OSCILLATIONS ANALYSIS IN NONLINEAR VARIABLE-STRUCTURE SYSTEMS WITH SECOND-ORDER SLIDING-MODES AND DYNAMIC ACTUATORS

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Abstract: A class of uncertain systems nonlinear in the input variable and driven by a dynamic actuator device is dealt with. We give sufficient conditions under which the feedback controller based on the "Sub-optimal" second-order sliding-mode control algorithm can guarantee the attainment of a boundary layer of the sliding manifold. The relationship between the actuator parameters and the size of the boundary layer is investigated. We discuss about the possible ways for estimating, or even shaping, the parameters of the periodic limit cycles that may occur in the steady-state within the boundary layer. A simulation example is given that confirm the results of the proposed analysis

Keywords: Sliding-mode control, variable structure systems, limit cycles, uncertain systems, nonlinear systems.

1. INTRODUCTION

It is known (Anosov, 1959) that in the presence of parasitic dynamics (provisionally associated with actuator devices) the standard 1-SMC approach leads to high-frequency oscillations around the sliding manifold (chattering). This conclusion were confirmed via the describing function analysis (Shtessel and Young-Ju, 1996), the "locus of a perturbed relay system" (LPRS) analysis (Boiko, 2003), Tsypkin locus (Tzipkin, 1984), and singular perturbation analysis (Fridman, 2001).

Second order sliding mode control (2-SMC) theory has been an area of activity of many researchers and practitioners over the last decade (see (Levant, 2003), (Shtessel, 2003), and references therein).

2-SMC algorithms performance with dynamic actuators was analyzed in recent years. The "Twisting" algorithm in the system composed of a linear plant and a linear unmodelled actuator was analyzed in (Boiko, 2004). Necessary conditions for the existence of periodic motions in a small boundary layer of the sliding manifold, and a methodology for computing the parameters of those motions, were obtained via the describing function analysis and the "modified Tsypkin locus".

In the present work we make reference to uncertain systems nonlinear in the input variable and a class of non-linear dynamic actuators. The aim of this paper is to present sufficient conditions under which the trajectory of the system enters an invariant vicinity of the sliding manifold. We also discuss about possible ways for estimating, or even shaping by proper setting of the controller, the parameters of the periodic limit cycles that may occur in the steady-state within the boundary layer. Simulation results are given.

2. PROBLEM FORMULATION

We consider the nonlinear single-input system

$$\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x}, z_1) \tag{1}$$

with state vector $\mathbf{x} \in X \subset R^n$ and scalar control input $z_1 \in Z_1 \subset R$ not accessible for direct modification due to the dynamical fast actuator

$$\mu\dot{\mathbf{z}} = \mathbf{b}(\mathbf{z}, \mathbf{x}, u), \tag{2}$$

where $\mathbf{z} = [z_1, z_2, \ldots, z_m] \in Z \subset R^m$ and $u \in U \subset R$ are the actuator′s state and input, respectively, and $\mu > 0$.

Let $\mathbf{a} : X \times Z_1 \rightarrow R^n$ and $\mathbf{b} : Z \times X \times U \rightarrow R^m$ be unknown vector-fields satisfying proper growth and smoothness constraints to be specified.

We consider, as control task, the finite-time vanishing of the output variable

$$s_1 = s_1(\mathbf{x}) : X \rightarrow R \tag{3}$$

Assume that the following conditions hold globally

$$\frac{\partial}{\partial z_1}\dot{s_1} = 0, \frac{\partial}{\partial z_1}\ddot{s_1} \neq 0 \tag{4}$$

which imply that the "sliding variable" s_1 has a well–defined relative degree $r = 2$ with respect to the plant input variable z_1.

Then, it is always possible (Isidori, 1995) to define a vector $w \in W \subset \mathbf{R}^{n-2}$ and a map $\Phi : R^n \rightarrow R^n$ such that

$$\mathbf{x} = \mathbf{\Phi}(\mathbf{w}, s_1, \dot{s}_1) \tag{5}$$

is a diffeomorfism on $W \times R^2$ preserving the origin. The w dynamics are generally referred to as the "internal dynamics" (Isidori, 1995).

The input-output, internal and actuator's dynamics form the following overall dynamical system

$$\dot{s}_1 = s_2 \tag{6}$$

$$\dot{s}_2 = \mathbf{f}(\mathbf{w}, s_1, s_2, z_1) \tag{7}$$

$$\dot{\mathbf{w}} = \mathbf{g}(\mathbf{w}, s_1, s_2) \tag{8}$$

$$\mu\dot{\mathbf{z}} = \mathbf{h}(\mathbf{z}, \mathbf{w}, s_1, s_2, u) \tag{9}$$

where functions $\mathbf{f} : W \times S \times Z_1 \rightarrow R$, $\mathbf{g} : W \times S \rightarrow R^{n-2}$ and $\mathbf{h} : Z \times W \times U \rightarrow R^m$ are smooth functions of their arguments. Note that if $n = 2$ there are no internal dynamics.

Assume what follows:

Assumption 1: *The internal dynamics* (8) *are input-to-state stable* (*ISS*)

Assumption 2: *The zero-dynamics* $\dot{\mathbf{w}} = \mathbf{g}(\mathbf{w}, 0, 0)$ *are globally asymptotically stable (GAS) at the origin.*

Assumption 3: *There exist positive constants* z^*, $\overline{F}$, $\underline{F}$, *with* $\overline{F} < 3\underline{F}$, *such that if* $|z_1| = z^*$ *then, whenever* $(\mathbf{w}, s_1, s_2) \in (W \times R^2)$, *the following conditions hold*

$$0 < \underline{F} \leq sign(z_1)\mathbf{f}(\mathbf{w}, s_1, s_2, z_1) \leq \overline{F} \tag{10}$$

Assumption 4: *Let* u^* *be an arbitrary constant. If* $u(t) = u^*$ *at* $t \geq t_0$ *then* $\forall \varepsilon > 0$ *there exists* $N = N(\varepsilon, u^*)$ *s.t.*

$$|z_1 - u^*| \leq \varepsilon \qquad \forall t \geq t_0 + N\mu \tag{11}$$

Assumption 5: *The magnitude of the actuator output* z_1 *is globally bounded by the constant* $\overline{Z}$.

Assumption 3 guarantees that the plant input z_1 affects the sign of $\dot{s}_2$ and has a sufficient "dominance" over the system uncertain dynamics. It gives semi-global validity to the attained result since in order to evaluate a proper value for z^* bounds on the magnitude of system variables must be known a-priori. Assumption 4 establishes that the actuator step-response has a settling time that is directly proportional to the μ parameter.

We focus this work on the analysis of the closed loop system (6)-(9) with the Suboptimal 2-SMC algorithm (Bartolini and Ferrara, 2001)

$$u(t) = -z^* sign\,(s_1 - \beta s_{M1}) \qquad \beta = \frac{1}{2} \tag{12}$$

where s_{M1} is the last "singular point" of s_1 (i.e. the frozen value of s_1 at the most recent time instant at which $\dot{s}_1 = 0$)

3. MAIN RESULT

We show that under Assumptions 1-5 the trajectories of the considered class of plants with the Sub-optimal 2-SMC control algorithm (12) enter in finite time a bounded invariant domain containing the so-called "second order sliding set" $s_1 = s_2 = 0$. We investigate, in particular, how the

μ parameter appearing in the actuator dynamics affect the size of the attracting, invariant, domain.

Theorem 1. *Consider system* (6)-(9), *satisfying Assumptions 1-5, driven by the Sub-Optimal controller* (12). *Then, the closed loop system trajectories enter in finite time the domain*

$$O_\mu \equiv \left\{(s_1, s_2) : |s_1| \le \rho_0^* \mu^2, |s_2| \le \rho_1^* \mu\right\} \quad (13)$$

where ρ_0^* and ρ_1^* are positive constants independent of μ, to stay there afterwards.

Proof. Let $t = t_0$ be the initial time instant (we get $t_0 = 0$ without loss of generality). The proof is organized in three steps:

STEP 1 There exist a time instant $t = t_{M1} \ge 0$ at which $s_2(t_{M1}) = 0$. The point $P_1 \equiv (s_1(t_{M1}), s_2(t_{M1})) \equiv (s_1^*, 0)$ is referred to as the first "singular point" on the $s_1 - s_2$ phase plane.

By the definition of s_{M1} it follows that whenever $0 \le t < t_{M1}$ then

$$s_{M1} = s_1(0), \quad sign(\dot{s}(t)) = sign(\dot{s}(0)) \quad (14)$$

and the actuator's input u is given by

$$u = -z^* sign\left(s_1 - \frac{1}{2}s_2(0)\right), \quad 0 \le t < t_{M1} \quad (15)$$

Define

$$V_1 = |s_1| \qquad V_2 = |s_2| \quad (16)$$

If $s(0)\dot{s}(0) \ge 0$ then taking into account (14) it follows that in the time interval $0 \le t < t_{M1}$ condition $\dot{V}_1 > 0$ holds, which implies that $|s_1|$ is increasing so that s_1 will not change sign and, moreover, condition $s_1(t) = \frac{1}{2}s_1(0)$ is never satisfied. Thus we get

$$u = -z^* sign\left(s(0)\right) \qquad 0 \le t < t_{M1} \quad (17)$$

Considering the Assumption 4 it can be concluded there exist $k_1 \ge 0$ such that

$$\dot{V}_2 \le -k^2 \qquad k_1\mu \le t < t_{M1} \quad (18)$$

where k is a nonzero constant. Condition (18) implies that $|\dot{s}|$ is decreasing and, in turns, the existence of t_{M1}.

If $s(0)\dot{s}(0) < 0$ then from (14) it follows that $\dot{V}_1 < 0$ in the time interval $0 \le t \le t_{c0}$, where $t_{c0} \in [0, t_{M1})$ is the first time instant at which condition $s_1(t_{c0}) = (1/2)s_1(0)$ is met (existence of t_{c0} is proved below). Thus, the actuator input u is undergoing a discontinuity (of both its sign and magnitude) at $t = t_{c0}$. More precisely

$$u = -z^* sign\left(s(0)\right) \qquad 0 \le t < t_{c0} \quad (19)$$
$$u = z^* sign\left(s(0)\right) \qquad t_{c0} \le t < t_{M1} \quad (20)$$

Thus, again considering the Assumption 4, it follows that there exists $k_2 > 0$ such that

$$\dot{V}_2 \ge k^2 \qquad k_2\mu \le t \le t_{c0}$$

which implies, in turns, the existence of t_{c0}. Condition $s_1(t) = (1/2)s(0)$ will be never satisfied whenever $t \in (t_{c0}, t_{M1})$, then there exists $k_3 > 0$ such that for any $t \in (t_{c0} + k_3\mu, t_{M1})$

$$\dot{V}_2 \le -k^2 \qquad t_{c0} + k_3\mu \le t \le t_{M1},$$

which guarantees the existence of t_{M1} and concludes the proof of step 1.

STEP 2 A sequence of singular points (i.e points of the type $P_i \equiv (s_{1i}^*, 0)$, $i = 1, 2, \ldots$) is enforced at the time instants $t = t_{M_i}$, $i = 1, 2, \ldots$

Let $s_{1i} = s_1(t_{Mi})$, $\ddot{s}_{1i} = \ddot{s}_1(t_{Mi})$ and $z_{1i} = z_1(t_{Mi})$. By the definition of t_{Mi} the following condition holds

$$sign(s_2(t)) = sign(s_2(t_{M1}^+)) \;\; t_{M1}^+ \le t < t_{M2} \quad (21)$$

Let us assume, without loss of generality, that the first singular point P_1 lies in the right half-plane (i.e. $s_{11} > 0$). If z_{11} is such that $\ddot{s}_{11} > 0$ then $s_2(t_{M1}^+) > 0$ and the actuator input u is given by

$$u = -z^* \qquad t_{M1} \le t < t_{M2} \quad (22)$$

Considering Assumption 4 it follows from (22) that there exist $k_4 > 0$ such that

$$\dot{V}_2 \ge k^2 \qquad t_{M1} \le t < t_{M1} + k_4\mu \quad (23)$$
$$\dot{V}_2 \le -k^2 \qquad t_{M1} + k_4\mu \le t \le t_{M2} \quad (24)$$

that implies the convergence to zero of s_2, i.e. the finiteness of t_{M2}. If $z_{11} < 0$ then the actuator input u is given by

$$u = -z^* \qquad t_{M1} \le t < t_{c1} \quad (25)$$

where $t_{c1} > t_{M1}$ is the first time instant subsequent t_{M1} such that condition $s_1(t_{c1}) = (1/2)s_{11}$ holds.

From this point on condition $s_1(t) = (1/2)s_1(0)$ will be never satisfied whenever $t \in (t_{c1}, t_{M2})$. Thus we get

$$u = z^* \qquad t_{c1} \le t < t_{M2} \quad (26)$$

so that there exists $k_6 > 0$ such that

$$\dot{V}_2 < -k^2 \qquad t_{c1} + k_6\mu \le t \le t_{M2}$$

which implies, in turns, the existence of t_{M2}. By iteration the Step 2 is demonstrated.

STEP 3 There exist $\rho_0^* > 0$ such that whenever $|s_{1i}| > \rho_0^* \mu^2$ the next singular point P_{i+1} will satisfy the contraction condition

$$|s_{1,i+1}| \leq \gamma |s_{1i}| \tag{27}$$

Let (with no loss of generality) $s_{1i} > 0$, then since $z_{1i} < 0$ the worst-case for the purpose of our analysis is to consider the value

$$z = -z^* - \varepsilon \qquad \varepsilon > 0 \qquad t_{Mi} \leq t < t_{ci,} \tag{28}$$

which gives rise to an upper bound for the actual value of for $|s_2(t_{ci})|$. By integrating (6)- (7), taking into account Assumption 3, it follows that

$$|s_2(t_{ci})| \leq \sqrt{s_{1i}\overline{F}} \tag{29}$$

At the switching time instant $t = t_{ci}$ the sign of u changes from negative to positive according to

$$u = z^* \qquad t_{ci} \leq t < t_{M,i+1} \tag{30}$$

Considering (30) and (14), it follows that there exists $k_7 > 0$ such that

$$\dot{V}_1 < 0 \qquad t_{ci} \leq t < t_{M,i+1}, \tag{31}$$

$$\dot{V}_2 \geq k^2 \qquad t_{ci} + k_7\mu \leq t \leq t_{M,i+1} \tag{32}$$

The worst-case for the z_1 time evolution is

$$\begin{array}{ll} z_1 = -\overline{Z} & t_{ci} \leq t \leq t_{ci} + k_7\mu, \\ z_1 = z^* - \varepsilon & t_{ci} + k_7\mu \leq t \leq t_{M,i+1} \end{array} \tag{33}$$

Now considering the trajectories of (7) using, as input z_1, the worst-case values in (33), after some algebraic manipulations, it follows that the "arrival" point $s_{1,i+1}$ is such that

$$|s_{1,i+1}| \leq \frac{\overline{F}}{3\underline{F}}|s_{1i}| + \xi_0\mu\sqrt{|s_{1i}|} + \xi_1\mu^2 \tag{34}$$

for some positive ξ_0, ξ_1. It readily follows that condition $\overline{F} < 3\underline{F}$ is sufficient to guarantee the satisfaction of the "contraction condition" (27) within the domain $|s_{1i} \leq \rho\mu^2|$ for some $\rho > 0$. To show that $|s_2|$ will remain confined to a $O(\mu)$ domain, it suffices to note that when s_{1i} is $O(\mu^2)$ then $|s_2(t'_{ci})|$, which constitutes the maximum value of $|s_2|$ along the trajectory between the two successive singular points, is $O(\mu)$.

Condition (27), in turns, guarantees that the system trajectory crosses the axis $s_2 = 0$ of the $s_1 - s_2$ plane closer and closer to the origin, which implies the convergence toward the vicinity (13).

Due to the guaranteed contraction (27) occurring at each step, analogous arguments as those that were used in (Bartolini and Ferrara, 2001) allow us to establish that the sequence the convergence process towards the attracting set O_μ is guaranteed to take pace in a finite time. $\triangleright$

4. DF ANALYSIS OF THE STEADY-STATE TRAJECTORIES

One could argue that as it often happens in conventional relay-based control systems periodic limit cycles might take place within the boundary layer O_μ.

In the nonlinear setting the formal investigation of periodic trajectories existence and stability is often prohibitive due to the complex nonlinear implicit Poincare$'$ maps that arise, which constitute, unfortunately, the unique available exact analysis tool.

One can observe, however, that if in the steady state the nonlinear system has been steered close to some operating point, and the local linearization has low-pass characteristics, then it is possible to achieve useful informations regarding the periodic behaviours by means of a DF analysis of the linearized system dynamics. DF analysis (Atherton, 1984; Tzipkin, 1984) is a very simple method that gives, in most cases, sufficient informations regarding the existence of limit cycles and also gives the possibility to evaluate, approximately, the parameters (amplitude and frequency) of the limit cycle (if any).

Consider the feedback interconnection between a stable transfer function $W(s)$, that represents the linearized plant plus actuator dynamics, and the Sub-optimal controller (Fig. 1). The Sub-optimal 2-SMC (12) can be represented by an active hysteretic relay with time-varying hysteresis thickness $\frac{1}{2}s_{1M}$. If the actual steady-state behavior of the system is a periodic motion then s_{1M} would be an alternating (ringing) series of positive and negative values, so that the hysteresis thickness remains constant. Thus, classical frequency-based methods (Atherton, 1984; Tzipkin, 1984) become applicable for the analysis of the closed-loop system.

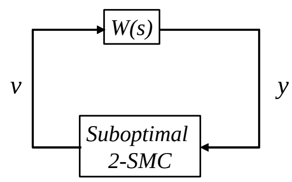

Fig. 1. The considered closed-loop control system

The expression of the DF of the negative-hysteresis relay is (Atherton, 1984)

$$q(A_y) = \frac{4c}{\pi A_y}\sqrt{1 - \frac{b^2}{A_y^2}} + j\frac{4bc}{\pi A_y^2} \tag{35}$$

where b is a half of the hysteresis ($b = s_{1M}/2$), c is the relay amplitude ($c = z_1^*$), and A_y is the amplitude of the harmonic input to the relay. Replacing b with $0.5A_y$, the following expression for the DF of the sub-optimal algorithm can be obtained:

$$q(A_y) = \frac{2c}{\pi A_y}\left(\sqrt{3}+j\right) \tag{36}$$

The periodic solution can be found if on the complex plane the negative reciprocal of the DF (36) intersects at some point the Nyquist plot of the transfer function W (Atherton, 1984). The negative reciprocal of the DF (36) is:

$$-\frac{1}{q} = -\frac{\pi A_y}{8c}\left(\sqrt{3}-j\right) \tag{37}$$

The locus (37) is depicted on the complex plane as a straight line backing from the origin with a 150^0 angle with the horizontal axis, as in Fig. 2. The periodic motion occurs if at some frequency $\omega = \overline{\omega}$ the phase characteristic of the actuator-plant transfer function is equal to -210^0. In that case the frequency and amplitude of the periodic solution can be derived from the "cross-over frequency" $\overline{\omega}$ and from the magnitude of vector $\overline{OA}$ in Fig. 2, respectively. It is easy to conclude that the intersection point A may only exist if the overall relative degree of the actuator and plant is higher than two.

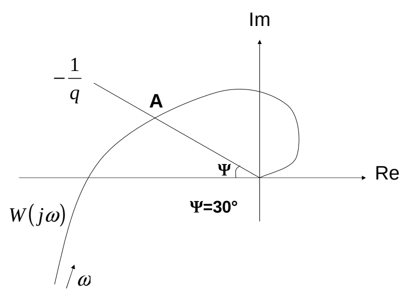

Fig. 2. Classical DF-analysis on the complex plane

If the so-called "Generalized Suboptimal" algorithm, i.e. the control law in (12) with $\beta \in [0.5, 1)$, is considered, then the DF (36) changes as follows:

$$q(A_y) = \frac{4c}{\pi A_y}\left(\sqrt{1-\beta^2}+j\beta\right) \tag{38}$$

which means that the Φ angle in Fig. 2 is now a function of β,

$$\Phi(\beta) = arcsin(\beta) \tag{39}$$

Note that $\Phi(1/2) = 30^0$. Even in those cases when the linearization error makes the actual oscillation parameters be very different from those predicted by DF analysis, we can derive a qualitative tuning guideline according to the following reasoning: if the plant has a low-pass characteristics, and the Nyquist plot of W is sufficiently "regular", then one can reasonably expect that increasing the anticipation factor β, i.e rotating the negative reciprocal of the DF (38) in the clock-wise direction towards the vertical axis, the frequency of the periodic oscillation increases and at the same time the amplitude decreases. The effectiveness of this tuning guideline is tested in the simulation section.

5. SIMULATION EXAMPLE

To validate the present analysis consider the following second-order system nonlinear in the input variable

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{x_2}{1+x_2^2} + [x_1 + (2+cos(z_1+x_2)z_1]^3 \end{aligned} \tag{40}$$

with the second-order actuator

$$\begin{gathered} \mu\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} + \mathbf{D}u, \qquad \mathbf{z} = [z_1, z_2]^T \\ \mathbf{C} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{gathered} \tag{41}$$

Let only x_1 be available for measurement. If the sliding variable is defined as $s_1 = x_1$ then it has relative degree two with respect to the plant input z_1 and, moreover, there are no zero-dynamics. It yields that the overall plant plus actuator dynamics meets Assumptions 1-5. Note that since both s_1 and its derivative s_2 are steered to a neighbourhood of zero (this is main advantage of the 2-SMC approach), it is possible to achieve the practical stabilization of the full state vector without measuring explicitly x_2.

The initial conditions are $[x_1(0), x_2(0)] = [1, 1]$ and $[z_1(0), z_2(0)] = [0, 0]$. The chosen discontinuous control magnitude of the Sub-optimal controller (12) is $z^* = 2$. In a first test (Test 1) the actuator's time constant was set to $\mu = 10^{-2}s$. Figs. 3 show that the sliding variable x_1 and its derivative x_2 converge to a bounded neighborhood of the second-order sliding domain $x_1 = x_2 = 0$.

To check whether the accuracy order (13) is actually achieved we performed a second test in which the actuator's time constant is reduced by a factor 2 ($\mu = 0.005s$). By comparing Figs. 3 with Figs. 4 it is apparent that the sliding accuracy is improved by a factor 4, as for the sliding variable $s = x_1$, and by a factor 2 as for its derivative $\dot{s} = x_2$, and this is according to the expected precision order.

We repeated the Test 1 using two different values of the Sub-optimal "Anticipation factor" β. The expected effects of increasing the oscillation

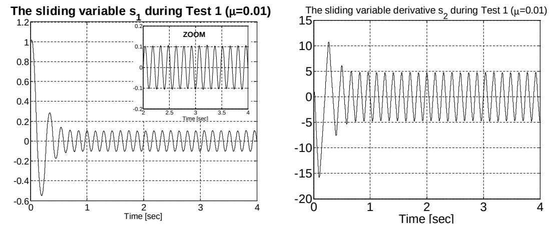

Fig. 3. The sliding variable s_1 (left) and its derivative s_2 (right) in Test 1 ($\mu = 0.01$)

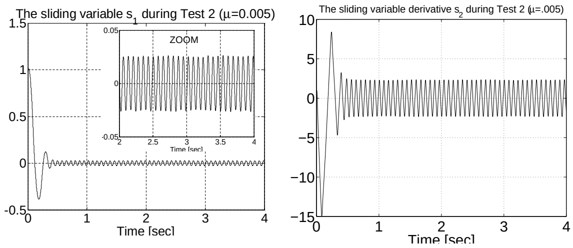

Fig. 4. The sliding variable s_1 (left) and its derivative s_2 (right) in Test 2($\mu = 0.05$)

frequency and, at the same time, reducing its amplitude is apparent from the analysis Fig. 5.

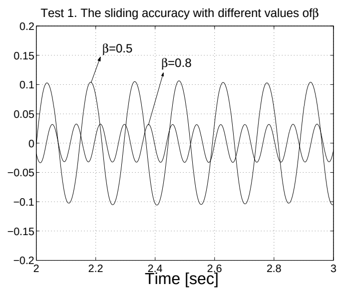

Fig. 5. The sliding variable s_1 during Test 1 with different values of β.

6. CONCLUSIONS

Stability and periodic limit cycles of nonlinear control systems with with dynamic actuators and the Suboptimal sliding-mode control (2-SMC) algorithm have been investigated. Sufficient conditions were given guaranteeing that the oscillations converge to a small vicinity of the second order sliding mode domain. It has been shown that a sensible way to infer indications about the periodic limit cycles existence is to make a local linearization followed by proper DF-like analysis. On the basis of qualitative considerations it is argued, and checked by simulations, that increasing the anticipating factor of the generalized suboptimal algorithm near to the unit value one can reduce the magnitude, and increase the frequency, of the steady-state periodic limit cycle. It must be noted that the larger the value of β the slower the reaching transient, hence the on-line adaptation of β needs to be investigated.

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