NUMERICAL SOLUTION OF FILTERING PROBLEM WITH MULTIMODAL DENSITIES

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Abstract: Numerical solution of filtering problem for nonlinear stochastic systems is treated. The aim is to improve the point-mass method for multimodal probability density functions of state. The main innovation items concern grid update, namely covering a nonnegligible probability density function support and merging grids in multigrid design. Comparing to the standard point-mass algorithm, the new boundary-based grid placement technique maintains estimation quality and the merging technique decreases computational demands for multimodal densities.

Keywords: stochastic systems, state estimation, nonlinear filters, probability density function, estimation algorithms

1. INTRODUCTION

A general solution of the state estimation problem for discrete-time stochastic nonlinear non-Gaussian system is completely described by conditional probability density functions (pdfs) of state and it is given by the Bayesian recursive relations. A closed-form solution of the relations is known only for linear Gaussian systems and a few special cases, e.g. (Sorenson, 1988; Söderström, 1994).

Existing global nonlinear filtering methods are based on the following approaches: analytical (Sorenson and Alspach, 1971), numerical (Bucy and Senne, 1971), and Monte Carlo (Liu and Chen, 1998) approaches. Each of these approaches approximates pdfs and/or state space differently.

This paper deals with the numerical approach, namely the point-mass (PM) method. This method was introduced by Bucy and Senne (1971) and it is based on covering the state space by a grid of isolated points. Values of conditional pdf's are computed only at these grid points. The main advantages of the PM method are relative theoretical simplicity of filter design and natural discrete approximation of state space. On the other hand, the original version of the method suffered from enormous computational demands, the method was focused on prediction problem only, a procedure for setting the number of the grid points was not specified, and the method was not appropriate for cases with multimodal pdfs. The multimodality may arise in Bayesian inference as a result of prior pdf, nonlinear functions in state or measurement equations, and multimodal pdf of state and/or measurement noise, which may represent e.g. abrupt changes of state or parameters and measurement outliers, respectively.

The PM method was elaborated by Sorenson (1988) and Kramer and Sorenson (1988) where the formal presentation of the method was improved by introducing the p-vector approach and piece-wise representation of pdfs and both filtering and prediction steps of estimation process were expressed. The complete solution of filtering, prediction and smoothing by PM approach was presented by Královec and Šimandl (2004). Further, Šimandl *et al.* (2002) designed an adaptive technique for setting the number of grid points, partially eliminating also numerical demands of the PM method. Grid design techniques for solution of filtering problems with multimodal pdfs by means of PM approach were proposed by Šimandl and Královec (2003). These techniques included merging and splitting of grids and allowed an effective treating of multimodal pdf's with separable modes.

The PM method has been successfully applied in a number of practical problems, e.g. in tracking and navigation (Bergman, 1997).

The goal of this paper is to improve the grid design for multimodal pdfs from (Šimandl and Královec, 2003), namely the merging technique, and introduce a new general grid design procedure which can substitute the standard design technique. The new procedure should be more effective in covering state pdf support and appropriate for multimodal pdfs with non-separable modes for which existing techniques are ineffective or may cause divergence of state estimate.

The paper is organized as follows. The PM approach to state estimation is described and the goal of the paper is stated in Section 2. A general multigrid PM algorithm is introduced in Section 3. A new boundary-based grid placement for PM algorithm is derived in Section 4 and an improved technique for handling multiple grids is provided in Section 5. The results of the paper are illustrated by a numerical example in Section 6.

2. PROBLEM STATEMENT

2.1 Point-Mass Approach to State Estimation

Consider the nonlinear stochastic system

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + \mathbf{w}_k, \quad k = 0, 1, 2, \ldots \tag{1}$$

$$\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k, \quad k = 0, 1, 2, \ldots \tag{2}$$

where the vectors $\mathbf{x}_k \in \mathbb{R}^n$, $\mathbf{z}_k \in \mathbb{R}^m$ represent the state of the system and the measurements at time k, respectively, $\mathbf{f}_k : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{h}_k : \mathbb{R}^n \to \mathbb{R}^m$ are known vector functions, and $\mathbf{w}_k \in \mathbb{R}^n$, $\mathbf{v}_k \in \mathbb{R}^m$ are state and measurement zero-mean white noise sequences with positive definite covariance matrices $\mathbf{Q}_k$, $\mathbf{R}_k$, respectively, mutually independent and independent of $\mathbf{x}_0$. The pdf of the initial state $p(\mathbf{x}_0)$ is assumed to be known, as well as the pdf's of the noises $p(\mathbf{w}_k)$, $p(\mathbf{v}_k)$.

It is well-known that the filtering pdf $p(\mathbf{x}_k|\mathbf{z}^k)$ and predictive pdf $p(\mathbf{x}_{k+1}|\mathbf{z}^k)$ are given by the Bayesian recursive relations

$$p(\mathbf{x}_k|\mathbf{z}^k) = \frac{p(\mathbf{x}_k|\mathbf{z}^{k-1})\, p(\mathbf{z}_k|\mathbf{x}_k)}{\int p(\mathbf{x}_k|\mathbf{z}^{k-1})\, p(\mathbf{z}_k|\mathbf{x}_k)\, \mathrm{d}\mathbf{x}_k} \tag{3}$$

$$p(\mathbf{x}_{k+1}|\mathbf{z}^k) = \int p(\mathbf{x}_k|\mathbf{z}^k)\, p(\mathbf{x}_{k+1}|\mathbf{x}_k)\, \mathrm{d}\mathbf{x}_k \tag{4}$$

where $\mathbf{z}^k = \{\mathbf{z}_0, \ldots, \mathbf{z}_k\}$ and $p(\mathbf{x}_0|\mathbf{z}^{-1}) = p(\mathbf{x}_0)$. The transition pdf $p(\mathbf{x}_{k+1}|\mathbf{x}_k)$ can be expressed as $p(\mathbf{x}_{k+1}|\mathbf{x}_k) = p_{\mathbf{w}_k}\big(\mathbf{x}_{k+1} - \mathbf{y}_{k+1}\big)$ where $\mathbf{y}_{k+1} = \mathbf{f}_k(\mathbf{x}_k)$. The measurement pdf can be written using (2) as $p(\mathbf{z}_k|\mathbf{x}_k) = p_{\mathbf{v}_k}\big(\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}_k)\big)$.

The key idea of the PM method (Bucy and Senne, 1971) for generating conditional pdfs of state at kth instant is to substitute a nonnegligible continuous support of the pdf by a grid of N_k isolated points. Values of the pdf are computed only at these grid points and thus the solution of (3), (4) is performed numerically over the grid instead of the continuous support. Nonnegligible support is a region in the state space where the true state is probable to lie and hence values of the pdf are nonnegligible there. However, finding a nonnegligible support effectively is a complex task and particularly for multimodal pdfs an efficient procedure is missing. The crucial point is delimiting a support by setting boundary points of a grid as a proper support delimiting yields significant reduction of computational demands of the PM algorithm without a decline in estimation quality.

2.2 Problem Formulation

The aim of the paper is twofold. The first aim is to propose a novel technique of grid placement which should overcome the weaknesses of the standard state-covariance-based grid placement technique. The second aim is improving the multigrid handling techniques of splitting and merging (Šimandl and Královec, 2003) namely by utilizing the Mahalanobis distance for merging of grids.

3. MULTIGRID POINT-MASS ALGORITHM

The PM approach represents a pdf $p(\mathbf{x}_k)$ by a set of M_k rectangular grids of points $\Xi_k[\mu](N_k[\mu]) = \{\boldsymbol{\xi}_{ki}[\mu]; \boldsymbol{\xi}_{ki}[\mu] \in \mathbb{R}^n, i = 1, \ldots, N_k\}$, $\mu = 1, \ldots, M_k$, by volume masses $\Delta\boldsymbol{\xi}_k[\mu]$ for each grid and by a set of pdf values at the grid points, $\mathcal{P}_k[\mu] = \{P_{k,i}[\mu]; P_{k,i}[\mu] = p_{\mathbf{x}_k}(\boldsymbol{\xi}_{ki}[\mu]), \boldsymbol{\xi}_{ki}[\mu] \in \Xi_k[\mu](N_k[\mu])\}$. Each grid $\Xi_k[\mu](N_k[\mu])$ is assigned a weight $\omega_k[\mu]$ which represents probability of appearance of the state in the region covered by the μth grid:

$$\omega_k[\mu] = \Delta\boldsymbol{\xi}_k[\mu] \sum_{i=1}^{N_k[\mu]} P_{k,i}[\mu] \tag{5}$$

and it holds that $\sum_{\mu=1}^{M_k} \omega_k[\mu] = 1$.

This PM representation can be used for approximation of filtering and predictive pdf's $p(\mathbf{x}_k|\mathbf{z}^k)$, $p(\mathbf{x}_{k+1}|\mathbf{z}^k)$.

For a μth grid with corresponding filtering pdf values it is possible to compute local predictive mean vector $\hat{\boldsymbol{\eta}}_{k+1}[\mu]$ and local predictive covariance matrix $\mathbf{C}_{k+1}[\mu]$ as follows

$$\hat{\boldsymbol{\eta}}_{k+1}[\mu] = \frac{\Delta\boldsymbol{\xi}_k[\mu] \sum_{i=1}^{N_k[\mu]} \boldsymbol{\eta}_{k+1,i}[\mu]\, P_{k|k,i}[\mu]}{\omega_{k|k}[\mu]} \tag{6}$$

$$\mathbf{C}_{k+1}[\mu] = \frac{\Delta\boldsymbol{\xi}_k[\mu] \sum_{i=1}^{N_k[\mu]} \boldsymbol{\eta}_{k+1,i}[\mu]\, \boldsymbol{\eta}_{k+1,i}^{\mathrm{T}}[\mu]\, P_{k|k,i}[\mu]}{\omega_{k|k}[\mu]} - \hat{\boldsymbol{\eta}}_{k+1}[\mu]\, \hat{\boldsymbol{\eta}}_{k+1}^{\mathrm{T}}[\mu] + \mathbf{Q}_k\ . \tag{7}$$

Global predictive mean $E(\mathbf{x}_{k+1}|\mathbf{z}^k)$ and covariance matrix $\mathrm{cov}(\mathbf{x}_{k+1}|\mathbf{z}^k)$ can be approximated using the local moments as follows:

$$\hat{\boldsymbol{\eta}}_{k+1} = \sum_{\mu=1}^{M} \omega_{k|k}[\mu]\, \hat{\boldsymbol{\eta}}_{k+1}[\mu] \tag{8}$$

$$\mathbf{C}_{k+1} = \sum_{\mu=1}^{M} \omega_{k|k}[\mu] \left(\mathbf{C}_{k+1}[\mu] + \hat{\boldsymbol{\eta}}_{k+1}[\mu]\, \hat{\boldsymbol{\eta}}_{k+1}^{\mathrm{T}}[\mu]\right) - \hat{\boldsymbol{\eta}}_{k+1}\, \hat{\boldsymbol{\eta}}_{k+1}^{\mathrm{T}}\ . \tag{9}$$

Moments of the filtering pdf can be enumerated analogously.

The algorithm of the PM method can be written in the following steps.

Algorithm of the PM method

Initialization: Define initial grids $\Xi_0[\mu](N_0[\mu])$ in $\mathbb{R}^n$ for the prior pdf $p(\mathbf{x}_0|\mathbf{z}^{-1})$: $\Xi_0[\mu](N_0[\mu]) = \{\boldsymbol{\xi}_{0i}[\mu]; i = 1,\ldots,N_0[\mu]\}$, volumes $\Delta\boldsymbol{\xi}_0[\mu]$ and sets of pdf values $\mathcal{P}_{0|-1}[\mu] = \{P_{0|-1,i}[\mu]; i = 1,\ldots,N_0[\mu]\}$ for $\mu = 1,\ldots,M_0$.

Then proceed for $k = 0, 1, \ldots$.

Step 1: At time k compute values of the approximate filtering pdf at points of grids $\Xi_k[\mu](N_k[\mu])$ using (3), for $i = 1,\ldots,N_k[\mu]$, $\mu = 1,\ldots,M_k$.

$$P_{k|k,i}[\mu] = c_k^{-1}\, P_{k|k-1,i}[\mu]\, p_{\mathbf{v}_k}\big(\mathbf{z}_k - \mathbf{h}_k(\boldsymbol{\xi}_{ki}[\mu])\big) \tag{10}$$

where the normalizing constant c_k must be enumerated by the sum over all grids as

$$c_k = \sum_{\mu=1}^{M_k} \Delta\boldsymbol{\xi}_k[\mu] \sum_{i=1}^{N_k[\mu]} \big[P_{k|k-1,i}[\mu] \cdot p_{\mathbf{v}_k}\big(\mathbf{z}_k - \mathbf{h}_k(\boldsymbol{\xi}_{ki}[\mu])\big)\big]\ . \tag{11}$$

Step 2: Transform $\Xi_k[\mu](N_k[\mu])$ to a grid $H_{k+1}[\mu](N_k[\mu]) = \{\boldsymbol{\eta}_{k+1,i}[\mu]; i = 1,\ldots,N_k[\mu]\}$ by the system dynamics for $\mu = 1,\ldots,M_k$

$$\boldsymbol{\eta}_{k+1,i}[\mu] = \mathbf{f}_k(\boldsymbol{\xi}_{ki}[\mu]) \tag{12}$$

Step 3: Redefine each grid $H_{k+1}[\mu](N_k[\mu])$, $\mu = 1,\ldots,M_k$, to obtain a new grid $\Xi_{k+1}[\mu](N_{k+1[\mu]})$, $\mu = 1,\ldots,M_{k+1}$, for state $\mathbf{x}_{k+1}$ with the same structural properties as the original grids: $\Xi_{k+1}[\mu](N_{k+1}[\mu]) = \{\boldsymbol{\xi}_{k+1,j}[\mu]; j = 1,\ldots,N_{k+1}[\mu]\}$.

Step 4: Compute values of the approximate predictive pdf for the new grids $\Xi_{k+1}[\mu](N_{k+1}[\mu])$ using (4)

$$P_{k+1|k,j}[\mu] = \Delta\boldsymbol{\xi}_k[\mu] \sum_{i=1}^{N_k[\mu]} \big[P_{k|k,i}[\mu] \cdot p_{\mathbf{w}_k}(\boldsymbol{\xi}_{k+1,j}[\mu] - \boldsymbol{\eta}_{k+1,i}[\mu])\big] \tag{13}$$

for $j = 1,\ldots,N_{k+1}[\mu]$, $\mu = 1,\ldots,M_k$. □

The Step 3 of the PM algorithm can be realized by *anticipative grid design technique* which was presented by Šimandl *et al.* (2002). The technique consists of two main parts. The first one treats placing a grid in the state space, i.e. delimits a nonnegligible pdf support, and it is important for the subject of this paper. The second part deals with setting a number of grid points and positioning of points within a grid which is not necessary to be discussed in this paper.

The delimiting of nonnegligible pdf support is based on the following steps:

Step 3a) Compute estimates of local predictive means $\hat{\boldsymbol{\eta}}_{k+1}[\mu]$ by (6) and predictive covariance matrices $\mathbf{C}_{k+1}[\mu]$ by (7) for $\mu = 1,\ldots,M_k$.

Step 3b) Perform Jordan decomposition of $\mathbf{C}_{k+1}[\mu]$: $\mathbf{C}_{k+1}[\mu] = \mathbf{T}_{k+1}[\mu]\boldsymbol{\Lambda}_{k+1}[\mu]\mathbf{T}_{k+1}^{\mathrm{T}}[\mu]$ where $\boldsymbol{\Lambda}_{k+1}[\mu] = \mathrm{diag}\{\lambda_{k+1}^{(\ell)}[\mu]\}_{\ell=1}^{n}$ and transform the state noise covariance matrix as $\bar{\mathbf{Q}}_{k+1}[\mu] = \mathbf{T}_{k+1}^{\mathrm{T}}[\mu]\mathbf{Q}_{k+1}\mathbf{T}_{k+1}[\mu]$.

Step 3c) Design transformed axis grid supports $\bar{I}_{k+1}^{(\ell)}[\mu] = \left[-b\sqrt{\lambda_{k+1}^{(\ell)}[\mu]}, b\sqrt{\lambda_{k+1}^{(\ell)}[\mu]}\right]$ and the support $\Omega_{k+1}[\mu]$ of pdf $p(\mathbf{x}_{k+1}|\mathbf{z}^k)$

$$\Omega_{k+1}[\mu] = \{\mathbf{x}_{k+1}; \mathbf{x}_{k+1} = \mathbf{T}_{k+1}[\mu]\bar{\mathbf{x}}_{k+1} + \hat{\boldsymbol{\eta}}_{k+1}[\mu], \forall \bar{\mathbf{x}}_{k+1} \in \bar{I}_{k+1}^{(1)}[\mu] \times \ldots \times \bar{I}_{k+1}^{(n)}[\mu]\} \tag{14}$$

where $\times$ denoted Cartesian product and $b > 0$ is a design parameter. □

In the anticipative grid design technique, an estimated predictive covariance matrix of state $\mathbf{C}_{k+1} \approx \mathrm{cov}(\mathbf{x}_{k+1}|\mathbf{z}^k)$ plays a key role for grid design. The grid is rotated according to eigenvectors of the matrix $\mathbf{C}_{k+1}$ and thus respect a shape of the predictive pdf $p(\mathbf{x}_{k+1}|\mathbf{z}^k)$. A bounding box of the grid $\Xi_{k+1}(N_{k+1})$ (i.e. a box in n-dim space determined by border points of the grid) is given by eigenvalues of $\mathbf{C}_{k+1}$. However, using eigenvalues for grid placement is suitable only for Gaussian or Gaussian-like pdfs where an analogy with the 3-sigma or 4-sigma rule, for instance, can be applied. If, for example, the predictive pdf was multimodal, the eigenvalue technique can generally create a grid that will leave some nonnegligible subspaces of the state space uncovered by a grid, or on the other hand, that will cover too large negligible areas by grid points. Note that distribution of predictive pdf is not actually known, so the both described undesirable cases cannot be detected in a simple way and grid design technique should eliminate them automatically.

4. BOUNDARY-BASED GRID PLACEMENT

The goal of this chapter is to present an improved procedure of nonnegligible support delimiting given by Steps 3a–3c of the anticipative grid design technique.

The idea of the new grid placement will be first explained by 1-dim case. Boundary points of a grid $\Xi_{k+1}(N_{k+1}) = \{\xi_{k+1,i}; i = 1, \ldots, N_{k+1}\}$ are given by boundary points of the deformed grid $H_{k+1}(N_k) = \{\eta_{k+1,j}; j = 1, \ldots, N_k\}$. Let the boundary points of $H_{k+1}(N_k)$ be denotes as

$$\eta_{k+1,\min} = \min_{j=1,\ldots,N_k} \eta_{k+1,j} \tag{15}$$

$$\eta_{k+1,\max} = \max_{j=1,\ldots,N_k} \eta_{k+1,j} \ . \tag{16}$$

A nonnegligible interval I_{k+1} in the state space, which will be covered by a grid $\Xi_{k+1}(N_{k+1})$, is then given by the interval $\left[\eta_{k+1,\min}, \eta_{k+1,\max}\right]$ enlarged by involving the state noise w_k influence:

$$I_{k+1} = \left[\eta_{k+1,\min} - a\sigma_k, \eta_{k+1,\max} + a\sigma_k\right] \tag{17}$$

where $\sigma_k^2 = \mathrm{var}(w_k)$.

Since $\eta_{k+1,\min}$, $\eta_{k+1,\max}$ represent a boundary of the nonnegligible domain of $p\big(f_k(x_k)|z^k\big)$, the interval I_{k+1} does not exclude any significant nonnegligible subsets of $p(x_{k+1}|z^k)$ domain and it depends on the *known* variance of the state noise.

The described technique will be called *boundary-based* grid design, as the grid is designed by means of the boundary of the grid $H_{k+1}(N_k)$.

A nonnegligible area of n-dim state space is approximated by a grid $\Xi_{k+1}[\mu]$ rotated according to eigenvectors of the corresponding local predictive covariance matrix $\mathbf{C}_{k+1}[\mu]$ as described above. Borders of the grid are set as follows (grid index $[\mu]$ will be omitted in the sequel of the section for notational convenience).

The deformed grid $H_{k+1}(N_k) = \{\boldsymbol{\eta}_{k+1,j}; j = 1, \ldots, N_k\}$ is first transformed to the basis of the state space of $\bar{\mathbf{x}}_{k+1}$, i.e. $\bar{\boldsymbol{\eta}}_{k+1,j} = \mathbf{T}_{k+1}^{\mathrm{T}} \boldsymbol{\eta}_{k+1,j}$, $\bar{H}_{k+1} = \{\bar{\boldsymbol{\eta}}_{k+1,j}; j = 1, \ldots, N_k\}$. Minima and maxima in all coordinates $\ell = 1, \ldots, n$ of grid points $\bar{\boldsymbol{\eta}}_{k+1,j}$ are found as

$$\bar{\eta}_{k+1,\min}^{(\ell)} = \min_{j=1,\ldots,N_k} \bar{\eta}_{k+1,j}^{(\ell)} \tag{18}$$

$$\bar{\eta}_{k+1,\max}^{(\ell)} = \max_{j=1,\ldots,N_k} \bar{\eta}_{k+1,j}^{(\ell)} \tag{19}$$

and they delimit borders of significant intervals

$$\bar{I}_{k+1}^{(\ell)} = \left[\bar{\eta}_{k+1,\min}^{(\ell)} - a\sqrt{\bar{Q}_{k+1}^{(\ell)}}, \bar{\eta}_{k+1,\max}^{(\ell)} + a\sqrt{\bar{Q}_{k+1}^{(\ell)}}\right] \tag{20}$$

where $\bar{Q}_{k+1}^{(\ell)}$ is ℓth diagonal element of transformed state noise covariance matrix $\bar{\mathbf{Q}}_{k+1} = \mathbf{T}_{k+1}^{\mathrm{T}} \mathbf{Q}_{k+1} \mathbf{T}_{k+1}$. The predictive pdf support Ω_{k+1} can be then computed by (14).

The proposed boundary-based grid placement technique ensures comprehensive approximation of state space by a rectangular grid also for non-Gaussian pdfs, including multimodal pdfs.

5. APPROXIMATION OF MULTIMODAL DENSITIES

5.1 Grid Splitting by Marginal Densities

Probability of certain areas of the state space may drop significantly by including new information at the filtering step, causing that the pdf can be taken for multimodal. For that reason splitting of the grid should be considered at the filtering step. Moreover, it is possible to utilize the advantageous regular shape of the grid. Thus, the grids $\Xi_k[\mu](N_k[\mu])$, $\mu = 1, \ldots, M_k$ with the filtering pdf values $\mathcal{P}_{k|k}[\mu]$ are examined for splitting.

It is necessary to find out if a grid $\Xi_k[\mu]$ covers separable nonnegligible areas. To avoid searching separable areas in $\mathbb{R}^n$, which would be an extremely computationally expensive process, Šimandl and Královec (2003) proposed examining separable areas individually for each state component by means of marginal pdfs and axis grids.

5.2 Grid Merging by Mahalanobis Distance

The development of grids is determined by the system dynamics in Step 2 of the general PM method algorithm. This transformation may cause more grids to overlap. To keep the algorithm effective it is advantageous to use only one grid for covering each nonnegligible region of the state space. Contrary to the prediction step, overlapping of grids cannot happen at the filtering step because grid points are fixed in this step.

Šimandl and Královec (2003) formed a decision rule for merging two grids based on a comparison of the distance of grids' centers with a limiting distance dependent on the least eigenvalues of the grids' local covariance matrices. In this paper, a more simple and straightforward criterion is presented, based on the *Mahalanobis distance*.

The Mahalanobis distance $D(\mathbf{x}, \mathbf{y}; \mathbf{C})$ of the points $\mathbf{x}$, $\mathbf{y}$ with respect to a positive-definite symmetric matrix $\mathbf{C}$ is defined as:

$$D^2(\mathbf{x}, \mathbf{y}; \mathbf{C}) = (\mathbf{x} - \mathbf{y})^{\mathrm{T}} \, \mathbf{C}^{-1} \, (\mathbf{x} - \mathbf{y}) \qquad (21)$$

The following set of grids is considered

$$\left\{ H_{k+1}[\mu](N_k[\mu]); \mu = 1, \ldots, M_k \right\} , \qquad (22)$$

where

$$H_{k+1}[\mu](N_k[\mu]) = \left\{ \boldsymbol{\eta}_{k+1,i}[\mu]; i = 1, \ldots, N_k[\mu] \right\}$$

and points of these grids were obtained by (12).

For each grid $H_{k+1}[\mu](N_k[\mu])$, $\mu = 1, \ldots, M_k$, local mean $\hat{\eta}_{k+1}[\mu]$ and local covariance matrix $\mathbf{C}_{k+1}[\mu]$ are computed by (6) and (7), respectively.

The Mahalanobis-distance decision rule for merging two grids can be stated as follows. Grids $H_{k+1}[\mu](N_k[\mu])$ and $H_{k+1}[\nu](N_k[\nu])$ should be merged if one of the Mahalanobis distances of the grids' local centers is less than a threshold $\delta > 0$:

$$\left[(\hat{\boldsymbol{\eta}}[\mu] - \hat{\boldsymbol{\eta}}[\nu])^{\mathrm{T}} \, \mathbf{C}^{-1}[\mu] \, (\hat{\boldsymbol{\eta}}[\mu] - \hat{\boldsymbol{\eta}}[\nu]) \right]^{\frac{1}{2}} < \delta$$

$$\text{or}$$

$$\left[(\hat{\boldsymbol{\eta}}[\mu] - \hat{\boldsymbol{\eta}}[\nu])^{\mathrm{T}} \, \mathbf{C}^{-1}[\nu] \, (\hat{\boldsymbol{\eta}}[\mu] - \hat{\boldsymbol{\eta}}[\nu]) \right]^{\frac{1}{2}} < \delta \quad (23)$$

The threshold is recommended to be set as $\delta = 4$ analogously to the 4-sigma rule. The time index was omitted for notational convenience in (23).

Criterion (23), which decides about merging two grids and replaces the merging criterion from (Šimandl and Královec, 2003), is quite simple but it does not cover all possible situations when two grids overlap. Simplicity of the rule is thus preferred to its generality because full-range verifying of grid overlapping would be computationally burdensome and algorithm efficiency would be decreased.

The merging of grids is realized by creating index sets $\mathcal{M}_\nu$, each of which contains indexes of grids that should be merged. A new grid is thus defined by grid indexes from one index set. Index sets are constructed by the following algorithm (index sets have vector structure; time index $k+1$ will be omitted in $\boldsymbol{\eta}$ and $\mathbf{C}$ for notational convenience).

Grid merging algorithm

1. $\nu := 1$, $\mu := 1$
2. $\mathcal{M}_\nu := [\mu]$
3. $\mu := \mu + 1$, $i := 0$
4. $i := i + 1$, $j := 0$
5. $j := j + 1$
6. if $D^2(\hat{\boldsymbol{\eta}}[\mu], \hat{\boldsymbol{\eta}}[\mathcal{M}_i(j)]; \mathbf{C}[\mu]) < \delta$
   or $D^2(\hat{\boldsymbol{\eta}}[\mu], \hat{\boldsymbol{\eta}}[\mathcal{M}_i(j)]; \mathbf{C}[\mathcal{M}_i(j)]) < \delta$
   $\mathcal{M}_i := \left[\mathcal{M}_i^{\mathrm{T}}, \mu \right]^{\mathrm{T}}$, go to 11
7. if $j < \dim(\mathcal{M}_i)$, go to 5
8. if $i < \nu$, go to 4
9. $\nu := \nu + 1$
10. $\mathcal{M}_\nu := \mu$
11. if $\mu < M_k$, go to 3
12. end

The symbol := in the algorithm stands for the assignment statement. Let the number of the index sets be denoted as M_{k+1}, i.e. $\nu = 1, \ldots, M_{k+1}$. The number of index sets is equal to the number of new grids $\Xi_{k+1}[\nu]$. Note that the algorithm ensures that $\mathcal{M}_{\nu_1} \cap \mathcal{M}_{\nu_2} = \emptyset$ for $\nu_1 \neq \nu_2$. The index μ of such a grid $H_{k+1}[\mu](N_k[\mu])$ which cannot merge with any other grid belongs to a one-element set $\mathcal{M}_\nu = [\mu]$.

New grids $\Xi_{k+1}[\nu](N_{k+1}[\nu])$ are designed separately by Step 3 of the PM method algorithm.

Finally, values of predictive pdf are enumerated for the points of each new grid $\Xi_{k+1}[\nu](N_{k+1}[\nu])$, $\nu = 1, \ldots, M_{k+1}$, using grids $\Xi_k[\mu](N_k[\mu])$ specified by the index set $\mathcal{M}_\nu$

$$P_{k+1|k,j}[\nu] = \sum_{\mu \in \mathcal{M}_\nu} \Delta\boldsymbol{\xi}_k[\mu] \sum_{i=1}^{N_k[\mu]} \left[P_{k|k,i}[\mu] \right.$$
$$\left. \cdot \, p_{\mathbf{w}_k} \left(\boldsymbol{\xi}_{k+1,j}[\nu] - \boldsymbol{\eta}_{k+1,i}[\mu] \right) \right] \quad (24)$$

for $j = 1, \ldots, N_{k+1}[\nu]$. The relation (24) substitutes (13) in Step 4 of the PM algorithm if merging is applied.

The grid merging technique thus consists of two procedures. The first one (Grid merging algorithm) selects groups of overlapping grids and the second one (24) produces pdf values at grid points of newly created grids.

6. NUMERICAL ILLUSTRATION

Consider the following stochastic system

$$\begin{aligned} x_{k+1}^{(1)} &= x_k^{(1)} x_k^{(2)} + w_k^{(1)} \\ x_{k+1}^{(2)} &= x_k^{(1)} + w_k^{(2)} \\ z_k &= 0.2 \big(x_k^{(2)} \big)^2 + v_k \end{aligned}$$

where $\mathbf{x}_k = [x_k^{(1)} \; x_k^{(2)}]^{\mathrm{T}}$, both state noise $\mathbf{w}_k = [w_k^{(1)} \; w_k^{(2)}]^{\mathrm{T}}$ and measurement noise v_k are zero-mean white Gaussian noises with $\mathrm{cov}(\mathbf{w}_k) = \mathbf{Q} = \mathrm{diag}\{0.25, 10^{-4}\}$, and $\mathrm{cov}(v_k) = R = 1$. Initial state is given by Gaussian pdf with mean $E\{\mathbf{x}_0\} = [0 \; 0.5]^{\mathrm{T}}$ and covariance matrix $\mathrm{cov}(\mathbf{x}_0) = \mathrm{diag}\{16, 0.02\}$.

The state is estimated by PM method with boundary-based grid design and splitting/merging technique. Results of the estimation are presented in Figure 1. The algorithm is started with one grid covering initial Gaussian pdf of state $\mathbf{x}_0$. Due to the quadratic nonlinearity in the measurement equation the grid is splitted just after the filtering step at time $k = 0$ to cover a support of two separable modes of the filtering pdf, as shown in the first graph of Figure 1. The two grids then develop separately until they overlap and fulfill the Mahalanobis distance criterion (23). Grids are merged at time $k = 12$. Filtering pdfs for $k = 11$ (before the merging) and for $k = 12$ (after the merging) are depicted in Figure 1 as well.

The computational demand of the algorithm was about three times lower than for the standard PM algorithm which used single grid and covariance-based grid design.

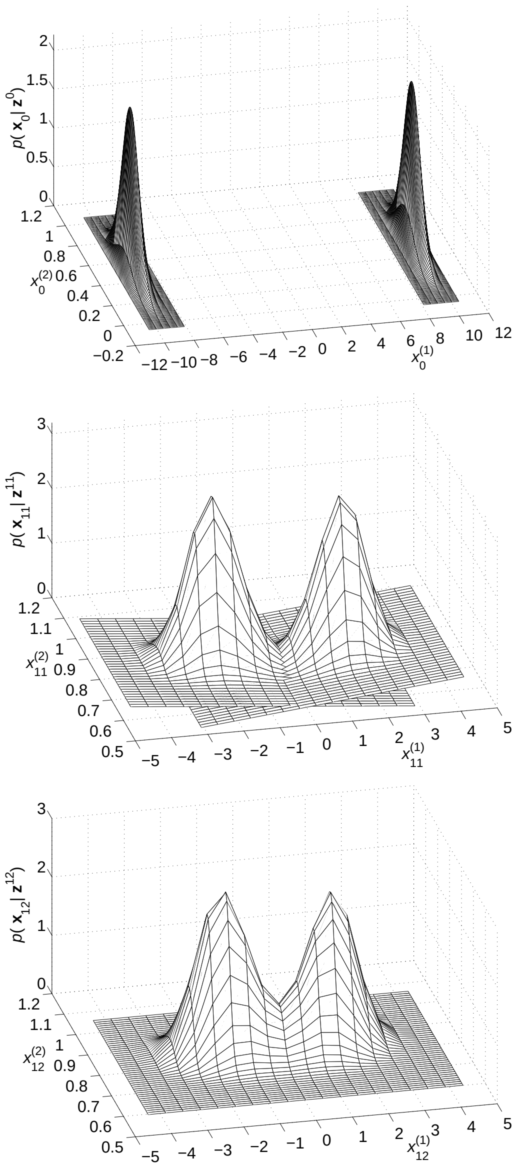

Fig. 1. Splitting and merging of pdfs.

7. CONCLUSIONS

A new boundary-based grid design for point-mass method was presented, which was more robust and for a general pdf it ensured a more effective covering of nonnegligible subspace of the state space than the standard technique based on estimation of predictive covariance matrix of state. Further, handling multiple grids in the state space was improved by introducing Mahalanobis distance criterion for merging of grids. The presented techniques enable an effective usage of PM method for systems with multimodal pdfs of state.

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