8 shows that Equation 32 may be written as

$$\delta \le f \tag{33}$$

This demonstrates that, for general $\mathbf{K}_0$ and $\boldsymbol{\Xi}$, the number of gain matrix entries to be reduced to zero may not exceed the number of available degrees of freedom. Note that the satisfaction of Equation 33 is not sufficient for the existence of a solution to Equation 15, and that the satisfaction of Equation 26 is still required.

Assuming, then, that $\mathbf{U}$ has been selected to meet Equation 26, the solution to Equation 15 is given by substitution into Equation 21:

$$\begin{aligned}\text{vec}\mathbf{Z} = &-\left(\mathbf{U}\Xi\right)^\dagger \mathbf{U}\text{vec}\mathbf{K}_0 \\ &+ \left(\mathbf{I} - \left(\mathbf{U}\Xi\right)^\dagger \mathbf{U}\Xi\right)\text{vec}\tilde{\mathbf{Z}}\end{aligned} \tag{34}$$

where $\tilde{\mathbf{Z}}$ is a matrix characterising any remaining free parameters. A solution for $\mathbf{K}$ may now be found. From Equation 3,

$$\begin{aligned}\mathbf{K} &= \mathbf{K}_0 + \left(\mathbf{I} - \mathbf{X}^\dagger\mathbf{X}\right)\mathbf{Z}\left(\mathbf{I} - \mathbf{YY}^\dagger\right) &(35)\\ \text{vec}\mathbf{K} &= \text{vec}\mathbf{K}_0 + \Xi\text{vec}\mathbf{Z} &(36)\end{aligned}$$

Substituting Equation 34, we may obtain

$$\begin{aligned}\text{vec}\mathbf{K} &= \text{vec}\mathbf{K}_0 - \Xi\left(\mathbf{U}\Xi\right)^\dagger \mathbf{U}\text{vec}\mathbf{K}_0 \\ &\quad + \Xi\left(\mathbf{I} - \left(\mathbf{U}\Xi\right)^\dagger \mathbf{U}\Xi\right)\text{vec}\tilde{\mathbf{Z}} &(37)\\ &= \text{vec}\mathbf{K}_0 - \Xi\left(\mathbf{U}\Xi\right)^\dagger \mathbf{U}\text{vec}\mathbf{K}_0 \\ &\quad + \Xi\text{vec}\tilde{\mathbf{Z}} - \Xi\left(\mathbf{U}\Xi\right)^\dagger \mathbf{U}\Xi\text{vec}\tilde{\mathbf{Z}} &(38)\end{aligned}$$

Equation 38 may be simplified by noting (from Equations 5 and 7) that

$$\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\dagger} \quad (39)$$
$$= \left(\mathbf{A}^{\dagger}\mathbf{A}\right)^{\star}\mathbf{A}^{\dagger} \quad (40)$$
$$= \mathbf{A}^{\star}\left(\mathbf{A}^{\dagger}\right)^{\star}\mathbf{A}^{\dagger} \quad (41)$$

and therefore that

$$\mathbf{A}^{\dagger} \in \text{range}\left(\mathbf{A}^{\star}\right) \quad (42)$$

It may therefore be seen that

$$(\mathbf{U}\boldsymbol{\Xi})^{\dagger} \in \text{range}\left(\boldsymbol{\Xi}^{\star}\mathbf{U}^{\star}\right) \quad (43)$$
$$\in \text{range}\left(\boldsymbol{\Xi}\right) \quad (44)$$

and therefore that

$$\boldsymbol{\Xi}\left(\mathbf{U}\boldsymbol{\Xi}\right)^{\dagger} = \left(\mathbf{U}\boldsymbol{\Xi}\right)^{\dagger} \quad (45)$$

since $\boldsymbol{\Xi}$ is idempotent.

So, using Equation 45, Equation 38 becomes

$$\text{vec}\mathbf{K} = \text{vec}\mathbf{K}_0 - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\text{vec}\mathbf{K}_0 + \Xi\text{vec}\tilde{\mathbf{Z}} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\text{vec}\tilde{\mathbf{Z}} \quad (46)$$
$$= \left(\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\left(\text{vec}\mathbf{K}_0 + \Xi\text{vec}\tilde{\mathbf{Z}}\right) \quad (47)$$

It must be noted that it is not possible to recover a matrix formulation for $\mathbf{K}$ since the vec operator has no effective inverse. The gain matrix must instead be derived in vector form as above, and reconstituted numerically into a matrix afterwards.

6. SENSITIVITY OF THE GAIN MATRIX

There exists a value of $\mathbf{Z}$ for which $|\mathbf{K}|_F$ is minimal.(The Frobenius matrix norm is the square root of the sum of squared gain matrix entries.) It is safe to assume that the value of $\mathbf{Z}$ found via Equation 47 will not equal this 'optimal' value. Therefore, nulling individual gain matrix entries will, in general, raise $|\mathbf{K}|_F$.

The extent to which $|\mathbf{K}|_F$ is affected will depend upon the elements of the gain matrix selected by the permutation matrix $\mathbf{U}$. A mechanism for determining the increase in $|\mathbf{K}|_F$ would be a useful tool when attempting to determine which elements should be set to zero.

Since the formulation for $\mathbf{K}$ in Equation 47 contains a free parameter matrix, it is important to find the value of the free parameter matrix that minimises the Frobenius norm of the final gain matrix.

6.1 Minimum Frobenius Norm

From earlier examination of the properties of $\boldsymbol{\Xi}$, we may now determine the value of $\tilde{\mathbf{Z}}$ that leads to the minimum $|\mathbf{K}|_F$. Because vec$\mathbf{K}$ is simply a rearrangement of entries,

$$|\mathbf{K}|_F = \|\text{vec}\mathbf{K}\| \quad (48)$$

and therefore

$$|\mathbf{K}|_F = \left\|\left(\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\left(\text{vec}\mathbf{K}_0 + \Xi\text{vec}\tilde{\mathbf{Z}}\right)\right\| \quad (49)$$
$$= \left\|\left(\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\text{vec}\mathbf{K}_0 + \left(\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\Xi\text{vec}\tilde{\mathbf{Z}}\right\| \quad (50)$$

The problem is now one of solving an equation of the form

$$\frac{d}{d\mathbf{x}}\|\mathbf{A}\mathbf{x} + \mathbf{y}\| = 0 \quad (51)$$

to find a minimum. It may be shown that

$$\frac{d}{d\mathbf{x}}\|\mathbf{A}\mathbf{x} + \mathbf{y}\| = \frac{\mathbf{A}'\left(\mathbf{A}\mathbf{x} + \mathbf{y}\right)}{\|\mathbf{A}\mathbf{x} + \mathbf{y}\|} \quad (52)$$

and therefore we must solve

$$\frac{d}{d\tilde{\mathbf{Z}}}\left\|\left(\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\text{vec}\mathbf{K}_0 + \left(\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\right)\Xi\text{vec}\tilde{\mathbf{Z}}\right\| = \mathbf{0} \quad (53)$$

i.e.

$$\left(\Xi - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\right)'\left((\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U})\text{vec}\mathbf{K}_0 + (\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U})\Xi\text{vec}\tilde{\mathbf{Z}}\right) = \mathbf{0} \quad (54)$$

so,

$$\left(\Xi - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\right)\left((\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U})\text{vec}\mathbf{K}_0 + (\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U})\Xi\text{vec}\tilde{\mathbf{Z}}\right) = \mathbf{0} \quad (55)$$

since $\mathbf{A}^{\dagger}\mathbf{A}$ and $\boldsymbol{\Xi}$ are both symmetrical.

Equation 55 may be simplified further:

$$\left(\Xi - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\right)\left(\text{vec}\mathbf{K}_0 - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\text{vec}\mathbf{K}_0 + \Xi\text{vec}\tilde{\mathbf{Z}} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\text{vec}\tilde{\mathbf{Z}}\right) = \mathbf{0} \quad (56)$$

so

$$\Xi\text{vec}\mathbf{K}_0 - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\text{vec}\mathbf{K}_0 + \Xi\text{vec}\tilde{\mathbf{Z}} - (\mathbf{U}\Xi)^{\dagger}\mathbf{U}\Xi\text{vec}\tilde{\mathbf{Z}} = \mathbf{0} \quad (57)$$

hence

$$\left(\Xi - (\mathbf{U}\Xi)^{\dagger}\,\mathbf{U}\Xi\right)\mathrm{vec}\mathbf{K}_0 + \left(\Xi - (\mathbf{U}\Xi)^{\dagger}\,\mathbf{U}\Xi\right)\mathrm{vec}\tilde{\mathbf{Z}} = \mathbf{0} \tag{58}$$

Equation 58 has the simple solution:

$$\hat{\tilde{\mathbf{Z}}} = -\mathbf{K}_0 \tag{59}$$

where $\hat{\tilde{\mathbf{Z}}}$ is the value of $\tilde{\mathbf{Z}}$ which minimises $|\mathbf{K}|_F$.

6.2 Increase in Minimum Norm

The minimum $|\mathbf{K}|_F$ may be simply calculated by substituting Equation 59 into Equation 47:

$$\min|\mathbf{K}|_F = \min\|\mathrm{vec}\mathbf{K}\| = \left\|\left(\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\,\mathbf{U}\right)\left(\mathrm{vec}\mathbf{K}_0 + \Xi\mathrm{vec}\hat{\tilde{\mathbf{Z}}}\right)\right\| \tag{60}$$

$$= \left\|\left(\mathbf{I} - (\mathbf{U}\Xi)^{\dagger}\,\mathbf{U}\right)(\mathbf{I} - \Xi)\,\mathrm{vec}\mathbf{K}_0\right\| \tag{61}$$

The minimum $|\mathbf{K}|_F$ achievable prior to nulling any gain matrix entries may be found by setting $\mathbf{U} = \mathbf{0}$.

So,

$$\min|\mathbf{K}|_F = \min\|\mathrm{vec}\mathbf{K}\| = \|(\mathbf{I} - \Xi)\,\mathrm{vec}\mathbf{K}_0\| \tag{62}$$

Comparison of Equations 61 and 62 shows the effect of nulling a subset of the gains. This information can be used to determine which gains may most easily be reduced to zero whilst maintaining the lowest possible gains elsewhere.

7. CONCLUSIONS

The design freedom remaining at the end of multi-stage eigenstructure assignment algorithms (Clarke *et al.*, 2003) has use beyond the retro-assignment stage offered by Clarke and Griffin (2004). Specifically, this freedom may be used to reduce individual entries in the gain matrix to zero, thereby imposing a structure upon the resulting controller.

An algorithm for nulling a subset of gain matrix entries has been presented, and it has been demonstrated that the maximum number of entries that may be nulled is, in general, equal to the number of available degrees of freedom. In addition, the effect upon the remaining entries of the gain matrix has been considered, and an expression generated for the minimum Frobenius norm of the gain matrix both before and after the nulling of entries.

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