CONTROL OF OSCILLATIONS OF MAGNETO-SENSITIVE ELASTIC SPHERE

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Abstract. Magneto-sensitive (MS) elastomers are a class of smart materials whose mechanical properties change instantly by the application of a magnetic field. The control of spherically symmetric oscillations of initially inflated MS elastic sphere by a homogeneous magnetic field is considered. The presented results can be used for generation of spherically symmetric acoustic waves.

Keywords: magneto-sensitive elastomers, generation and damping of oscillations, Lyapunov functions.

1. INTRODUCTION

Magneto-sensitive elastomers are materials that respond to an applied magnetic field with an instantaneous change in the mechanical behavior. These materials typically consist of micron-sized ferrous particles (2–3 μm) dispersed in an elastomer. An improved understanding of MS elastomers is demanded by the prospect to provide simple, reliable and rapid-response interfaces between controls laws and mechanical systems. It is now well recognized that MS elastomers have the potential to improve the design of electromechanical devices and their operation. For example, an elastomer with field dependent properties may be used as a device with a variable stiffness. Therefore, this wide range of potential applications and associated economic benefits are the reason for the intense research on these materials in recent years (Kordonsky, 1993, Carlson and Jolly, 2000).

In this paper we consider the spherically symmetric deformation of an isotropic elastic MS sphere in a homogeneous magnetic field. We present the appropriate differential equation of dynamics for sphere inflated by the inside pressure, where the magnetic field is an external parameter changing the stiffness of the sphere. The appropriate problem of the parametric control and damping of oscillations for the dynamical system is solved by the Lyapunov method (Fradkov, 1999). The presented theoretical results are based on the theorem about asymptotic stability in reference to the part of variables (Rumyantsev and Oziraner, 1987) and the Barbashin-Krasovski theorem.

2. BASIC EQUATIONS

We consider an isotropic elastic sphere of the inside radius a_0 and outside radius b_0 in the reference configuration. The inside pressure $P > 0$ spherically symmetric inflates the sphere to sizes (a, b). From the incompressibility of MS elastomers (Brigadnov and Dorfmann, 2003) it follows that the volume of the sphere is constant, i.e.

$$b^3 - a^3 = b_0^3 - a_0^3 = D_0 \ . \tag{1}$$

The spherically symmetric mapping is described by the relation

$$\mathbf{x}(t, r) = \mathbf{X}(r) + U(t, r)\mathbf{e}_r \ ,$$

where $\mathbf{x}$ and $\mathbf{X}$ are the actual and reference radius-vectors as functions of the time t and radius r only, U is the field of the full radial displacements of the sphere and $\mathbf{e}_r$ is the radial basic vector of the Euler spherical coordinates (r, θ, φ). We assume that $U(t, r) = a(t) - a_0 + u(t, r)$, where u is the relative field of radial displacments such that $u(t, a_0) \equiv 0$. From the incompressibility condition (1) the current outside radius is easily found as $b = \left(a^3 + D_0\right)^{1/3}$.

From the local condition of incompressibility $\mathrm{div}\,(U\mathbf{e}_r) = 0$ with the condition $u(t, a_0) \equiv 0$ we obtain

$$U(t, r) = (a(t) - a_0)\left(\frac{a_0}{r}\right)^2 \ . \tag{2}$$

In the considered problem the full displacements are finite but deformations are small, therefore, for description of the actual configuration we can use the linear elasticity theory. As a result, for the Cauchy strain tensor having in the spherical coordinates the form

$$\boldsymbol{\varepsilon}(t, r) = (a(t) - a_0)\,\frac{a_0^2}{r^3}\left(-2\mathbf{e}_r\mathbf{e}_r + \mathbf{e}_\theta\mathbf{e}_\theta + \mathbf{e}_\varphi\mathbf{e}_\varphi\right) \ , \tag{3}$$

and the Cauchy stress tensor $\boldsymbol{\sigma}$, the Hooke law for incompressible materials is true

$$\boldsymbol{\sigma} = \frac{2}{3}E\boldsymbol{\varepsilon} - p\mathbf{I} \ , \tag{4}$$

where E is the Young modulus, p is the hydrostatic pressure and $\mathbf{I}$ is the identity second order tensor.

Neglecting the weight of the sphere, in a homogeneous magnetic field the equilibrium equation has the classical form

$$\rho_0 \frac{\partial^2 \mathbf{x}}{\partial t^2} = \mathrm{div}\,\boldsymbol{\sigma} \ , \tag{5}$$

where ρ_0 is the density of a MS elastomer.

Substituting relations (2)–(4) into the equation (5) we obtain the condition $\sigma_{\theta\theta} = \sigma_{\varphi\varphi}$ and the differential equation for functions $a(t)$ and $p(t, r)$

$$\rho_0 \ddot{a}\left(\frac{a_0}{r}\right)^2 + \frac{\partial p}{\partial r} = 0 \ , \tag{6}$$

which is true for every radius $r \in (a_0, b_0)$ and time $t > 0$.

On the inside surface of the sphere we have the surplus pressure $P > 0$ and the outside surface is free, therefore, the following boundary conditions are true

$$\sigma_{rr}(t, a_0) = -P \ , \qquad \sigma_{rr}(t, b_0) = 0 \ . \tag{7}$$

After integration of the equation (6) in r and taking into account the boundary conditions (7) we obtain the differential equation for the current inside radius of the sphere

$$\ddot{a} + c\,(a - a_0) = f \ , \tag{8}$$

where c and f are the reduced stiffness and external force, respectively, having the following form:

$$c = \frac{4}{3}\left(\frac{E}{\rho_0}\right)\frac{D_0}{a_0^2 b_0^2\,(b_0 - a_0)} > 0 \ ,$$

$$f = \left(\frac{P}{\rho_0}\right)\frac{b_0}{a_0\,(b_0 - a_0)} > 0 \ .$$

For a constant inside pressue $P_0 > 0$ the inflated sphere has the inside and outside radiuses

$$a_* = a_0\left[1 + \frac{3}{4}\left(\frac{P_0}{E}\right)\frac{b_0^3}{D_0}\right] > a_0 \ , \tag{9}$$

$$b_* = \left(a_*^3 + D_0\right)^{1/3} > b_0 \ .$$

For example, the sphere with initial radiuses $a_0 = 0.1$ and $b_0 = 0.11$ (m) prepared from the elastomer with the Young modulus $E \approx 1.8\ MPa$ inflates to sizes $a_* \approx 0.15$ and $b_* \approx 0.155$ (m).

It was proven (Brigadnov and Dorfmann, 2003) that in a magnetic field the Young modulus of MS elastomers only increases according to the following relation

$$E = E_0 \left(1 + \eta B^2\right) \ , \tag{10}$$

where E_0 is the Young modulus corresponding to zero magnetic flux density B, $\eta > 0$ is the MS coefficient. From experiments it is well known that $\eta = \eta(\lambda)$, where $\lambda \in [0, 0.5]$ is the MS particle volume fractions in an elastomer (Kordonsky, 1993, Carlson and Jolly, 2000). For commercially available MS elastomers $\eta(0) = 0$ and $\eta(0.5) \approx 0.5$ (Brigadnov and Dorfmann, 2003).

For example, the examined above initially inflated sphere from a MS elastomer with $\lambda \approx 0.3$ in a homogeneous magnetic field $B \approx 0.8\ Tl$ shrinks to sizes $a_1 \approx 0.14$ and $b_1 \approx 0.146$ (m).

In the following section we will consider oscillations of a MS elastic sphere near the static equilibrium state (9). After the standard replacements $y = a - a_*$ and $\tau = \sqrt{c_0}\, t$, where c_0 is the the reduced stiffness corresponding to zero magnetic field, the equation (8) is transformed into the following simple form

$$\ddot{y} + \left(1 + \eta B^2\right) y = 0 \ . \tag{11}$$

3. STATIONARY OSCILLATIONS

Consider the problem of generation of harmonic oscillations with the desired frequency ω_d and the desired amplitude A_d in system (11). Define the desired coefficient of the reduced stiffness by the formula $c_d = \omega_d^2$. The multiplicator with y is represented as

$$1 + \eta B^2 = c_d + u_g \tag{12}$$

where u_g is new control input. Let B_{max} be the magnetic saturation of the MS elastomer, typically B_{max} is about 1 Tl (Brigadnov and Dorfmann, 2003). Then

$$1 < c_d < 1 + \eta B_{max}^2 \tag{13}$$

or

$$c_d = 1 \tag{14}$$

The last conditions impose restrictions on the desired frequency. Introduce the function of energy $E = \frac{1}{2}(\dot{y}^2 + c_d y^2)$ and the desired energy $E_d = \frac{1}{2} c_d A_d^2$. It is easy to check that $\dot{E} = -y\dot{y}u$. Consider the Lyapunov function $V = \frac{1}{2}(E - E_d)^2$ and synthesize the control input from the condition of decreasing the Lyapunov function on the trajectories of the closed loop system (Fradkov, 1999). Differentiating we get that $\dot{V} = -y\dot{y}(E - E_d)u$. Under the condition (13) choose as control input

$$u_g = F(y\dot{y}(E - E_d)) \tag{15}$$

where $F(t)$ is continuous, strictly increasing function, $F(0) = 0$. It may be bounded. It is reasonable to choose F so, that

$$1 - c_d < F < 1 - c_d + \eta B_{max}^2 \ .$$

Under the condition (14) choose as control

$$u_g = F_+(y\dot{y}(E - E_d)) \tag{16}$$

where $F_+(t)$ is continuous function, $F_+(t) = 0$ for $t \leq 0$, and it strictly increases for positive t. It may be bounded.

For the deviation of energy the following equation holds

$$(E - E_d)^{\cdot} = -\dot{y} y F(y\dot{y}(E - E_d)) \tag{17}$$

Proposition 1. The closed loop system (11), (12), (13) (or (14), (15), (16)) has the desired harmonic oscillation as the solution. The extended system (11), (12), (13) (or (14)), (15), (17), (16), (17) is asymptotically stable in reference to the variable $E - E_d$. The closed loop system has also the equilibrium $y = 0, \dot{y} = 0$, which is unstable.

Proof. The asymptotic stability follows from the application of theorem on asymptotic stability in reference to the part of variables (Rumyantsev and Oziraner, 1987) to the function V. Because $\dot{E} \geq 0$ in small environ of the point $(0, 0)$ then this equilibrium is unstable in accordance to the Chetaev theorem on instability.

Consider also the problem of oscillation damping. Construct the control input from the condition of decreasing E. Represent the multiplicator with y as

$$1 + \eta B^2 = c + u_0 \tag{18}$$

where $c = 1$ or $c > 1$. In first case define as control input

$$u_0 = F_+(y\dot{y}) \tag{19}$$

and in the second case

$$u_0 = F(y\dot{y}) \tag{20}$$

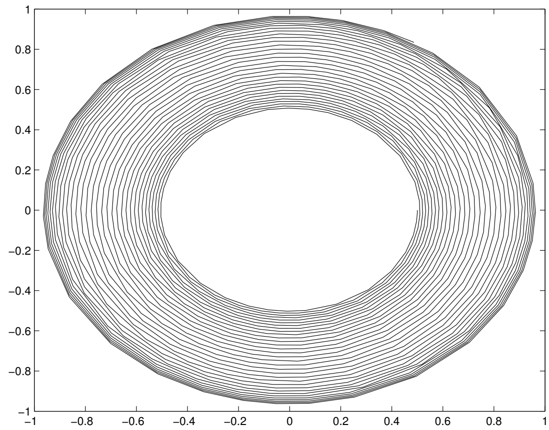
where the functions F and F_+ were mentioned above.

Proposition 2. The closed loop system (11), (18), (19) (or (20)) is globally asymptotically stable in reference to the variables $y, \dot{y}$.

Proof. It follows from the application of the Barbashin-Krasovski theorem to the function E.

Modelling

The closed loop system described in Proposition 1 was inegrated numerically with the following parameters: $c_d = A_d = 1, \quad F(p) = \arctan(p)/\pi$ on time interval $[0, 100]$ with the different initial data $(y_0, \dot{y}_0)$: $(0.5, 0)$, $(0.8, -1)$, $(1, 1)$, $(2, 1)$, $(2, 0)$. In first figure there is the phase portrait for initial data (0.5, 0) and time interval [0,200]. Also it was integrated the system described in Proposition 2 on time interval $[0, 200]$ with the parameter $c = 1.5$ and the different initial data $(1, 0)$ and $(1, 1)$. In all cases there were the desired convergencies.



4. NONSTATIONARY OSCILLATIONS

Here solve the problem of inverted dynamics. Rewrite the equation (11) as

$$\ddot{y} + Uy = 0 \ , \tag{21}$$

where U is a new control. Try to find the desired oscillations as

$$y_d(t) = A(t) \sin(\varphi_p(t)) \ ,$$

where $A(t)$, $\varphi_p(t)$ are sufficiently smooth functions. Differentiate and substitute in (21), then the coefficients with the harmonics consider equals zero. It may be get the following differential equations

$$2\dot{A}\dot{\varphi}_p + A\ddot{\varphi}_p = 0 \ ,$$

$$\ddot{A} - A\dot{\varphi}_p^2 + UA = 0 \ .$$

Under condition of representing the desired frequency as exponential function of time

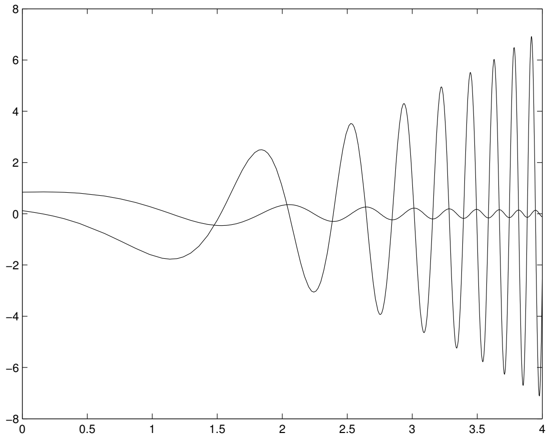
$$\varphi_p = \kappa \exp(kt) + \mu, \quad \kappa, k \neq 0 \ ,$$

these differential equations have the following solutions

$$A = c \exp(-\frac{k}{2}t), \quad c = const \ ,$$

$$U(t) = k^2\kappa^2 t \exp(2kt) - \frac{k^2}{4}.$$

In second figure there are graphics of the functions $y(t)$, $\dot{y}(t)$ for the parameters $\kappa = 1$, k=1, $\mu = 0$, $y(0) = \sin(1)$, $\dot{y}(0) = \cos(1) - \sin(1)/2$ under the abovementioned control.



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