POSSIBILISTIC ROBUST CONTROL FOR FUZZY PLANTS: CONTROLLING PERFORMANCE DEGRADATION.

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Abstract: One of the main problems of classical robust control approaches is that if a set of specifications are not fulfilled for a family of plants being considered, they must be relaxed. However, not all the plants in the family will be equally possible in practice. In this paper, a fuzzy set in a linear plant space will be used to associate a possibility to each member in the family. Then, a controller will be designed so that a fuzzy set of specifications is achieved. The core of this set will define the *hard* specifications to be achieved by the most possible plants, while the support will represent the minimum specifications required to the whole family of plants. Intermediate cuts define how performance is allowed to degrade. Several design methodologies are presented.

Keywords: fuzzy systems, robust control, parametric uncertainty, fuzzy specifications.

1. INTRODUCTION.

Fuzzy control has been a successful technique in the application field, mainly due to the parallelism with human reasoning schemes. Fuzzy or neuro-fuzzy systems usually act as universal function approximators (UFA) used in implementation of nonlinear controllers (White and Sofge, 1992; Wang, 1994; Brown and Harris, 1994).

Robust control (RC) (Zhou *et al.*, 1995), based on classical foundations, has also reached maturity as a technique for linear systems capable of dealing with uncertainty in multivariable models under quite general assumptions. Quantitative Feedback Theory (Horowitz, 1993) deals with structured frequency-dependent uncertainty. Parametric Robust Control (Bhattacharyya *et al.*, 1995) considers interval parameters.

One problem RC designs have is conservativeness. Apart from the conservativeness of some of the solutions provided by standard techniques (the non-conservative solutions in a general μ-synthesis case may be NP-hard with no *a priori* bound on the controller complexity), at the core of the techniques lies the issue of trying to ensure satisfactory performance for a whole family of systems. Sometimes a solution cannot be found because the set of possible plants is too large. In that case, specifications have to be degraded or the family of systems must be reduced leaving out infrequent cases, without performance guarantees for them.

This work stems from previous papers by the authors (Bondia *et al.*, 2004*a*; Bondia and Picó, 2003; Bondia *et al.*, 2004*b*), setting up the main ideas in an unified approach and also putting forward some tools based on interval arithmetic. The focus of this research is to pose the problem of differentiating worst-case performance from *most-cases* performance and trying to solve simultaneously both problems, for the particular case of SISO linear uncertain-parameter systems. Fuzzy sets will be used as a convenient tool to express both the parameter space and the specifications. On one hand, fuzzy plant parameters will be assumed to be the result of identification (the core of the fuzzy set is expected to contain a good approximation to the true plant in the majority of cases and the support will be assumed to contain all possible values of them); on the other hand, on the specification space, the core of the set will denote the requested *most-cases* behaviour, the support of the set will denote the *worst-case* limit of acceptability (for example very low stability margins).

The problem can be cast as requesting inclusion of all α-cuts $0 \leq \alpha \leq 1$ of the closed-loop plant into the corresponding cut of the desired specifications. A simpler case could be considering only the core and support of the sets, (similar to a *rough set* approach).

2. FUZZY OPEN-LOOP MODELS

In this paper, fuzzy models are used to represent *uncertainty*, departing from using them as UFA with no associated uncertainty.

Fuzzy open-loop models can be the result of various ID algorithms. For example, in black-box ID, there are prior assumptions that constitute the tuning knobs, such as noise bounds, variances, etc. not precisely known which may be represented as fuzzy sets. So, a fuzzy set model will result from the application of ID algorithms.

A related situation is the design of controllers for a family of plants with some spread in parameter values. There are two extreme situations: one of them is identifying each of the systems and tuning individual optimised controllers and, on the other hand, relaxing the specifications so that a unique controller can cope with all plants. If there is a wide parameter variation, the level of performance quality loss in *all* units can be unacceptable so that a compromise needs to be reached.

So, although the ideal case would be to find a fixed controller able to perform satisfactorily in all plants in the above family, if it cannot be found (maybe due to conservativeness of the chosen design techniques), a *fuzzy* set of plant parameters can be described so that when mapped to a *fuzzy set of specifications* it will yield good performance in the majority of the possible plants (those belonging to the user-defined core, $\mu = 1$), but with a reduced fraction of them, significantly off-specifications, it will yield a suitably defined second-class performance (support, $\mu > 0$), unless individual retuning is carried out.

3. PROBLEM STATEMENT

Once a plant model with fuzzy uncertainty, $\tilde{P}$, is available, a control design must be pursued. The designed controller will be a deterministic dynamical system K, obviously with no associated uncertainty.

For each plant $P \in \mathbb{P}$, the action of the controller K in closed-loop produces a particular behaviour evaluated in terms of a performance measure $J(P, K)$. Letting K being constant, the performance measure defines an evaluation map: $J_K : \mathbb{P} \rightarrow \mathbb{S}$ where $\mathbb{S}$ is a specification space.

A target specification will be defined as a fuzzy subset of $\mathbb{S}$, denoted as $\tilde{S}$. For instance, the evaluation map might map a process to a real number, this number being any optimal-control related cost index. A (crisp) target specification would be a desired cost in an interval $[0, \gamma]$. This is, indeed, a common setup (Zhou *et al.*, 1995). Other specification spaces are analysed in section 4.

Given a fuzzy plant $\tilde{P}$, the fuzzy image set $\tilde{J} = J_K(\tilde{P})$ will be defined as:

$$\pi_{\tilde{J}}(s) = \max_{P \in \mathbb{P}, J_K(P)=s} \pi_{\tilde{P}}(P) \qquad (1)$$

In this context, the control design problem can be cast as an inclusion problem.

Design problem.- Given a plant $\tilde{P}$ and a specification set $\tilde{S}$, design a fixed controller K such that

$$J_K(\tilde{P}) \subset \tilde{S}. \qquad (2)$$

The inclusion must be understood in terms of α-cuts. Denoting by $\tilde{P}_\alpha$ and $\tilde{S}_\alpha$ the corresponding α-cuts of the fuzzy sets,

$$J_K(\tilde{P}_\alpha) \subset \tilde{S}_\alpha$$

In this way, a *fuzzy* set of plants must be mapped by the controller to a subset of a *fuzzy set of target specifications*, indicating that it will yield good performance in the majority of the prototypes (core) but with a reduced fraction of them, unlikely or off-specification situations, it will yield a user-defined degradation of performance.

4. DEFINING FUZZY SPECIFICATIONS.

The design methodology employed to solve (2) will depend on the selected specification space. Currently, two specification spaces have been explored: reference models and characteristic polynomials.

4.1 Fuzzy reference models.

Let $\mathbb{S}$ be the set of transfer functions in a given class of reference models. The fuzzy set $\tilde{S}$ defines a fuzzy family of reference models. The cut S_α will define the family of *target* reference models for the corresponding cut of the plant, so that for any plant in P_α, the controller must map the closed-loop to an element in S_α.

For instance, in many cases the desired behaviour corresponds to that of a first or second order

$$M(s, \tilde{\mathbf{r}}) = \frac{\tilde{k}}{1 + \tilde{\tau} s}, \tag{3}$$

$$M(s, \tilde{\mathbf{r}}) = \frac{\tilde{k}}{1 + 2\tilde{\xi}\frac{s}{\tilde{\omega}_n} + \left(\frac{s}{\tilde{\omega}_n}\right)^2} \tag{4}$$

Modal interval arithmetic (Gardeñes *et al.*, 1985; SIGLA/X and Sáinz, 2001) can be applied to obtain the fuzzy parameters of (3)-(4) from classical specifications such steady-state error, overshoot and settling time, as shown in the following example.

Example 4.1. Consider that a second order-like response is sought with steady-state error $\tilde{e}_p[\%] = tri(-1, 0, 1)$, overshoot $\tilde{\delta}[\%] = tri(0, 0, 10)$ and settling time $\tilde{t}_s[s] = trap(1, 1.5, 4, 4)$.

The specifications will be assured by the second order reference model (4) with gain, damping factor and natural frequency α-cuts:

$$k_\alpha = 1 + \frac{e_{p,\alpha}}{100} = 1 + \frac{[\alpha - 1, -\alpha + 1]}{100}$$

$$\xi_\alpha = \sqrt{\frac{1}{1 + \frac{\pi^2}{\ln^2(0.01\delta_\alpha)}}} = \sqrt{\frac{1}{1 + \frac{\pi^2}{\ln^2([0, -0.1\alpha + 0.1])}}}$$

$$\omega_{n,\alpha} = \frac{4}{t_{s,\alpha} du(\xi_\alpha)} = \frac{4}{[0.5\alpha + 1, 4] du(\xi_\alpha)}$$

where $du([a, b]) = [b, a]$. For instance, for $\alpha = 0$:

$$k_0 = [0.99, 1.01],\ \xi_0 = [0.592, 1],\ \omega_{n,0} = [1.692, 4]$$

From the practical point of view this may be too restrictive. Usually, the order of the closed-loop is not so important as to behave like the family of reference models in some sense. For instance, in terms of the frequency response of the closed-loop plant, which is required to be included for each frequency into the frequency response of the family of reference models (figure 1). In this case, $\tilde{S}$ will be the frequency response of a fuzzy set of reference models.

In other cases, a desired frequency response may be built by adding to a nominal target closed loop a particular used-defined tolerance $\delta(\omega)$, in

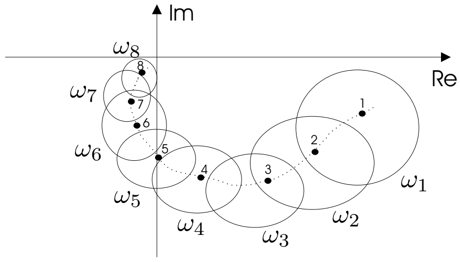

Fig. 1. Feasible frequency response for a given α-cut.

the spirit of conventional robust control (Zhou *et al.*, 1995). In this case, however $\delta(\omega)$ may be a fuzzy set.

The assumption is made that any frequency response inside the proposed envelope will yield a satisfactory time response, even if the resulting loop is not a first- or second-order model (3), (4). Experience with the results from the tools described in Section 5 shows that this is true in most situations.

4.2 Fuzzy characteristic polynomials.

Alternatively, $\mathbb{S}$ may be defined as the coefficient's space of a polynomial. In this case, the fuzzy specifications $\tilde{S}$ will correspond to a fuzzy characteristic polynomial

$$\tilde{p}(s) = \sum_{i=0}^{n} \tilde{a}_i s^i \tag{5}$$

defining where the closed-loop poles are desired to be located. As the coefficients of (5) are considered to be independent, the Edge Theorem (Bhattacharyya *et al.*, 1995) can be applied to plot the boundary the poles location of each α-cut of interest, as shown in the next example.

Example 4.2. Figure 2 shows the boundary of the poles of the polynomial

$$s^3 + [3, 5]s^2 + [6, 11]s + [9, 12] \tag{6}$$

This can be evaluated from the following twelve root locus problems:

$$1 + [3, 5]\frac{s^2}{s^3 + a_1 s + a_0},\ a_1 \in \{6, 11\}, a_0 \in \{9, 12\}$$

$$1 + [6, 11]\frac{s}{s^3 + a_2 s^2 + a_0},\ a_2 \in \{3, 5\}, a_0 \in \{9, 12\}$$

$$1 + [9, 12]\frac{1}{s^3 + a_2 s^2 + a_1 s},\ a_2 \in \{3, 5\}, a_1 \in \{6, 11\}$$

5. CONTROLLER DESIGN.

Once defined the fuzzy set of specifications $\tilde{S}$, a controller K must be designed so that

$$J_K(\tilde{P}_\alpha) \subset \tilde{S}_\alpha \tag{7}$$

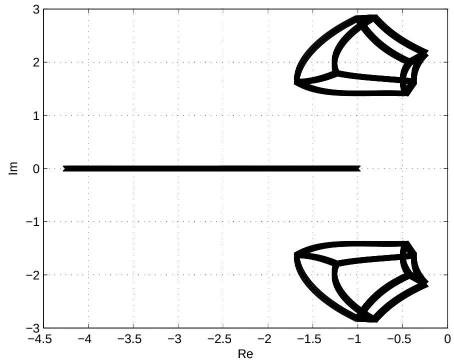

Fig. 2. Poles of the family of polynomials (6)

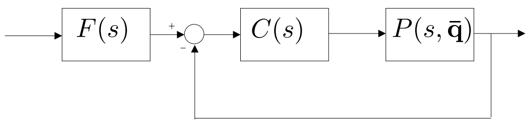

Fig. 3. TDF control structure.

where $J_K(\cdot)$ is the mapping of the open-loop plant in the specifications space induced by the controller K.

Denoting by $\mathcal{K}^{\alpha_r,\alpha_p}$ the set of controllers so that $J_K(\tilde{P}_{\alpha_p}) \subset \tilde{S}_{\alpha_r}$, the solution to (7) is given by

$$\mathcal{K} := \bigcap_{\alpha\in[0,1]} \mathcal{K}^{\alpha,\alpha} \quad (8)$$

Evidently, if a solution $\mathcal{K}^{1,0}$ exists then it is a solution for the problem since $\mathcal{K}^{1,0} \subset \mathcal{K}^{0,0}$ because $core(\tilde{S}) \subset support(\tilde{S})$.

To obtain $\mathcal{K}$, a discretisation on α will have to be done, obtaining for each α-cut under consideration the set of controllers solving the problem. To achieve this, interval methods (Jaulin *et al.*, 2001; COCONUT, 2001) are particularly suitable for two reasons: (a) for an α-cut the fuzzy problem reduces to an interval problem, and (b) interval methods are oriented towards obtaining *sets of solutions*, as in (8). Furthermore, the solutions obtained are guaranteed to fulfill the specifications.

Next, some design examples are given introducing two different interval approaches: branch-and-prune algorithms and algebraic solutions by means of modal interval arithmetic.

Example 5.1. (**Frequency-response inclusion**). In (Bondia, 2002; Bondia and Picó, 2003) a methodology for dealing with interval parametric uncertainty is presented, to tune the frequency response of an uncertain set of continuous plants so that it is included in a specified set of specifications in the Nyquist diagram by using a LTI two-degree of freedom configuration (figure 3).

Specifications are given in terms of an uncertain reference model $M(s,\bar{\mathbf{r}})$, where $\bar{\mathbf{r}}$ is an interval parameter vector. This reference model captures the desired closed-loop frequency response $\mathcal{M}(j\omega) := \{M(j\omega,\mathbf{r}) \mid \mathbf{r}\in\bar{\mathbf{r}}\}$.

The problem to be solved is obtaining F and C so that the inclusion $\mathcal{G}_{lc}(j\omega) \subseteq \mathcal{M}(j\omega)$ holds for $\omega\in\Omega$, where $\mathcal{G}_{lc}$ denotes the family of closed-loop frequency responses: $\mathcal{G}_{lc}(j\omega) := \{FCP(\mathbf{q})/(1+CP(\mathbf{q})) \mid \mathbf{q}\in\bar{\mathbf{q}}\}$.

Under some mild considerations, the problem can be formulated as a constraint satisfaction problem (CSP) on the controller parameters $\bar{\boldsymbol{\theta}}$ (Bondia *et al.*, 2004*b*; Bondia *et al.*, 2004*a*). This CSP may be solved by the algorithm SIVIA (*Set Inversion via Interval Analysis*) (Bondia *et al.*, 2004*b*; Jaulin *et al.*, 2001; COCONUT, 2001). As result, two subpavings consisting of an internal and external approximation of the solution set is obtained. All the elements in the internal subpaving will be a guaranteed solution of the CSP, whereas the external subpaving may contain no solutions. An important advantage of the algorithm is that, if the external subpaving is empty, it is guaranteed that the CSP has no solution, indicating that other specifications must be used.

As an example, consider the plant

$$P(s,\tilde{\mathbf{q}}) = \frac{\tilde{b}}{(s+\tilde{a})(s+\tilde{c})} \quad (9)$$

with trapezoidal fuzzy sets:

$$\begin{aligned}\tilde{a} &= \mathrm{trap}(0.75, 1, 1.5, 1.8),\\ \tilde{b} &= \mathrm{trap}(0.75, 1, 2, 2),\\ \tilde{c} &= \mathrm{trap}(9.5, 10.10.5, 11).\end{aligned}$$

Hard specifications are given by the reference model

$$M(s,\bar{\mathbf{r}}) = \frac{\bar{K}}{\left(\left(\frac{s}{\bar{\omega}_n}\right)^2 + \frac{2\bar{\zeta}}{\bar{\omega}_n}s + 1\right)(10s+1)} \quad (10)$$

with $\bar{K} = [0.99, 1.01]$, $\bar{\zeta} = [0.45, 2]$, $\bar{\omega}_n = [0.5926, 2]$. A PI controller $C(s,\boldsymbol{\theta}) = K_p + K_i\frac{1}{s}$ is considered without prefilter, $N(s) = 1$.

Frequency response inclusion is sought for $\Omega := \{0.001, 0.01, 0.1, 1, 10\}$ rad/s. For $\omega > 10$ rad/s, inclusion may not hold. However, stability will be guaranteed by means of an interval Routh-Hurwitz test (Barmish, 1994).

The above specifications cannot be fulfilled by the whole family of plants (the external approximation of the solution set of the corresponding CSP is empty). Thus, the following soft specifications are defined:

$$\bar{K} = [0.9, 1.1],\ \bar{\zeta} = [0.3, 5],\ \bar{\omega}_n = [0.5333, 8] \quad (11)$$

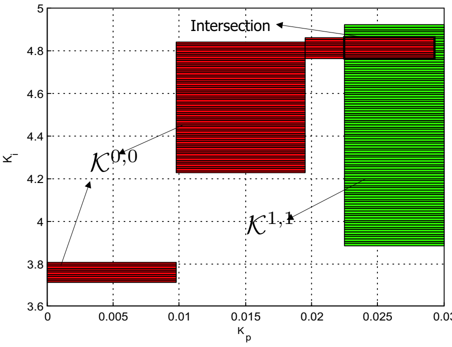

Fig. 4. Internal approximations of $\mathcal{K}^{0,0}$ and $\mathcal{K}^{1,1}$.

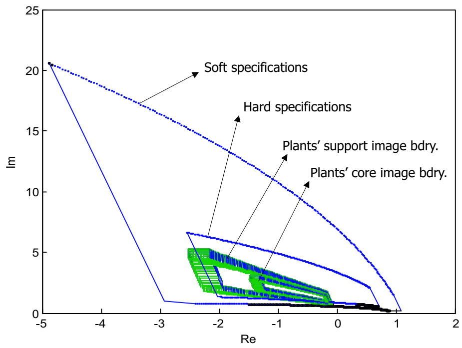

Fig. 5. Frequency response inclusion for $\omega = 1$ rad/s (inverse Nyquist plot).

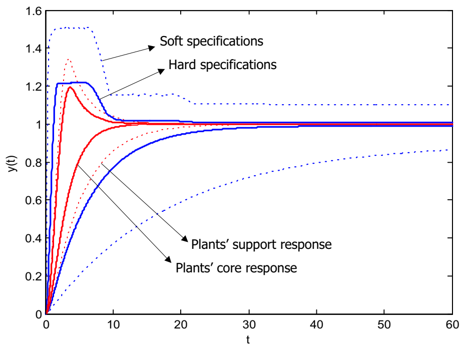

Fig. 6. Time response.

Figure 4 shows the internal approximation obtained for $\mathcal{K}^{0,0}$ and $\mathcal{K}^{1,1}$ for a precision of 0.01 in the SIVIA algorithm. Any controller in the intersection will guarantee the fulfillment of the hard specifications for the core of plants, whereas for the whole family of plants, a performance no worse than the soft specifications will be achieved. For instance, a feasible regulator is:

$$C(s) = 0.026 + \frac{4.8}{s} \tag{12}$$

Figure 5 shows the frequency response inclusion for a frequency of 1 rad/s. Figure 6 shows the obtained time response for illustration (envelopes of time responses of a grid of the core/support family of plants/specifications).

Example 5.2. (**Fuzzy pole-placement**). Let the second order plant

$$P(s, \tilde{\boldsymbol{q}}) = \frac{\tilde{k}}{s^2 + \tilde{a}_1 s + \tilde{a}_0} \tag{13}$$

be given. Consider

$$\begin{aligned} \tilde{k} &= \text{trap}(1.1, 1.2, 1.3, 1.5) \\ \tilde{a}_1 &= \text{trap}(0.3, 0.4, 0.7, 0.8) \\ \tilde{a}_0 &= \text{trap}(0.02, 0.03, 0.05, 0.07) \end{aligned} \tag{14}$$

and a PD controller $C(s, \boldsymbol{\theta}) = k_p + k_d s$. A desired closed-loop characteristic polynomial is defined as

$$p(s, \tilde{\boldsymbol{r}}) = s^2 + \tilde{r}_1 s + \tilde{r}_0 \tag{15}$$

where $\tilde{r}_1 = \text{trap}(2.5, 6, 8, 10)$ and $\tilde{r}_0 = \text{trap}(10, 10, 16, 60)$.

Closed-loop coefficients are, for fixed α, $a_{1,\alpha} + k_\alpha k_{d,\alpha}$ and $a_{0,\alpha} + k_\alpha k_{p,\alpha}$, so they must verify:

$$\begin{aligned} a_{1,\alpha} + k_\alpha k_{d,\alpha} &\subseteq r_{1,\alpha} \\ a_{0,\alpha} + k_\alpha k_{p,\alpha} &\subseteq r_{0,\alpha} \end{aligned} \tag{16}$$

A fuzzy box $k_{d,\alpha} \times k_{p,\alpha}$ will be a solution to the system of equations (16) if and only if

$$\begin{aligned} &\forall k_d \in k_{d,\alpha}, \forall a_1 \in a_{1,\alpha}, \exists r_1 \in r_{1,\alpha} \mid a_1 + k_d = r_1 \\ &\forall k_p \in k_{p,\alpha}, \forall a_0 \in a_{0,\alpha}, \exists r_0 \in r_{0,\alpha} \mid a_0 + k_p = r_0 \end{aligned}$$

Solving for the controller parameters in (16)

$$k_{d,\alpha}^{\forall} = \frac{r_{1,\alpha}^{\exists} - a_{1,\alpha}^{\forall}}{k_\alpha^{\forall}}, \qquad k_{p,\alpha}^{\forall} = \frac{r_{0,\alpha}^{\exists} - a_{0,\alpha}^{\forall}}{k_\alpha^{\forall}} \tag{17}$$

where the corresponding modality $(\exists, \forall)$ has been assigned to each variable as superscript. As they are rational uni-incident functions, modal interval arithmetic (Gardeñes *et al.*, 1985; SIGLA/X and Sáinz, 2001) can be applied to obtain the desired semantics. It corresponds to the following interval evaluation

$$k_{d,\alpha} = \frac{r_{1,\alpha} - du(a_{1,\alpha})}{du(k_\alpha)}, \quad k_{d,\alpha} \text{ proper} \tag{18}$$

$$k_{p,\alpha} = \frac{r_{0,\alpha} - du(a_{0,\alpha})}{du(k_\alpha)}, \quad k_{p,\alpha} \text{ proper} \tag{19}$$

where an interval $[a, b]$ is proper if $a \leq b$. Figure 7 shows the result obtained for the defined plant and specifications.

Figures 8 and 9 show the closed-loop poles for the support and kernel respectively considering the controller $k_d = 5$, $k_p = 11.5$. In both plots the dark boundary indicates the poles of the desired characteristic polynomial whereas the light boundary indicates the closed-loop poles for the chosen controller. As it can be seen, specifications are fulfilled.

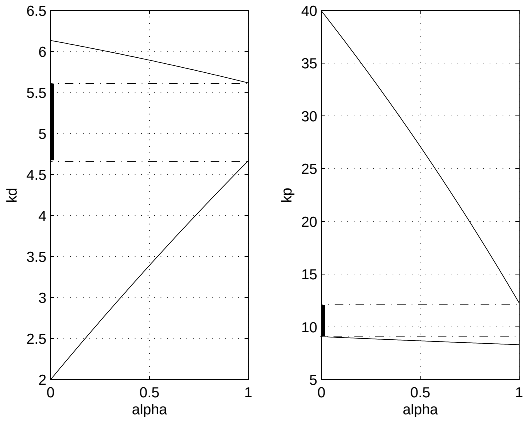

Fig. 7. Solution set for each α-cut.

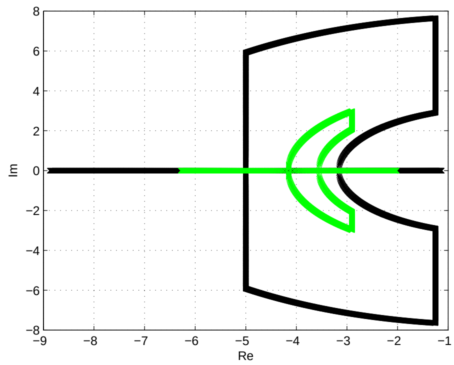

Fig. 8. Closed-loop poles for $C(s) = 11.5 + 5s$. Support.

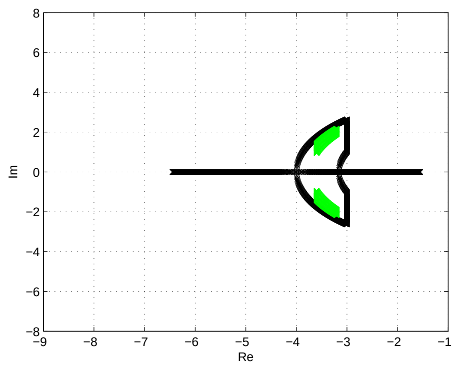

Fig. 9. Closed-loop for $C(s) = 11.5 + 5s$. Kernel.

6. CONCLUSIONS.

In this paper, the problem of achieving good performance for a set of likely plants while controlling its degradation for a wider family of them has been discussed. The framework uses fuzzy plant models (from identification or fuzzy uncertainty in physical parameters), and defines a fuzzy specification set.

Different fuzzy specifications are analyzed such as fuzzy reference models representing the desired time/frequency response and fuzzy characteristic polynomials under the framework of fuzzy pole placement. In both cases, interval analysis is used to design the controller. Refinement and generalisation of the procedures here presented by examples is under research.

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