A COMPARISON OF TWO ALGORITHMS FOR THE STATIC $\mathcal{H}_\infty$ LOOP SHAPING PROBLEM

E. Prempain *

* *Department of Electrical Engineering & Electronics, University of Liverpool, Brownlow Hill, Liverpool L69 3GJ UK*

Abstract: This paper compares two different linear matrix inequality (LMIs) algorithms for the static $\mathcal{H}_\infty$ loop shaping synthesis problem. One effective and popular algorithm to solve fixed-order control problems is the so-called Cone Complementary Linearization algorithm (CCL). The CCL algorithm is guaranteed to produce, at each iteration, a reduced order controller. The algorithm proposed in this paper is quite different in its nature as it is based on sufficient, potentially conservative, LMI conditions. Our algorithm is compared to the Cone Complementary algorithm on a collection of plants taken from the benchmark library $COMPl_eib$. The numerical experiments indicate that our algorithm is computationally more attractive and more efficient than the cone complementary algorithm.

Keywords: Reduced-order control, Linear Matrix Inequality Optimization, H_∞ Optimization, Linear Multivariable Systems.

1. INTRODUCTION

This paper compares two different linear matrix inequality (LMIs) algorithms for the static version of the $\mathcal{H}_\infty$ loop shaping design procedure of McFarlane and Glover (1992). The solution for static or reduced order $\mathcal{H}_\infty$ problems involves minimizing the rank of a matrix variable subject to linear matrix inequalities constraints. Solving this rank minimization problem is in general very difficult see e.g. (Iwasaki *et al.*, 1994), (Syrmos *et al.*, 1997), (Arzelier and Peaucelle, 2002), (Leibfritz, 2001). Over the recent years, simple heuristics have been developed to handle rank minimization problems in the LMI framework. One effective and popular algorithm is the so-called iterative Cone Complementary Linearization algorithm (CCL) proposed by El Ghaoui *et al.* (1997), where the rank minimization problem is approximated by a sequence of semi-definite programs involving the minimization of the trace of a certain semi-definite matrix variable.

Alternatively, and at the expense of conservatism, the rank constraint can be directly incorporated into the LMI formulation of the problem. The algorithm proposed in this paper is based on that principle and leads, in a number of applications, to a conservative but nevertheless very efficient algorithm. The new algorithm proposed in this paper and the Cone Complementary algorithm are compared on a collection of plants taken from the benchmark library $COMPl_eib$ (Leibfritz and Lipinski, 2004). This library includes a number of examples collected from real-world applications as well as pure academic problems. The current version of $COMPl_eib$ contains about 80 plants which are static output feedback stabiliz-

* E. Prempain. Tel. +44 151 794 4515. Fax +44 151 794 4540. E-mail ep26@liv.ac.uk

able. The numerical experiments carried out with these plants indicate that the sufficient condition algorithm proposed in this paper is computationally more attractive than the cone complementary algorithm.

The paper is organized as follows. Section 2 introduces the background material for static output feedback stabilization. Section 3 presents the sufficient LMI conditions for static $\mathcal{H}_\infty$ loop shaping control synthesis. Sections 4 reviews the cone complementarity algorithm and introduces an algorithm based on the sufficient LMI conditions introduced in section 3. Section 5 provides numerical results. Finally, section 6 ends the paper with some conclusions.

2. PRELIMINARIES

Let us consider a linear time invariant system described by state-space equations

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \qquad (1)$$

where $A \in R^{n \times n}$, $u \in R^m$ is the control input, $y \in R^p$ is the measured output. The pairs (A, B) and (A, C) are assumed to be, respectively, stabilizable and detectable, we assume that C and B are full rank.

Our aim is to compute a static output feedback law $u = Ky$ that ensures the stability of the closed-loop system $A_{cl} = A + BKC$.

Lemma 1. (Finsler's lemma) Let X be a given symmetric matrix and let Z be a matrix such that

$$\xi^T X \xi < 0$$

for all nonzero vector ξ such that $Z\xi = 0$. Then there exists a constant $\sigma > 0$ such that

$$X - \sigma Z^T Z < 0$$

Proof. See e.g. (Boyd *et al.*, 1994).

Lemma 2. The following statements are equivalent.

i) There exists a stabilizing static output feedback gain.
ii) There exists a positive-definite matrix $R > 0$ such that

$$N_B^T (AR + RA^T) N_B < 0,$$
$$N_C^T (R^{-1}A + A^T R^{-1}) N_C < 0,$$

where N_B and N_C denote bases of the null spaces of B^T and C, respectively.

Proof. See e.g. (Iwasaki *et al.*, 1994).

3. MAIN RESULTS

3.1 Static Output Feedback Stabilization

Lemma 3. The following statements are equivalent.

i) There exists a stabilizing static output feedback gain.
ii) There exist a positive-definite matrix $R > 0$, a matrix L of compatible dimension and a positive real number γ such that

$$AR + RA^T - \gamma BB^T < 0, \qquad (2)$$
$$(A + LC)R + R(A + LC)^T < 0. \qquad (3)$$

Proof. Using Finsler's lemma the conditions of statement ii) can be rewritten as:

There exist $R > 0$ and positive real numbers σ_1, σ_2 such that

$$AR + RA^T - \sigma_1 BB^T < 0, \qquad (4)$$
$$AR + RA^T - \sigma_2 RC^T CR < 0. \qquad (5)$$

Clearly, if (4) and (5) are satisfied for some positive real numbers σ_1 and σ_2, then they are also satisfied if one replaces σ_1 and σ_2 with $\gamma = \max(\sigma_1, \sigma_2)$. Now, let $L = -\gamma RC^T/2$. With such a matrix L, condition (3) reduces to condition (5) and lemma 3 is proven ■

For a given matrix L such that $A + LC$ is stable, the conditions of lemma 3 are linear in R and γ. This contrasts with the well-known static output feedback conditions of lemma 2 which involve R and R^{-1}. Also, when L is given, conditions (4) and (5) are sufficient for the existence of a static output feedback gain.

The feasibility of the linear conditions of lemma 3 strongly depends on the choice of the matrix L. There is infinite number of matrices L rendering $A + LC$ stable. However, there is no systematic rule for selecting the matrix L so that the LMI system (2) and (3) is feasible. Therefore, we are forced to use some heuristics to simplify the problem. A computationally attractive way to generate a scalar dependent set of matrices L such that $A + LC$ is stable is given by the following lemma.

Lemma 4. Let $L = -YC^T$ where $Y \geq 0$ is the stabilizing solution to

$$AY + YA^T - \alpha YC^T CY + BB^T = 0. \qquad (6)$$

For any $0 < \alpha < 2$ the matrix $(A + LC)$ is a stability matrix.

Proof. Since (A, B) is stabilizable and (C, A) is detectable, it is well-known that the above Riccati equation has a unique stabilizing positive semi-definite solution Y. First, note that α must be strictly positive to guarantee the existence of a semi-definite positive solution Y for (6). Now, $(A+LC)Y + Y(A+LC)^T := Q = -BB^T - (2-\alpha)YC^TCY < 0$. From (Zhou *et al.*, 1995), $A+LC$ is stable if $Y \geq 0$, $Q < 0$ and $(Q, (A+LC)^T)$ is detectable. Hence, if the pair $(A+LC, Q)$ is stabilizable, one can conclude on the stability of $A+LC$. Because, Q is a square and a full rank matrix, it is clear that the pair $(A+LC, Q)$ is stabilizable and therefore $A+LC$ is stable for any value of α in $]0, 2[$ ■

Remark: Lemma 4 guarantees the stability of $A+LC$ for $0 < \alpha < 2$. But, note that $A+LC$ can be stable even for $\alpha > 2$.

4. STATIC $\mathcal{H}_\infty$ LOOP SHAPING CONTROL

4.1 LMI formulation

Without loss of generality, a static controller is considered. Let G_s be a strictly proper plant of order n having a stabilizable and detectable state-space realization:

$$G_s := \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \tag{7}$$

with $A \in R^{n\times n}$,$B \in R^{n\times n_u}$, $C \in R^{n_y\times n}$. G_s can be considered the shaped plant in the Glover-McFarlane $\mathcal{H}_\infty$ loop shaping design procedure.

A minimal normalized left coprime factorization of $G_s = \tilde{M}^{-1}\tilde{N}$ is given by, (Zhou *et al.*, 1995), (Skogestad and Postlethwaite, 1997)

$$\left[\tilde{N}, \tilde{M}\right] = \left[\begin{array}{c|cc} A+LC & B & L \\ \hline C & 0 & I \end{array}\right], \tag{8}$$

where $L = -YC^T$ and the matrix Y is the unique symmetric positive semi-definite solution to the algebraic Riccati equation

$$AY + YA^T - YC^TCY + BB^T = 0 \tag{9}$$

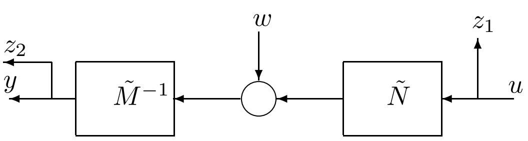

Fig. 1. Open loop Glover-McFarlane $\mathcal{H}_\infty$ loop shaping interconnection

Theorem 1. Let $L = -YC^T$ where $Y \geq 0$ is the stabilizing solution to (9). There exists a static loop shaping controller K such that

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + G_sK)^{-1}\tilde{M}^{-1} \right\|_\infty < \gamma \tag{10}$$

if $\gamma > 1$ and if and only if there exist two positive definite matrices R and S solving the inequalities

$$S(A+LC) + (A+LC)^TS - \gamma CC^T < 0 \tag{11}$$

$$W(R, L, \gamma) := \begin{pmatrix} AR + RA^T - \gamma BB^T & RC^T & -L \\ CR & -\gamma I_p & I_p \\ -L^T & I_p & -\gamma I_p \end{pmatrix} < 0 \tag{12}$$

$$\mathcal{K}(R, S) := \begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0 \tag{13}$$

and $rank(\mathcal{K}(R, S)) = n$.

Proof. See (Prempain and Postlethwaite, 2004).

It is well-known that the minimization of a rank constraint is hard to solve. Various heuristics have been developed to handle problems of this type. One simple heuristic, applicable when the matrix is symmetric positive semi-definite, is to minimize its trace in place of its rank. We have the following result:

Theorem 2. There exists a stabilizing static-output feedback controller if and only if the global minimum of the following optimization problem

$$\min trace(RS) \tag{14}$$

subject to (11), (12) and (13) is equal to n.

Proof. See e.g. (El Ghaoui *et al.*, 1997).

To solve such a problem, El Ghaoui *et al.* (1997) proposed a linear approximation of $trace(RS)$. More precisely, at point (R_0, S_0), a linear approximation of $trace(RS)$ is $trace(S_0R + R_0S) + constant$. This simple idea forms the basis of the CCL algorithm given in the next section.

4.2 Sufficient LMI conditions

A direct way to enforce the rank constraint (theorem 1) merely consists of replacing S by R^{-1}. Doing so, the inequality (11) becomes

$$R^{-1}(A+LC) + (A+LC)^TR^{-1} - \gamma CC^T < 0$$

or equivalently

$$(A+LC)R + R(A+LC)^T - \gamma RCC^TR < 0$$

Clearly, this last inequality is satisfied if

$$V(L,R) := (A+LC)R + R(A+LC)^T < 0$$

Hence, we have the following result:

Corollary 1. There exists a static loop shaping controller K such that

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I+G_sK)^{-1}\tilde{M}^{-1} \right\|_\infty < \gamma$$

if $\gamma > 1$ and if there exists a positive definite matrix R solving the inequalities

$$V(L,R) < 0 \quad (15)$$

$$W(L,R,\gamma) < 0 \quad (16)$$

Note that the conditions of this corollary are linear in R and γ. Hence, for a given matrix L, these conditions are amendable to LMI optimization.

5. ALGORITHMS FOR STATIC LOOP SHAPING SYNTHESIS

5.1 Algorithm 1: Cone Complementarity Algorithm (CCL)

To solve the optimization problem (14), a linear approximation of $trace(XS)$ takes the form

$$\phi_{lin}(R,S) = constant + trace(S_0R + R_0S)$$

From (17) the following iterative algorithm (El Ghaoui *et al.*, 1997) is:

(1) Find a feasible point S_0, R_0. If there are none, exit, set $k = 1$.
(2) Solve the LMI problem
minimize $O_k := trace(S_kR_{k-1} + R_kS_{k-1})/\epsilon + \gamma$ subject to (11), (12) and (13).
(3) Go to step 4 if $\|O_k - O_{k-1}\| < tol$ where tol is given positive number. Otherwise, set $k = k+1$ and go to step 2.
(4) Terminating phase. Try to reconstruct the controller K using the analytic formulae given in (Iwasaki and Skelton, 1994). The final feedback controller K_{ST} is then constructed using the static output feedback controller K with the shaping functions W_1 and W_2 such that $K_{ST} = W_1KW_2$. If the reconstruction fails, reduce ϵ and go back to step 2.

When γ is fixed, El Ghaoui *et al.* (1997) showed that the algorithm converges and finds, at every step k, a controller of order that is less or equal to $n - \max(n_u, n_y)$. This is quite a theoretical result. Sometimes, in practice, numerical difficulties prevent the reconstruction of a controller of that order.

It worth noting that the convergence property of the algorithm is not affected if, in step 2, γ is a decision variable of the LMI optimization problem.

5.2 Algorithm 2. Derived from the Sufficient LMI conditions of Corollary 1

As discussed earlier, there is no theoretical result enable us to determine a suitable matrix L to guarantee the feasibility of the LMI conditions (15) and (16). To overcome this problem, we suggest to generate an α-dependent set of matrices L (using Lemma 4) and then, for each L, to check the feasibility of the LMI conditions (15) and (16). As mentioned earlier, it is interesting to extend the range of α beyond 2, providing that $A + LC$ is stable. The algorithm is described as follows:

(1) Define $\Lambda = [\alpha_1, ..., \alpha_n]$ a row vector of n logarithmically equally spaced points representing various values of α. $n = 50$, $\alpha_1 = 0.01$ and $\alpha_n = 100$ suffice in practice.
(2) For $j = 1, 2, ...$, set $\alpha = \alpha_j$ in (6) and compute $L_j := -Y_jC^T$ where Y_j is the unique positive semi-definite solution to the Riccati equation (6). Then, solve the problem P_j : *minimize* γ_j *subject to* $V_j := V(L_j, R_j) < 0$, $W_j := W(L_j, R_j, \gamma_j) < 0$, $R_j > 0$. If P_j is feasible then reconstruct the controller K_j using, for instance, the analytic formulae given in (Iwasaki and Skelton, 1994).
(3) If for $j = 1, 2, ...$, the problem P_j is not feasible then stop (the method is not applicable).
(4) Terminating phase. Select the static gain K among the K_j leading to the best closed-loop γ attenuation (γ^*) (i.e. the attenuation obtained with the augmented plant including the normalized coprime factorization of the shaped plant). The final feedback controller K_{ST} is then constructed using the static output feedback controller K with the shaping functions W_1 and W_2 such that $K_{ST} = W_1KW_2$.

6. BENCHMARK EXAMPLES FROM $COMPL_EIB$ 1.0.

$COMPl_eib$ consists of examples collected from the engineering literature and contains models from real-wold application as well as pure academic models (Leibfritz and Lipinski, 2004). The current version of $COMPl_eib$ contains about 80 plants which are static output feedback stabilizable. The library is useful for testing linear and non-linear semi-definite optimization solvers such as *SeDuMi* (Sturm, 2001) and *PENBMI* (Kočvara and Stingl, 2003).

The static version of the Glover and McFarlane design procedure described in this paper has been tested on the Aircraft Models (AC), Helicopter Models (HE), Jet Engine models (JE), Reactor Models (REA) and Decentralized Interconnected Systems (DIS) which are available in $COMPl_eib$. This represents a collection of about 65 plants. For each plant, the static version of the Glover and McFarlane design procedure has been applied with W_1 and W_2 equal to the identity matrix as we were just interested in testing the ability of our algorithm to find a static feedback controller.

For every run, the following parameters $\epsilon = 10^{-7}$ and $tol = 0.1$ were used for the CCL algorithm. In addition, the maximum number of iterations for the CCL algorithm has been fixed to be 150. For algorithm 2, Λ has been selected as a vector of 80 logarithmically equally spaced points between 0.01 and 100. All the computations were performed using SeDuMi (Sturm, 2001). For both algorithms, the routine *klmi* from the *LMI MATLAB toolbox* (Gahinet *et al.*, 1995) is used for the controller reconstruction.

Figures 2 and 3 are typical of algorithms 1 and 2. The figures correspond to the aircraft model 'AC1' of $COMPl_eib$, a 5 states, unstable, minimum phase model with 3 inputs and 3 outputs.

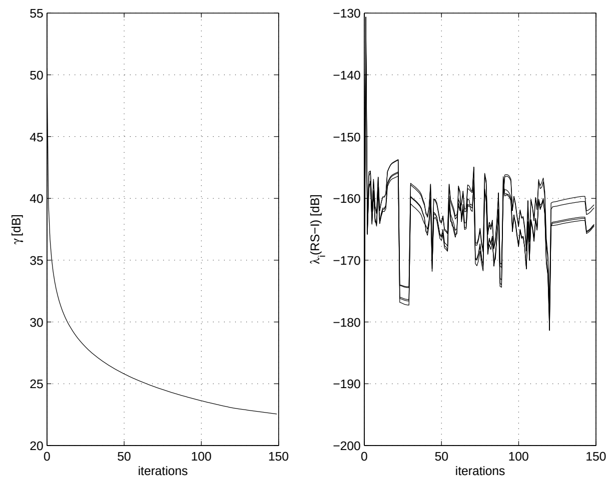

Fig. 2. CCL algorithm: left γ, right $\lambda_i(RS - I_n)$ for the plant AC1

Figure 2 shows the typical behaviour of the CCL algorithm. Clearly, the convergence of γ is slow and exceeds the maximum number of iterations ($N = 150$). The right plot of figure 2 shows the eigenvalues of $RS - I$ versus the iterations. This plot suggests that the plant is Static Output Feedback stabilizable. In this case, the routine *klmi* was able to reconstruct a static controller leading to a final closed-loop attenuation $\gamma = 7.79$.

Figure 3 shows the evolution of γ in terms of α obtained with the algorithm presented in section

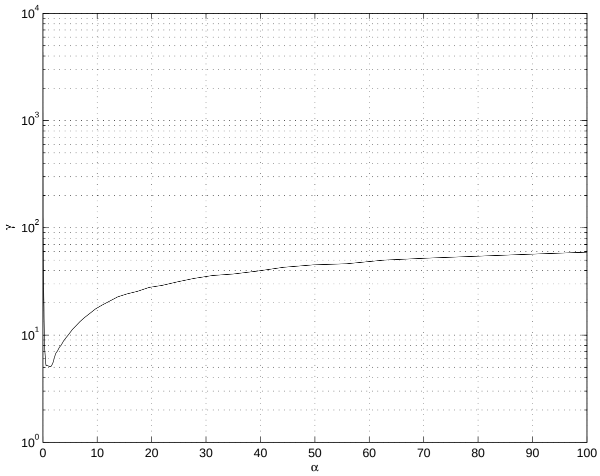

Fig. 3. Algorithm 2: γ versus α for the plant AC1

5.2 (algorithm 2). In this case, our algorithm works better than the CCL algorithm since it leads to a better closed-loop attenuation: $\gamma^* = 5.085$ for $\alpha = 1.3384$.

Table 6 compares the performance of our algorithm (algorithm 2) with the cone complementarity algorithm (algorithm 1). The plants AC_i are all static output feedback stabilizable. The first column contains the plant orders obtained after using the MATALB command *minreal* which computes minimum state space realizations. In fact, some plants in $COMPl_eib$ are not in minimal state-space form (e.g. AC13 and AC14). This may lead to numerical difficulties. The 3rd column contains the closed-loop gain attenuation achieved with the CCL algorithm. The algorithm is considered to fail if it is not possible to reconstruct a static output feedback controller with the matrices R and S returned at the end of the optimization process. The figures in parentheses indicate the order of the controller obtained in the case of algorithm failure. The table shows that our algorithm

Table 1. Algorithm performances for the aircraft models of $COMPl_eib$

AC_i	Plant order	γ (alg. 1, CCL)	α (alg. 2)	γ^* (alg. 2)
1	5	7.79	1.33	5.08
2	5	7.79	1.33	5.08
3	5	10.64	0.94	3.91
4	3	2.62	0.94	2.18
5	4	3421	1.06	2930
6	7	5.12	1.19	3.51
7	6	4.96	0.74	3.39
8	6	29.1 fails (5)	8.64	29.9
9	10	32.1 fails (9)	3.02	19.3
10	48	n/a	n/a	n/a
11	5	fails (2)	0.74	4.49
12	4	67.8 (3)	1.06	2.15
13	26	fails	0.8	45.65
14	26	fails	0.8	45.65
15	4	3.46	1.5	2.93
16	4	3.42	1.5	2.90
17	4	1.7	0.94	1.54
18	10	fails	100	8e3

performs significantly better than the CCL algorithm: it provides smaller closed-loop attenuation and, most importantly, it did not fail to find static controllers on this batch of plants.

Algorithm 2 has been tested on all the other plants of $COMPl_eib$ which are SOF and of order less than 40. This represents a collection of 67 plants. The cases where our algorithm was not able to find a solution are: TF3, NN1, NN5, NN6, NN7, NN10, NN12. Note that the SISO plants $NN3$ and $RE4$ have been excluded from the list since they are indeed not SOF (a simple root locus can be used to verify this). Hence, our algorithm is successful for 89% of the cases.

It is worth mentioning that, at each iteration, the two algorithms do not require the same amount of computer work. The cone complementary algorithm involves $n(n+1)+1$ decision variables while our algorithm involves only $n(n+1)/2+1$ decision variables, where n is the plant order. Hence, for a given problem and at each iteration, our algorithm requires approximately 8 times less computer work than the cone complementary algorithm. Also, a major advantage of our algorithm is that it virtually does not require any tuning parameters apart from α.

7. CONCLUSIONS

In this paper, a simple method for static output feedback control synthesis is presented. The method has been extended to the well-known McFarlane and Glover H_∞ design method. The effectiveness of this iterative synthesis method has been demonstrated on various numerical examples. Unlike previous algorithms, the algorithm does not require (A, B, C) to be a minimum phase plant (Syrmos *et al.*, 1997), (Garcia *et al.*, 2003). The numerical experiments carried out with the plants of $COMPl_eib$ indicate that the sufficient condition algorithm proposed in this paper is computationally more attractive than the cone complementary algorithm. It worth mentioning that the static version of the Glover-McFarlane design procedure, presented in this paper, has been used to design low order controllers for the BEll 205 helicopter. These controllers have been successfully implemented and flight tested (Prempain and Postlethwaite, 2004). Future work is necessary to clarify the relationship between the choice of the gain matrix L and the feasibility of the LMI conditions.

REFERENCES

Arzelier, D. and D. Peaucelle (2002). An iterative method for mixed H_2/H_∞ synthesis via static output-feedback. *Proceedings of the IEEE Conference on Decision and Control* **3**, 3464–3469.

Boyd, S., L. el Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory.* SIAM. Studies in Applied Mathematics.

El Ghaoui, L., F. Oustry and M. AitRami (1997). A cone complementary linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control* **42**, 1171–1176.

Gahinet, P., A. Nemirovski, A. J. Laub and M. Chilali (1995). LMI control toolbox. *The Math Works.*

Garcia, G., B. Pradin, S. Tarbouriech and F. Zeng (2003). Robust stanilization and guaranteed cost control for discrete-time linear systems by static output feedback. *Automatica* **39**, 1635–1641.

Iwasaki, T. and R. E. Skelton (1994). All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas. *Automatica* **30**(8), 1307–1317.

Iwasaki, T., R. E. Skelton and J. C. Geromel (1994). Linear quadratic suboptimal control with static output feedback. *System and Control Letters* **23**(6), 421–430.

Kočvara, M. and M. Stingl (2003). PENNON-a code for nonlinear and semidefinite programming. *Optimization Methods and Software* **8**(3), 317–333.

Leibfritz, F. (2001). An LMI-based algorithm for designing suboptimal static H_2/H_∞ output feedback controllers. *Journal on Control and Optimization* **39**(6), 1711–1735.

Leibfritz, F. and W. Lipinski (2004). Compleib 1.0 - user manual and quick reference. *Tech.-Report, University of Trier.*

McFarlane, D. C. and K. Glover (1992). A loop shaping design procedure using H_∞ synthesis. *IEEE Trans. Automat. Control* **37**(6), 759–769.

Prempain, E. and I. Postlethwaite (2004). Static H_∞ loop shaping control of a fly-by-wire helicopter. *CDC04.*

Skogestad, S. and I. Postlethwaite (1997). Multivariable feedback control. analysis and design. *Wiley.*

Sturm, J. F. (2001). SeDuMi-a Matlab* toolbox for optimization over symmetric cones. *Department of Econometrics, Tilburg University, The Netherlands.*

Syrmos, V. L., C. T. Abdallah, P. Dorato and K. Grigoriadis (1997). Static Output Feedback - A Survey. *Automatica* **33**(2), 125–137.

Zhou, K., J. Doyle and K. Glover (1995). *Robust and Optimal Control.* Prentice-Hall International, Inc.