POLYNOMIALLY PARAMETER-DEPENDENT LYAPUNOV FUNCTIONS FOR ROBUST $\mathcal{H}_\infty$ PERFORMANCE ANALYSIS

G. Chesi [*] A. Garulli [*] A. Tesi [**] A. Vicino [*]

[*] *Dipartimento di Ingegneria dell'Informazione*
Università di Siena
Email: {chesi,garulli,vicino}@dii.unisi.it

[**] *Dipartimento di Sistemi e Informatica*
Università di Firenze
Email: atesi@dsi.unifi.it

Abstract: The computation of robust $\mathcal{H}_\infty$ performance of linear systems subject to polytopic parametric uncertainty is known to be a difficult problem in robust control. In this paper, quadratic parameter-dependent Lyapunov functions, with polynomial dependence on the uncertain parameters, are exploited to provide upper bounds to the robust $\mathcal{H}_\infty$ performance. It is shown that such bounds can be computed via convex optimizations constrained by LMIs. Numerical examples show that the proposed technique is a powerful alternative to existing methods based on linearly parameter-dependent Lyapunov functions.

Keywords: Robust control, $\mathcal{H}_\infty$ performance, parametric uncertainty, Lyapunov function, LMI.

1. INTRODUCTION

Robust performance analysis of systems affected by real structured parametric uncertainty is a widely studied difficult problem in robust control. Lyapunov methods have been recognized since long time as a powerful tool for tackling such problem, and have gained a renewed interest in the last decade due to the development of efficient algorithms to solve LMI-based optimization problems.

A classic approach is based on common Quadratic Lyapunov Functions (QLFs). An upper bound to the robust $\mathcal{H}_\infty$ performance of a linear system with polytopic state space uncertainty, based on the existence of a common QLF, can be computed by solving an EigenValue Problem (EVP), which is a convex optimization with LMI constraints (Boyd *et al.*, 1994). On the other hand, it is well known that robustness performance evaluation based on quadratic stability can be quite conservative.

In order to reduce conservativeness, *parameter-dependent Lyapunov functions* have been considered. LMI-based tests for the computation of upper bounds to the robust $\mathcal{H}_\infty$ performance have been presented by several authors. The approaches proposed in the literature are characterized by the way the selected class of Lyapunov functions depends on the uncertain parameters. Lyapunov functions affine in the parameters have been used in (Gahinet *et al.*, 1996), and more recently in (de Oliveira *et al.*, 2004). Multi-affine dependence has been adopted in (Dettori and Scherer, 2000), in connection with parameter-dependent multipliers. Lyapunov functions in which the dependence on the parameters is expressed as a linear fractional transformation have been considered in (Peaucelle and Arzelier, 2001).

Recently, the class of Homogeneous Polynomially Parameter-Dependent Quadratic Lyapunov Functions (HPD-QLFs) has been introduced to study robust stability of polytopic systems (Chesi *et al.*, 2003*a*). The main feature of HPD-QLFs is that they are quadratic Lyapunov functions whose dependence on the uncertain parameters is expressed as a polynomial homogeneous form.
In this paper, HPD-QLFs are used to compute an upper bound to the robust $\mathcal{H}_\infty$ performance of a linear system affected by polytopic uncertainty. By exploiting a suitable square matricial representation of homogeneous forms, it is shown that the sought upper bound can be obtained via the solution of an EVP, for both continuous-time and discrete-time systems. Numerical examples are provided to demonstrate the potential of robustness analysis based on HPD-QLFs.

The paper is organized as follows. Section 2 formulates the robust $\mathcal{H}_\infty$ performance problem and provides preliminary material on matricial homogeneous forms. The proposed technique for the computation of upper bounds to the $\mathcal{H}_\infty$ performance is described in Section 3. Numerical examples are reported in Section 4, while Section 5 provides some concluding remarks.

2. PROBLEM FORMULATION AND PRELIMINARIES

Notation:

- 0_n: origin of $\mathbb{R}^n$;
- $\mathbb{R}^n_0$: $\mathbb{R}^n \setminus \{0_n\}$;
- $0_{m\times n}$: origin of $\mathbb{R}^{m\times n}$;
- I_n: identity matrix $n \times n$;
- A': transpose of matrix A;
- $A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix A;
- $A \otimes B$: Kronecker's product of matrices A and B;
- $\mathrm{sq}([p_1, p_2, \ldots, p_q]')$: $[p_1^2, p_2^2, \ldots, p_q^2]'$;
- $\mathrm{sqr}([p_1, p_2, \ldots, p_q]')$: $[\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_q}]'$, for $p_i \geq 0$, $i = 1, \ldots, q$.

Consider the continuous-time state space model

$$\left\{ \begin{array}{rcl} \dot{x}(t) &=& A(p)x(t) + B(p)w(t) \\ y(t) &=& C(p)x(t) + D(p)w(t) \\ p &\in& \mathcal{P} \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^r$ is the input, $y \in \mathbb{R}^g$ is the output and $p = [p_1, p_2, \ldots, p_q]' \in \mathbb{R}^q$ is the uncertain parameter vector, constrained in the unit simplex

$$\mathcal{P} = \left\{ p \in \mathbb{R}^q : \ \sum_{i=1}^{q} p_i = 1, \ p_i \geq 0 \right\}. \tag{2}$$

The matrices $A(p) \in \mathbb{R}^{n\times n}$, $B(p) \in \mathbb{R}^{n\times r}$, $C(p) \in \mathbb{R}^{g\times n}$ and $D(p) \in \mathbb{R}^{g\times r}$ are assumed linear functions of p according to

$$\begin{array}{rclrcl} A(p) &=& \displaystyle\sum_{i=1}^{q} p_i A_i, & B(p) &=& \displaystyle\sum_{i=1}^{q} p_i B_i \\ C(p) &=& \displaystyle\sum_{i=1}^{q} p_i C_i, & D(p) &=& \displaystyle\sum_{i=1}^{q} p_i D_i \end{array} \tag{3}$$

where $A_i, B_i, C_i, D_i, \ i = 1, \ldots, q$, are given real matrices of suitable dimensions.

For any $p \in \mathcal{P}$, the transfer matrix from w to y is given by

$$H(s,p) = C(p) \left(sI_n - A(p)\right)^{-1} B(p) + D(p). \tag{4}$$

For a fixed p, the $\mathcal{H}_\infty$ norm of $H(s,p)$ can be computed through the bounded real lemma (Boyd *et al.*, 1994), in the following way

$$\begin{array}{c} \|H(s,p)\|_\infty = \displaystyle\inf_{\gamma\in\mathbb{R}, P=P'\in\mathbb{R}^{n\times n}} \gamma \\ \text{s.t.} \\ \left\{ \begin{array}{rcl} P &>& 0 \\ E(P,p) + \dfrac{1}{\gamma^2} F(p) &<& 0 \end{array} \right. \end{array} \tag{5}$$

where

$$E(P,p) = \left[\begin{array}{cc} PA(p) + A(p)'P & PB(p) \\ B(p)'P & -I_r \end{array} \right] \tag{6}$$

$$F(p) = \left[\begin{array}{c} C(p)' \\ D(p)' \end{array} \right] [C(p), D(p)] \tag{7}$$

The problem dealt with in this paper is to compute the worst case $\mathcal{H}_\infty$ norm over $\mathcal{P}$, i.e.

$$\gamma^* = \sup_{p\in\mathcal{P}} \|H(s,p)\|_\infty. \tag{8}$$

The key step to address the above problem is the construction of a Homogeneous Polynomially Parameter-Dependent Quadratic Lyapunov Function (simply abbreviated as HPD-QLF)

$$v_m(x,p) = x' P_m(p) x \tag{9}$$

where $P_m(p) = P_m(p)' \in \mathbb{R}^{n\times n}$ is a *homogeneous matricial form* of degree m, i.e., a matrix whose entries are (real q-variate) homogeneous polynomial forms of degree m. Specifically, we define the $\mathcal{H}_\infty$-cost guaranteed by a HPD-QLF of degree m as

$$\gamma_m^* = \inf_{\gamma\in\mathbb{R},P_m(p)=P_m(p)'\in\mathbb{R}^{n\times n}} \gamma$$
$$\text{s.t.}$$
$$\begin{cases} P_m(p) > 0 \quad \forall p\in\mathcal{P} \\ E(P_m(p),p) + \dfrac{1}{\gamma^2}F(p) < 0 \quad \forall p\in\mathcal{P} \end{cases} \tag{10}$$

where $E(P_m(p),p)$ is as in (6) with $P = P_m(p)$. Clearly, $\gamma_m^* \geq \gamma^*$ for all $m \geq 0$.

2.1 Parameterization of homogeneous matricial forms

Let $C_{2m}(p) \in \mathbb{R}^{n\times n}$ be a homogeneous matricial form of degree $2m$ in $p \in \mathbb{R}^q$. Then, $C_{2m}(p)$ can be parameterized according to the Square Matricial Representation (SMR) of homogeneous forms introduced in (Chesi *et al.*, 2003*b*) as

$$C_{2m}(p) = \left(p^{\{m\}}\otimes I_n\right)' \bar{C} \left(p^{\{m\}}\otimes I_n\right) \tag{11}$$
$$\doteq \Delta\left(n, p^{\{m\}}, \bar{C}\right) \tag{12}$$

where $p^{\{m\}} \in \mathbb{R}^{\sigma(q,m)}$ is a vector containing all monomials of degree m in p, and $\bar{C}$ is a square matrix of dimension $n\sigma(q,m)$ being

$$\sigma(q,m) = \frac{(q+m-1)!}{(q-1)!m!}. \tag{13}$$

Such a matrix, denoted hereafter as an SMR matrix of $C_{2m}(p)$ and also known as Gram matrix (Choi *et al.*, 1995), is not unique. Indeed, all the matrices $\bar{C}$ describing $C_{2m}(p)$ are given by

$$\bar{C} + \bar{U}, \quad \bar{U} \in \mathcal{U}_{n,m} \tag{14}$$

where

$$\begin{aligned}\mathcal{U}_{n,m} = \Big\{&\bar{U} = \bar{U}' \in \mathbb{R}^{n\sigma(q,m)\times n\sigma(q,m)} : \\ &\Delta\left(n, p^{\{m\}}, \bar{U}\right) = 0_{n\times n} \quad \forall p\in\mathbb{R}^q\Big\}.\end{aligned} \tag{15}$$

In (Chesi *et al.*, 2003*a*) it has been shown that the set $\mathcal{U}_{n,m}$ is a linear space of dimension

$$\begin{aligned}u(n,m) = \frac{1}{2}n\Big\{&\sigma(q,m)[n\sigma(q,m)+1] \\ &-(n+1)\sigma(q,2m)\Big\}.\end{aligned} \tag{16}$$

Let $\bar{U}(\alpha)$, $\alpha \in \mathbb{R}^{u(n,m)}$, be a linear parameterization of $\mathcal{U}_{n,m}$. The Complete SMR (CSMR) of $C_{2m}(p)$ is hence given by

$$C_{2m}(p) = \Delta\left(n, p^{\{m\}}, \bar{C} + \bar{U}(\alpha)\right). \tag{17}$$

The computation of the CSMR of homogeneous matricial forms can be performed by applying algebraic procedures similar to those reported in (Chesi *et al.*, 2003*b*) for scalar homogeneous forms.

3. ROBUST $\mathcal{H}_\infty$ NORM COMPUTATION

In this section it is shown how upper bounds to γ_m^* can be computed through convex LMI optimizations. The first condition to be satisfied is the positive definiteness of the HPD-QLF matrix $P_m(p)$ within the set $\mathcal{P}$, i.e. the first constraint in (10). In this respect, a parameterization of positive definite matrices $P_m(p)$ is provided next.

Lemma 1. The condition

$$P_m(p) > 0 \quad \forall p \in \mathcal{P} \tag{18}$$

holds if and only if

$$P_m(\mathrm{sq}(p)) > 0 \quad \forall p \in \mathbb{R}_0^q. \tag{19}$$

Proof See (Chesi *et al.*, 2003*a*). □

Observe that $P_m(\mathrm{sq}(p))$ can be written as

$$P_m(\mathrm{sq}(p)) = \Delta\left(n, p^{\{m\}}, \bar{S}\right) \tag{20}$$

for some suitable matrix $\bar{S} \in \mathcal{S}_m$ where

$$\begin{aligned}\mathcal{S}_m = \Big\{&\bar{S} = \bar{S}' \in \mathbb{R}^{n\sigma(q,m)\times n\sigma(q,m)} : \\ &\Delta\left(n, p^{\{m\}}, \bar{S}\right) \text{ does not contain} \\ &\text{entries } p_1^{i_1}p_2^{i_2}\dots p_q^{i_q} \text{ with any odd } i_j\Big\}.\end{aligned} \tag{21}$$

Then, for such a matrix $\bar{S}$, $P_m(p)$ can be obtained according to

$$P_m(p) = \Delta\left(n, \mathrm{sqr}(p)^{\{m\}}, \bar{S}\right). \tag{22}$$

It is not difficult to show that the set $\mathcal{S}_m$ is a linear space of dimension

$$\begin{aligned}s(m) = \frac{1}{2}n\Big\{&\sigma(q,m)[n\sigma(q,m)+1] \\ &-(n+1)[\sigma(q,2m)-\sigma(q,m)]\Big\}.\end{aligned} \tag{23}$$

Let $\bar{S}(\beta)$, $\beta \in \mathbb{R}^{s(m)}$, be a linear parameterization of $\mathcal{S}_m$. Clearly, this induces a corresponding linear parameterization of the family of candidate HPD-QLF matrices in (22), namely

$$P_m(p,\beta) = \Delta\left(n, \mathrm{sqr}(p)^{\{m\}}, \bar{S}(\beta)\right) \tag{24}$$

which depends linearly on the parameterization β of $\mathcal{S}_m$. Following the above reasoning, one has the next result, which is the key step for the formulation of the sufficient condition to solving the robust stability problem.

Lemma 2. Let $\bar{S}(\beta)$ belong to $\mathcal{S}_m$ in (21). Then,

$$\bar{S}(\beta) > 0 \Rightarrow P_m(p,\beta) > 0 \quad \forall p \in \mathcal{P}. \tag{25}$$

Now, the aim is to provide LMI-based conditions which guarantee that also the second constraint in (10) is satisfied. To this purpose, let us introduce the homogeneous matricial form of degree $m+1$

$$\begin{aligned} Q_{m+1}(p,\beta,\zeta) \;=\; & E(P_m(p,\beta),p) + N_{m+1}(p) \\ & +\zeta F(p)\left(\sum_{i=1}^{q} p_i\right)^{m-1} \end{aligned} \quad (26)$$

where

$$N_m(p) = \mathrm{diag}\left\{0_{n\times n}, \left[1-\left(\sum_{i=1}^{q} p_i\right)^{m}\right] I_r\right\}. \quad (27)$$

Notice that, due to the constraint (2), one has

$$Q_{m+1}(p,\beta,\frac{1}{\gamma^2}) = E(P_m(p,\beta),p) + \frac{1}{\gamma^2}F(p)$$

$\forall p \in \mathcal{P}$. Let us write $Q_{m+1}(\mathrm{sq}(p),\beta,\zeta)$ as

$$Q_{m+1}(\mathrm{sq}(p),\beta,\zeta) = \Delta\left(l, p^{\{m+1\}}, \bar{R}(\beta,\zeta)\right) \quad (28)$$

where $l = n+r$, and $\bar{R}(\beta,\zeta) \in \mathbb{R}^{l\sigma(q,m+1)\times l\sigma(q,m+1)}$ is any SMR matrix of $Q_{m+1}(\mathrm{sq}(p),\beta,\zeta)$ (observe that $\bar{R}(\beta,\zeta)$ depends affinely on β and ζ).

Theorem 1. Let $\bar{U}(\alpha)$ be any linear parameterization of $\mathcal{U}_{l,m+1}$ and define

$$\hat{\gamma}^*_m = \frac{1}{\sqrt{\zeta^*}} \quad (29)$$

where ζ^* is the solution of the EVP

$$\begin{aligned} \zeta^* = & \sup_{\zeta\in\mathbb{R},\beta\in\mathbb{R}^{s(m)},\alpha\in\mathbb{R}^{u(l,m+1)}} \zeta \\ & \text{s.t.} \\ & \begin{cases} \bar{S}(\beta) > 0 \\ \bar{R}(\beta,\zeta)+\bar{U}(\alpha) < 0 \end{cases} \end{aligned} \quad (30)$$

Then, $\hat{\gamma}^*_m \geq \gamma^*_m$.

Proof. First, let $P_m(p,\beta)$ be defined as in (24). Then, from (30) and Lemma 2 one has that $P_m(p,\beta) > 0\, \forall p \in \mathcal{P}$, and hence the first condition in (10) holds. Second, let us observe that $\bar{R}(\beta,\zeta)+\bar{U}(\alpha)$ is the CSMR matrix of $Q_{m+1}(\mathrm{sq}(p),\beta,\zeta)$ in (28). Hence, (30) implies that $Q_{m+1}(\mathrm{sq}(p),\beta,\zeta) < 0\ \forall p \in \mathbb{R}^q_0$. From Lemma 1 it turns out that $Q_{m+1}(p,\beta,\zeta) < 0\ \forall p \in \mathcal{P}$. By observing that $Q_{m+1}(p,\beta,\zeta) = E(P_m(p,\beta),p) + \zeta F(p)$, $\forall p \in \mathcal{P}$, one has that also the second condition in (10) holds for $\frac{1}{\gamma^2} = \zeta$. Therefore, $\hat{\gamma}^*_m \geq \gamma^*_m$. □

Table 1 shows the number of free parameters in the EVP (30), amounting to $s(m)+u(l,m+1)+1$, for different values of n,m,q.

A question that naturally arises is whether there exists a relationship between the families of HPD-QLFs of degree m and $m+1$. The following result clarifies that, if the sufficient condition of Theorem 1 is satisfied for m, then it is satisfied also for $m+1$.

Table 1. Number of free parameters in the EVP (30)

$q=2, r=1$			$q=3, r=1$		
$n \setminus m$	1	2	$n \setminus m$	1	2
2	23	52	2	94	349
3	44	100	3	178	658
4	72	164	4	289	1066

Theorem 2. Let m be a nonnegative integer. Then, $\hat{\gamma}^*_m \geq \hat{\gamma}^*_{m+1}$.

Proof. Let $\tilde{S}(\tilde{\beta})$ and $\tilde{U}(\tilde{\alpha})$ be linear parameterizations of $\mathcal{S}_{m+1}$ and $\mathcal{U}_{l,m+2}$ respectively, and let $\tilde{R}(\tilde{\beta},\zeta) \in \mathbb{R}^{l\sigma(q,m+2)\times l\sigma(q,m+2)}$ be any SMR matrix of $Q_{m+2}(\mathrm{sq}(p),\tilde{\beta},\zeta)$ where $Q_{m+2}(p,\tilde{\beta},\zeta)$ follows from (26), with

$$P_{m+1}(p,\tilde{\beta}) = \Delta\left(n, \mathrm{sqr}(p)^{\{m+1\}}, \tilde{S}(\tilde{\beta})\right). \quad (31)$$

Suppose that there exist ζ, $\alpha \in \mathbb{R}^{u(l,m+1)}$ and $\beta \in \mathbb{R}^{s(m)}$ such that the LMIs in (30) are satisfied. Then, $\hat{\gamma}^*_m \geq \hat{\gamma}^*_{m+1}$ if there exist $\tilde{\alpha} \in \mathbb{R}^{u(l,m+2)}$ and $\tilde{\beta} \in \mathbb{R}^{s(m+1)}$ such that

$$\begin{cases} \tilde{S}(\tilde{\beta}) > 0 \\ \tilde{R}(\tilde{\beta},\zeta)+\tilde{U}(\tilde{\alpha}) < 0 \end{cases} \quad (32)$$

From the proof of Theorem 1 we have that, $\forall p \in \mathcal{P}$, $P_m(p,\beta) > 0$ and $Q_{m+1}(p,\beta) < 0$. In order to select $\tilde{\beta}$, let us define $P_{m+1}(p) = P_m(p,\beta)\sum_{i=1}^{q} p_i$ and let us show that $P_{m+1}(\mathrm{sq}(p))$ admits a positive definite SMR matrix. Let K_{m+1} be the matrix satisfying

$$p \otimes p^{\{m\}} = K_{m+1}p^{\{m+1\}}\ \forall p \in \mathbb{R}^q. \quad (33)$$

Then,

$$\begin{aligned} & P_{m+1}(\mathrm{sq}(p)) = \left(\sum_{i=1}^{q} p_i^2\right)\Delta\left(n, p^{\{m\}}, \bar{S}(\beta)\right) \\ & = p'p\left(p^{\{m\}}\otimes I_n\right)'\bar{S}(\beta)\left(p^{\{m\}}\otimes I_n\right) \\ & = (\ldots)'\left(I_q\otimes\bar{S}(\beta)\right)\left(p\otimes p^{\{m\}}\otimes I_n\right) \\ & = (\ldots)'\left(I_q\otimes\bar{S}(\beta)\right)\left(K_{m+1}p^{\{m+1\}}\otimes I_n\right) \\ & = (\ldots)'(\ldots)'\left(I_q\otimes\bar{S}(\beta)\right)(K_{m+1}\otimes I_n) \\ & \quad \left(p^{\{m+1\}}\otimes I_n\right) \\ & = \Delta\left(n, p^{\{m+1\}}, S^*\right) \end{aligned} \quad (34)$$

where the notation $(\ldots)'AB$ stands for $B'AB$, and

$$S^* = (K_{m+1}\otimes I_n)'\left(I_q\otimes\bar{S}(\beta)\right)(K_{m+1}\otimes I_n). \quad (35)$$

Since $\bar{S} > 0$ and K_{m+1} is a matrix with full column rank, it follows that $S^* > 0$.

Now, let us select $\tilde{\beta}$ such that $\tilde{S}(\tilde{\beta}) = S^*$ (such a $\tilde{\beta}$ exists since $S^* \in \mathcal{S}_{m+1}$). We hence have that the first condition in (32) is satisfied and $P_{m+1}(p,\tilde{\beta}) = P_m(p,\beta)\sum_{i=1}^q p_i$.

Let us observe that $Q_{m+2}(p,\tilde{\beta},\zeta) = Q_{m+1}(p,\beta,\zeta)\sum_{i=1}^q p_i$. Following the same development as in (34), one gets

$$Q_{m+2}(\mathrm{sq}(p),\tilde{\beta},\zeta) = \Delta\left(l, p^{\{m+2\}}, \tilde{R}(\tilde{\beta},\zeta)\right) \quad (36)$$

where

$$\begin{aligned}\tilde{R}(\tilde{\beta},\zeta) &= (K_{m+2}\otimes I_l)' \\ &\left(I_q \otimes \left(\bar{R}(\beta,\zeta)+\bar{U}(\alpha)\right)\right)(K_{m+2}\otimes I_l)\,.\end{aligned} \quad (37)$$

Since $\bar{R}(\beta,\zeta)+\bar{U}(\alpha) < 0$ it follows that $\tilde{R}(\tilde{\beta},\zeta) < 0$. Therefore, $Q_{m+2}(\mathrm{sq}(p),\tilde{\beta},\zeta)$ admits the negative definite SMR matrix $\tilde{R}(\tilde{\beta},\zeta)+\tilde{U}(\tilde{\alpha})$ with $\tilde{\alpha} = 0_{u(l,m+2)}$, and also the second condition in (32) holds. □

The proposed technique can be applied also to discrete-time systems

$$\begin{cases} x(t+1) = A(p)x(t) + B(p)w(t) \\ y(t) = C(p)x(t) + D(p)w(t) \\ p \in \mathcal{P} \end{cases}$$

In particular, Theorem 1 provides the sought upper bound $\hat{\gamma}_m^*$ of γ^*, if one makes the following changes:

- $\bar{U}(\alpha)$ is any linear parameterization of $\mathcal{U}_{l,m+2}$ with $\alpha \in \mathbb{R}^{u(l,m+2)}$;
- $\bar{R}(\beta,\zeta) \in \mathbb{R}^{l\sigma(q,m+2)\times l\sigma(q,m+2)}$ is any SMR matrix of $Q_{m+2}(\mathrm{sq}(p),\beta,\zeta)$ where

$$\begin{aligned}Q_{m+2}(p,\beta,\zeta) &= E(P_m(p,\beta),p) + N_{m+2}(p) \\ &+\zeta\left(\sum_{i=1}^q p_i\right)^m F(p)\end{aligned}$$

and $E(P,p)$ is now defined as

$$E(P,p) = \begin{bmatrix} A(p)'PA(p)-P & A(p)'PB(p) \\ B(p)'PA(p) & B(p)'PB(p)-I_r \end{bmatrix}.$$

A result analogous to Theorem 2 can also be derived.

4. EXAMPLES

4.1 Example 1

Consider the parametric system described by (1)–(3) with $q = 2$ and

$$A_1 = \hat{A}_0 + \kappa\hat{A}_1,\ A_2 = \hat{A}_0 - \kappa\hat{A}_1$$

where $\kappa \in \mathbb{R}$ and

$$\hat{A}_0 = \begin{bmatrix} -2 & 1 & -1 \\ 2.5 & -3 & 0.5 \\ -1 & 1 & -3.5 \end{bmatrix},\ \hat{A}_1 = \begin{bmatrix} -0.7 & -0.5 & -2 \\ -0.8 & 0 & 0 \\ 1.5 & 2 & 2.4 \end{bmatrix}.$$

The matrices B_i, C_i and D_i are, for all $i = 1,2$,

$$B_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ C_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}',\ D_i = 0.$$

Upper bounds to the robust $\mathcal{H}_\infty$ performance γ^*, for some values of κ (the semi-length of the segment of matrices $A(p)$), have been computed via different methods. Table 2 shows the upper bounds provided by the approach proposed in (Gahinet *et al.*, 1996) (indicated by GAC), the approach proposed in (de Oliveira *et al.*, 2004) (denoted by OOLMP), and our technique with $m = 1$ (linear dependence) and $m = 2$ (quadratic dependence). Notice that the maximum value of κ for which $A(p)$ is Hurwitz is $\kappa = 3.552$ (Chesi *et al.*, 2003*a*).

Table 2. Results for Example 1

κ	GAC	OOLMP	$\hat{\gamma}_1^*$	$\hat{\gamma}_2^*$
1.0	1.515	1.515	1.515	1.515
1.2	1.631	1.567	1.567	1.567
1.4	2.247	1.567	1.567	1.567
1.6	4.261	1.567	1.567	1.567
1.8	342.3	1.567	1.567	1.567
2.0	∞	1.616	1.616	1.567
2.2	∞	2.261	2.261	2.261
2.4	∞	3.500	3.500	3.500
2.6	∞	5.255	5.255	5.232
2.8	∞	7.991	7.991	6.201
3.0	∞	15.50	15.50	6.201
3.2	∞	399.6	399.6	6.201
3.4	∞	∞	∞	6.201
3.5	∞	∞	∞	6.201

4.2 Example 2

Consider system (1)–(3) with $q = 3$ and

$$A_1 = \begin{bmatrix} -0.42 & -1.68 & -2.24 & 2.92 \\ -0.74 & -1.74 & -4.58 & 1.44 \\ -2.92 & 3.84 & -6.98 & 2 \\ -4.92 & -2.68 & -8.66 & -0.78 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.78 & 5.52 & 1.36 & 5.8 \\ -5.42 & -4.62 & -0.26 & -1.08 \\ 2.48 & 6 & -7.7 & -7.72 \\ -1.32 & 3.8 & 2.14 & 2.1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -4.2 & -3.12 & -2.96 & 1.84 \\ 4.48 & -1.02 & -2.78 & -7.38 \\ 1.22 & -0.12 & -2.66 & -0.34 \\ 2.1 & 4.52 & -1.28 & -1.5 \end{bmatrix}$$

$$B_i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ C_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}',\ D_i = 0,\ i = 1,2,3.$$

It turns out that the upper bound to γ^* provided by the OOLMP method is 8.952. On the other hand, our technique provides $\hat{\gamma}_1^* = 4.222$ for $m = 1$, and $\hat{\gamma}_2^* = 1.215$ for $m = 2$.

5. CONCLUSIONS

Polynomially parameter-dependent quadratic Lyapunov functions have been exploited to obtain upper bounds to the robust $\mathcal{H}_\infty$ performance of linear systems subject to polytopic parametric uncertainty, through convex optimizations constrained by LMIs. Numerical examples show that the proposed technique compares favorably to existing methods based on linearly parameter-dependent Lyapunov functions.

Future research will consider the possibility of extending the proposed technique to the synthesis of controllers minimizing the robust $\mathcal{H}_\infty$ performance. Another research line under investigation concerns the use of HPD-QLFs in connection with robustness analysis techniques based on parameter-dependent multipliers.

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