SEPARATION APPROACH FOR NUMERICAL SOLUTION OF THE FOKKER-PLANCK EQUATION IN ESTIMATION PROBLEM

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Abstract: The paper deals with a filter design for nonlinear continuous stochastic systems with discrete-time measurements. The general recursive solution is given by the Fokker-Planck equation (FPE) and by the Bayesian rule. The stress is laid on computation of the predictive conditional probability density function from FPE. A new usable numerical scheme is designed. In the scheme, the separation technique based on an upwind volume method and a finite difference method for hyperbolic and parabolic part FPE is used. The approach is illustrated in some numerical examples.

Keywords: stochastic systems, state estimation, nonlinear filters, Fokker-Planck equation, numerical solutions, finite volume method, finite difference method.

1. INTRODUCTION

The problem of state estimation for nonlinear continuous stochastic systems with discrete-time measurements is of special interest with respect to real continuous processes and the digital devices used to processing measurements. A general solution is completely described by conditional probability density functions (pdf's) of state and it is given by direct manipulation of the Fokker-Planck equation (FPE) and the Bayesian rule (BR). The FPE (Risken, 1984) is a partial differential equation (PDE) that governs the evolution of the predictive pdf between the measurement time instants, the BR represents a correction of the previous predictive pdf at the measurement times. A closed-form solution of the state estimation problem is known only for linear Gaussian systems (Kalman and Bucy, 1961; Jazwinski, 1970) and a few special cases, e.g. exact finite dimensional filters based on the exponential pdf family (Daum, 1988), exact infinite-dimensional filters (Kouritzin, 1998) and the Gaussian sum filters (Šimandl and Švácha, 2002).

In other cases it is necessary to employ analytical or numerical approximations of the system or pdf's, e.g. system approximation (Jazwinski, 1970; Schmidt, 1993), moment approximation method of the conditional distribution (Kushner and Budhiraja, 2000), decomposing the nonlinear filter into time-consuming off-line and computationally efficient on-line components (Lototsky and Rozovskii, 1998), using the sequential Monte Carlo methods (del Moral and Jacod, 2001). These approximations work well in many applications but can be unsatisfactory or even unusable with respect to their features. Therefore other approaches to the solution of the state estimation problem are still quested and developed.

The solution of the FPE can be seen as a cornerstone for whole recursive computation. Extensive numerical simulations of the FPE have been performed using finite element methods (FEM's) (Mirkovic, 1996) or using of the Monte Carlo simulation (Spencer and Bergman, 1993). Further the finite difference methods (FDM's) (Press *et al.*, 1986) belong to standard numerical approaches to the solution of PDE's and thus can be also applied to approximation of the FPE. Nevertheless most of these approaches are focused on specific physical processes and don't correspond to direct manipulation in estimation algorithm.

The goal of the paper is to present a new usable and alternative numerical solution of the FPE in state estimation problem based on separation of the FPE into two parts. The aim is to solve the first hyperbolic part by upwind finite volume methods (FVM's) (LeVeque, 2002) and the second parabolic part by the standard FDM's (Press *et al.*, 1986). It is supposed that separation of the FPE and choice of a suitable numerical method for each part should achieve better estimation quality comparing to application of a single numerical method to unseparated FPE.

The paper is organized as follows: The problem formulation and general solution of the considered estimation problem is presented in Section 2. Section 3 is focused on the new numerical solution of the FPE. The results of the paper are illustrated in some numerical examples in Section 4.

2. PROBLEM STATEMENT

Consider the problem of state estimation where the state $\mathbf{x}(t)$ evolves in continuous time according to the Itô stochastic differential equation (SDE)

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), t)dt + \mathbf{G}(t)d\mathbf{w}(t) \tag{1}$$

and the measurement $\mathbf{z}_k$ is given as

$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k, t_k) + \mathbf{v}_k \tag{2}$$

where t is time, t_k are time instants for $k = 0, 1, 2, \ldots$, $\mathbf{x}(t)$ is state vector with $\dim \mathbf{x}(t) = n$ (it is used short notation $\mathbf{x}_k = \mathbf{x}(t_k)$), $\mathbf{z}_k$ represents measurement vector at time t_k with $\dim \mathbf{z}_k = m$, $\mathbf{f}(\mathbf{x}(t), t)$ and $\mathbf{h}(\mathbf{x}_k, t_k)$ are known vector functions, and $\mathbf{G}(t)$ is known $n \times n$ matrix. The process noise, $\mathbf{w}(t)$, is a $\mathcal{R}^n$-valued Brownian motion with $\mathsf{E}(d\mathbf{w}, d\mathbf{w}^T) = \mathbf{I}dt$. The measurement noise $\mathbf{v}_k$ is white and Gaussian with $\dim(\mathbf{v}_k) = m$, $\mathsf{E}(\mathbf{v}_k) = 0$ and $\mathrm{cov}(\mathbf{v}_k) = \mathbf{R}_k$ thus $p(\mathbf{v}_k) = \mathcal{N}(\mathbf{v}_k : 0, \mathbf{R}_k)$. The noises $\mathbf{w}(t)$, $\mathbf{v}_k$ and the random variable $\mathbf{x}(t_0)$ are mutually independent.

The aim is to determine the conditional filtering pdf $p(\mathbf{x}_k|\mathbf{z}^k)$ and predictive pdf $p(\mathbf{x}(t)|\mathbf{z}^k)$ for $t \in I_{k,k+1} \triangleq (t_k, t_{k+1}\rangle$ (i.e. for the measurement times $t_k < t \leq t_{k+1}$), where $\mathbf{z}^k \triangleq [\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k]^T$.

The general recursive solution of the filtering problem can be given by the Bayesian approach. The filtering pdf $p(\mathbf{x}_k|\mathbf{z}^k)$ at the measurement times represents a correction (update) of the previous predictive pdf $p(\mathbf{x}_k|\mathbf{z}^{k-1})$ and has the following form

$$p(\mathbf{x}_k|\mathbf{z}^k) = \frac{p(\mathbf{x}_k|\mathbf{z}^{k-1})p(\mathbf{z}_k|\mathbf{x}_k)}{\int p(\mathbf{x}_k|\mathbf{z}^{k-1})p(\mathbf{z}_k|\mathbf{x}_k)d\mathbf{x}_k} \tag{3}$$

where $p(\mathbf{x}_0|\mathbf{z}^{-1})$ is the prior pdf of the initial state $\mathbf{x}_0$.

The predictive pdf $p(\mathbf{x}(t)|\mathbf{z}^k)$ for $t \in I_{k,k+1}$ is given by the FPE

$$\begin{aligned}\frac{\partial p(\mathbf{x}(t)|\mathbf{z}^k)}{\partial t} &= -\frac{\partial p(\mathbf{x}(t)|\mathbf{z}^k)}{\partial \mathbf{x}(t)}\mathbf{f}(\mathbf{x}(t), t) \\ &\quad - p(\mathbf{x}(t)|\mathbf{z}^k)\,\mathrm{tr}\left(\frac{\partial \mathbf{f}(\mathbf{x}(t), t)}{\partial \mathbf{x}(t)}\right) \\ &\quad + \frac{1}{2}\,\mathrm{tr}\left(\mathbf{Q}(t)\frac{\partial^2 p(\mathbf{x}(t)|\mathbf{z}^k)}{\partial \mathbf{x}^2(t)}\right)\end{aligned} \tag{4}$$

with the initial condition $p(\mathbf{x}_k|\mathbf{z}^k)$, where $\frac{\partial p(\mathbf{x}(t)|\mathbf{z}^k)}{\partial \mathbf{x}(t)}$ is the gradient of $p(\mathbf{x}(t)|\mathbf{z}^k)$ with respect to $\mathbf{x}(t)$, $\frac{\partial \mathbf{f}(\mathbf{x}(t),t)}{\partial \mathbf{x}(t)}$ is the Jacobian of $\mathbf{f}(\mathbf{x}(t), t)$ with respect to $\mathbf{x}(t)$, tr denotes "trace", $\frac{\partial^2 p(\mathbf{x}(t)|\mathbf{z}^k)}{\partial \mathbf{x}^2(t)}$ is the Jacobian of the transpose of the gradient $\frac{\partial p(\mathbf{x}(t)|\mathbf{z}^k)}{\partial \mathbf{x}(t)}$ and $\mathbf{Q}(t) = \mathbf{G}(t)\mathbf{G}(t)^T$.

The key idea of most numerical approaches for generation of conditional pdf's of the state is to substitute a nonnegligible continuous support of the pdf by a grid of cells. Values of the pdf are computed at these grid cells only and thus the solution of (3), (4) is performed numerically over the grid instead of the continuous support. The nonnegligible support is a region in the state space where actual state is probable to lie and hence values of the pdf are nonnegligible there.

The basic numerical scheme can be summarized in the following recursive algorithm:

Initialization: Define an initial grid in $\mathcal{R}^n$ for the prior pdf $p(\mathbf{x}_0|\mathbf{z}^{-1})$:

$$P'_0[i_1, \ldots, i_n] = \hat{p}_{\mathbf{x}_0|\mathbf{z}^{-1}}\left(\mathbf{x}[i_1, \ldots, i_n]|\mathbf{z}^{-1}\right) \tag{5}$$

where $P'_0[i_1, \ldots, i_n]$ represents approximate value of pdf at $\mathbf{x}[i_1, \cdots, i_2]$. The grid is rectangular with $N_1 \times N_2 \ldots \times N_n$ cells, $\Delta x_1, \ldots, \Delta x_n$ are sizes of the cells. The grid is parallel with coordinate axes.

Step 1: At time t_k compute values of the approximate filtering pdf $\hat{p}(\mathbf{x}_k|\mathbf{z}^k)$ at grid cells using

$$\begin{aligned}P_k[i_1, \ldots, i_n] &= \hat{p}_{\mathbf{x}_k|\mathbf{z}^k}\left(\mathbf{x}[i_1, \ldots, i_n]|\mathbf{z}^k\right) = \\ &= c'^{-1}_k P'_k[i_1, \ldots, i_n] p_{\mathbf{v}_k}\left(\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}[i_1, \ldots, i_n])\right)\end{aligned} \tag{6}$$

where

$$c'_k = \sum_{i_1=1}^{N_1} \cdots \sum_{i_n=1}^{N_n} \Delta\, \mathbf{x} P'_k[i_1,\ldots,i_n] \cdot \tag{7}$$
$$\cdot\; p_{\mathbf{v}_k}\left(\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}[i_1,\ldots,i_n])\right)$$

and $\Delta\mathbf{x} = \Delta x_1 \Delta x_2 \ldots \Delta x_n$.

Consider $t_j = t_k$.

Step 2: Define a new suitable grid in $\mathcal{R}^n$ similarly to initialization step for the predictive pdf $p(\mathbf{x}(t)|\mathbf{z}^k$ for $t_j \in I_{k,k+1}$, where $j = 0,\ldots,M$ (i.e. at the time instants $t_k, t_k+\Delta t, t_k+2\Delta t,\ldots,t_k+(M-1)\Delta t, t_{k+1}$), Δt is time step

$$P'_j[i_1,\ldots,i_n] = \hat{p}_{\mathbf{x}_j|\mathbf{z}^k}\left(\mathbf{x}[i_1,\ldots,i_n]|\mathbf{z}^k\right). \tag{8}$$

Step 3: Compute values $P'_j[i_1,\ldots,i_n]$ of the approximate predictive pdf $p(\mathbf{x}(t)|\mathbf{z}^k$ for $t_j \in I_{k,k+1}$ using a suitable numerical method for the FPE (4).

Let $k \longleftarrow k+1$ and continue with **Step 1**.

The given algorithm provides only basic frame. The solution of the FPE (Step 2 and Step 3) is the cornerstone for whole recursive computation. The next section is focused on these two steps of the recursive estimation algorithm within new usable solution of the FPE.

3. NEW NUMERICAL SOLUTION OF THE FOKKER-PLANCK EQUATION

The basic idea of the numerical solution of the FPE (4) is to see the FPE as a composition of a parabolic and a hyperbolic part, to consider them separately and subsequently to choose an efficient method for solution of each part. The first hyperbolic part

$$\frac{\partial p(\mathbf{x}(t)|\mathbf{z}^k)}{\partial t} = -\frac{\partial p(\mathbf{x}(t)|\mathbf{z}^k)}{\partial \mathbf{x}(t)}\mathbf{f}(\mathbf{x}(t),t) \tag{9}$$

is solved by upwind FVM's (LeVeque, 2002). Upwind schemes based on FVM's represent powerful class of numerical methods for hyperbolic PDE's. The second parabolic part

$$\frac{\partial p(\mathbf{x}(t)|\mathbf{z}^k)}{\partial t} = -\,p(\mathbf{x}(t)|\mathbf{z}^k)\,\mathrm{tr}\left(\frac{\partial \mathbf{f}(\mathbf{x}(t),t)}{\partial \mathbf{x}(t)}\right) + \frac{1}{2}\,\mathrm{tr}\left(\mathbf{Q}(t)\frac{\partial^2 p(\mathbf{x}(t)|\mathbf{z}^k)}{\partial \mathbf{x}^2(t)}\right) \tag{10}$$

takes results from (9) and is solved by implicit scheme FDM's (Press *et al.*, 1986).

Now the Steps 2 and 3 from the basic algorithm considered in Session 2 will be designed.

3.1 Separation approach for one-dimensional system

Step 2: Consider dividing the nonnegligeble support of the filtering pdf $p(x_k|z^k)$ into N intervals (grid cells) $x[i]$, $\Delta x = x[i+1/2] - x[i-1/2]$, $i = 1,\ldots,N$. The value $P_k[i]$ approximates the average of the pdf $p(x_k|z^k)$ over i-th interval at time t_k and also represents approximate value of the pdf at $x[i]$:

$$P_k[i] = \hat{p}_{x_k|z^k}\left(x[i]|z^k\right) = \tag{11}$$
$$= \frac{1}{\Delta x}\int_{x[i-1/2]}^{x[i+1/2]} p(x_k|z^k)dx_k$$

Step 3: The values $P_k[i]$ (11) are the initial condition for numerical solution of the FPE (4) for $t_j \in I_{k,k+1}$, where $j = 0,\ldots,M$.

An explicit algorithm for the hyperbolic part FPE (9) can be developed (LeVeque, 2002):

$$P'_{j+1}[i]^* = P'_j[i] - \frac{\Delta t}{\Delta x}\left(F_j[i+1/2] - F_j[i-1/2]\right) \tag{12}$$

where flux $F_j[i+1/2]$ is an approximation of

$$f(x[i],t)p(x[i+1/2],t|z_k)$$

over time step Δt:

$$F_j[i+1/2] \approx \tag{13}$$
$$\approx f\left(x[i],t\right)\frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} p\left(x[i+1/2],t|z^k\right)dt.$$

The values $P'_{j+1}[i]^*$ are modified at each time step t_j by (12) through the endpoints of the intervals. The specific variant of FVM depends on how are chosen $F_j[i+1/2]$ and $F_j[i-1/2]$, e.g.:

$$F_j[i+1/2] = \tag{14}$$
$$= \frac{1}{2}[f\left(x_j[i],t_j\right)\left(P'_j[i+1] + P'_j[i]\right)$$
$$-|f\left(x_j[i],t_j\right)|\left(P'_j[i+1] - P'_j[i]\right)],$$

$$F_j[i-1/2] = \tag{15}$$
$$= \frac{1}{2}[f\left(x_j[i],t_j\right)\left(P'_j[i] + P'_j[i-1]\right)$$
$$-|f\left(x_j[i],t_j\right)|\left(P'_j[i] - P'_j[i-1]\right)]$$

Then the scheme (12)-(15) is an upwind FVM with first-order accuracy. In order for the explicit scheme to be stable, the condition

$$|f(x_j[i],t_j)\frac{\Delta t}{\Delta x}| \le 1 \tag{16}$$

has to be satisfied for $i = 1,\ldots,N$ and $t_j \in I_{k,k+1}$. A more usable approximation of (15) can be found in LeVeque (2002).

Figures 1, 2 illustrate time and state discretisation (grid cells) for numerical solution of the FPE based on MFV's.

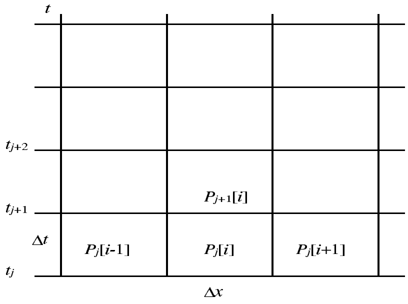

Fig. 1. Grid cells for numerical solution of the FPE.

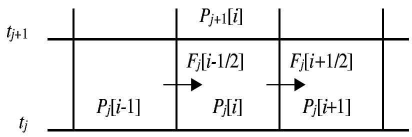

Fig. 2. Upwind finite volume method - updating the cell average by fluxes at the cell edges.

Finally, the classical FDM can be used for parabolic part (10) of the FPE. The discrete implicit scheme represents matrix equation of N-th order and is unconditionally stable

$$\frac{P'_{j+1}[i] - P'_{j+1}[i]^*}{\Delta t} = \tag{17}$$
$$-\left.\frac{\partial f\left(x(t), t\right)}{\partial x(t)}\right|_{x=x(t);t=t_{j+1}} P'_{j+1}[i]$$
$$+\frac{1}{2}Q(t_{j+1})\frac{P'_{j+1}[i+1] - 2P'_{j+1}[i] + P'_{j+1}[i-1]}{\Delta x^2}.$$

3.2 Separation approach for n-dimensional system

Given scheme for the 1-dimensional system can be extended to higher dimension:

Step 2: Consider dividing the nonnegligeble support of the filtering pdf $p(\mathbf{x}_k|\mathbf{z}^k)$ into $N_1 \times N_2 \ldots \times N_n$ grid cells $\mathbf{x}[i_1, \ldots, i_n]$. The value $P_k[i_1, \ldots, i_n]$ approximates the average of the pdf $p(\mathbf{x}_k|\mathbf{z}^k)$ over $[i_1, \ldots, i_n]$ cell at time t_k and also represents approximate value of the pdf at $\mathbf{x}[i_1, \ldots, i_n]$:

$$P_k[i_1, \ldots, i_n] = \hat{p}_{\mathbf{x}_k|\mathbf{z}^k}\left(\mathbf{x}[i_1, \ldots, i_n]|\mathbf{z}^k\right) = \tag{18}$$
$$= \frac{1}{\Delta \mathbf{x}} \int_{x[i_1-1/2, i_2, \ldots, i_n]}^{x[i_1+1/2, i_2, \ldots, i_n]} \cdots$$
$$\cdots \int_{x[i_1, i_2, \ldots, i_n-1/2]}^{x[i_1, i_2, \ldots, i_n+1/2]} p(\mathbf{x}_k|\mathbf{z}^k) d\mathbf{x}_k$$

Step 3: The values $P_k[i_1, \ldots, i_n]$ (18) are the initial condition for numerical solution of the FPE (4) for $t_j \in I_{k,k+1}$.

Figure 3 illustrates time discretisation and state discretisation (grid cells) for numerical solution of FPE ($n = 2$).

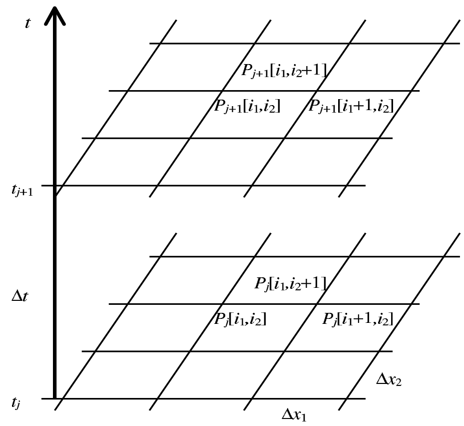

Fig. 3. Grid cells ($n = 2$) for numerical solution of the FPE.

An explicit algorithm for the hyperbolic part FPE (9) can be developed (LeVeque, 2002):

$$P'_{j+1}[i_1, \ldots, i_n]^* = P'_j[i_1, \ldots, i_n] - \tag{19}$$
$$-\sum_{l=1}^{n} \frac{\Delta t}{\Delta x_l}(F_j^l[i_1, \ldots, i_l + 1/2, \ldots, i_n] -$$
$$-F_j^l[i_1, \ldots, i_l - 1/2, \ldots, i_n])$$

where

$$F_j^l[i_1, \ldots, i_l + 1/2, \ldots, i_n] \approx \tag{20}$$
$$\approx f_l\left(\mathbf{x}[i_1, \ldots, i_l, \ldots, i_n], t\right) \cdot$$
$$\cdot \frac{1}{\Delta t} \int_{t_j}^{t_{j+1}} p\left(\mathbf{x}[i_1, \ldots, i_l + 1/2, \ldots, i_n], t|\mathbf{z}^k\right) dt$$

is an approximation.

Thus the values $P'_{j+1}[i_1, \ldots, i_n]^*$ are modified at each time step t_j by (19). For example simple numerical approximations of F_j^l are

$$F_j^l[i_1, \ldots, i_l + 1/2, \ldots, i_n] = \tag{21}$$
$$= \frac{1}{2}[f_1\left(x_j[i_1, \ldots, i_l, \ldots, i_n], t_j\right) \cdot$$
$$\cdot \left(P'_j[i_1, \ldots, i_l + 1, \ldots, i_n] + P'_j[i_1, \ldots, i_n]\right)$$
$$-|f_l\left(x_j[i_1, \ldots, i_l, \ldots, i_n], t_j\right)| \cdot$$
$$\cdot \left(P'_j[i_1, \ldots, i_l + 1, \ldots, i_n] - P'_j[i_1, \ldots, i_n]\right)]$$

$$F_j^l[i_1,\ldots,i_l-1/2,\ldots,i_n]= \tag{22}$$
$$=\frac{1}{2}[f_1\left(x_j[i_1,\ldots,i_l,\ldots,i_n],t_j\right)\cdot$$
$$\cdot\left(P_j'[i_1,\ldots,i_n]+P_j'[i_1,\ldots,i_l-1,\ldots,i_n]\right)$$
$$-|f_l\left(x_j[i_1,\ldots,i_l,\ldots,i_n],t_j\right)|\cdot$$
$$\cdot\left(P_j'[i_1,\ldots,i_n]-P_j'[i_1,\ldots,i_l-1,\ldots,i_n]\right)].$$

The scheme (19)-(22) is an upwind FVM with first-order accuracy. In order for the explicit scheme to be stable, the condition

$$\sum_{l=1}^{n}|f_l(\mathbf{x}_j[i_1,\ldots,i_n],t_j)|\frac{\Delta t}{\Delta x_l}\leq 1 \tag{23}$$

has to be satisfied for $i_1=1,\ldots,N_1$, $i_2=1,\ldots,N_2$ ... $i_n=1,\ldots,N_n$ and $t_j\in I_{k,k+1}$.

The classical FDM is used for parabolic part (10) of the FPE. The discrete implicit scheme represents matrix equation of N-th (where $N=N_1\cdot N_2\cdots N_n$) order and is unconditionally stable

$$\frac{P_{j+1}'[i_1,\ldots,i_n]-P_{j+1}'[i_1,\ldots,i_n]^*}{\Delta t}= \tag{24}$$
$$-\sum_{l=1}^{n}\frac{\partial f_l\left(\mathbf{x}(t),t\right)}{\partial x_l(t)}\bigg|_{\mathbf{x}=\mathbf{x}(t);t=t_{j+1}}P_{j+1}'[i_1,\ldots,i_n]$$
$$+\frac{1}{2}\sum_{l=1}^{n}\mathbf{Q}_{l,l}(t_{j+1})\cdot\frac{P_{j+1}'[i_1,\ldots,i_l+1,\ldots,i_n]-..}{\Delta x_l^2}$$
$$\frac{..-2P_{j+1}'[i_1,\ldots,i_n]+P_{j+1}'[i_1,\ldots,i_l-1,\ldots,i_n]}{}.$$

From (19) and (24) it is noticeable that computational complexity of the algorithm grows exponentially with increasing state dimension for given accuracy. Also suitable design of grid cells and condition of stability (23) are crucial for stable and time optimal computation at all time instants $t_j\in I_{k,k+1}$. Basic idea for efficient design of grid cells can be considered similarly as in the point-mass approach (Šimandl *et al.*, 2002) and then substantial reduction of numerical demand can be achieved.

4. NUMERICAL ILLUSTRATION

To show different performance of the FDM (Press *et al.*, 1986) and new separation approach (SA) the linear non-gaussian system is considered

$$dx(t)=0.1x(t)dt+dw(t)$$
$$z_k=2x_k+v_k$$

with t_k ($t_0=0s, t_1=0.1s, t_2=0.2s,\ldots$), the prior pdf $p(x_0|z^{-1})=\mathcal{N}(x_0:-2,1)$ and $p(v_k)=0.5\mathcal{N}(v_k:0.5,0.1)+0.5\mathcal{N}(v_k:2,0.2)$. The new SA and the classical implicit FDM filter with grid parameters ($\Delta x=0.1, x\in\langle-10,10\rangle, N=200$) and $\Delta t=0.01$ are designed. The aim is to compare quality of these filters with the exact filtering pdf produced by the Gaussian sum filter (Šimandl and Švácha, 2002).

Comparison of time evolution of

$$J(t)=\int(\hat{p}(x(t)|z^k)-p(x(t)|z^k))^2dx$$

for the FDM and the SA is in Figure 4. It can be seen that estimate quality of the SA is better than that of the FDM.

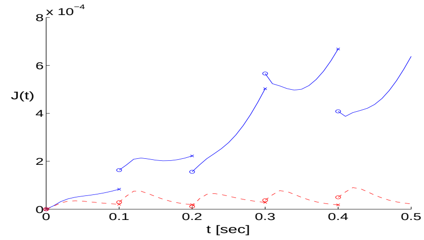

Fig. 4. Time evolution of $J(t)$ for the FDM (solid) and the SA (dashed), J_k (circle), J_k' (× mark).

4.1 Example 2

To show the extension of the SA to higher dimension, the following continuous stochastic process $\mathbf{x}(t)$ observed at discrete time instants t_k ($t_0=0s, t_1=0.2s, t_2=0.4s,\ldots$)

$$dx_1(t)=(-0.05x_1(t)+x_2(t))dt+dw_1(t)$$
$$dx_2(t)=(0.7x_1(t)-0.4x_2(t))dt+dw_2(t)$$
$$z_1(t_k)=x_1(t_k)+x_2(t_k)+v_1(t_k)$$
$$z_2(t_k)=x_2(t_k)+v_2(t_k)$$

with the pdf's

$$p(\mathbf{x}_0|\mathbf{z}^{-1})=0.5\mathcal{N}\left(\mathbf{x}_0:\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0.1&0\\0&0.1\end{bmatrix}\right)$$
$$+0.5\mathcal{N}\left(\mathbf{x}_0:\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}0.1&0\\0&0.1\end{bmatrix}\right),$$
$$p(\mathbf{v}_k)=0.5\mathcal{N}\left(\mathbf{x}_0:\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0.1&0\\0&0.1\end{bmatrix}\right)$$
$$+0.5\mathcal{N}\left(\mathbf{x}_0:\begin{bmatrix}-1\\1\end{bmatrix},\begin{bmatrix}0.1&0\\0&0.1\end{bmatrix}\right)$$

is considered. The new SA filter with grid parameters ($\Delta x_1=0.2, \Delta x_2=0.2, x_1\in\langle-2,5\rangle, x_2\in\langle-2,5\rangle, N_1=35, N_2=35$) and $\Delta t=0.005$ is designed. The development of the state estimate $\hat{\mathbf{x}}(t)$ and the filtering pdf $\hat{p}(\mathbf{x}_k|\mathbf{z}^k)$ generated by the SA filter are shown in Figures 5 and 6, respectively.

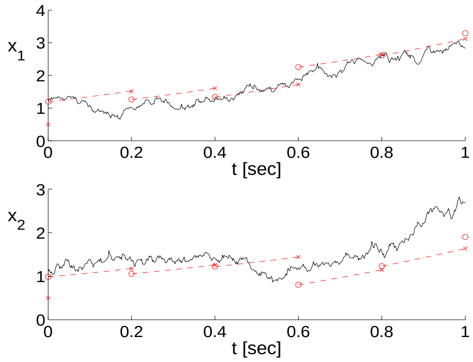

Fig. 5. Development of the state $\mathbf{x}(t)$ (solid) and its point estimates $\hat{\mathbf{x}}_k$ (circle), $\hat{\mathbf{x}}'_k$ ($\times$ mark), $\hat{\mathbf{x}}'(t)$ (dashed) for $t \in I_{k,k+1}$.

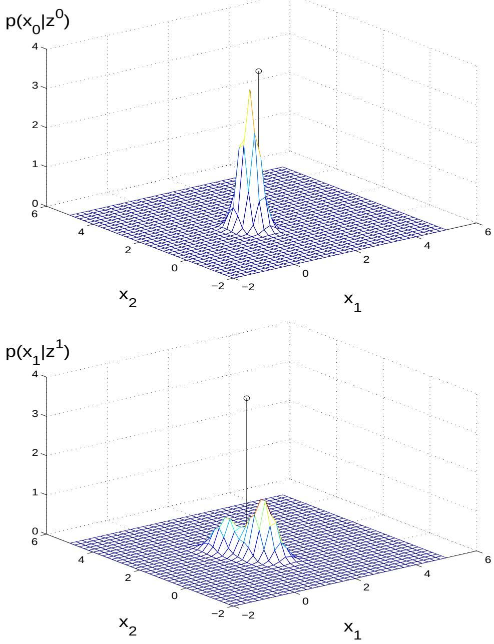

Fig. 6. Development of the filtering pdf $\hat{p}(\mathbf{x}_k|\mathbf{z}^k)$ for $k = 0, 1$ and $\mathbf{x}(t_k)$ (circle).

5. CONCLUSION

A new separation approach for numerical solution of the FPE was designed. The approach is based on separation of the FPE into hyperbolic and parabolic parts and application of efficient numerical methods to each of them. The hyperbolic part is solved by an explicit FVM and the results are used in an implicit FDM for the parabolic part. The scheme has simple implementation and the extension to higher dimension is straightforward. Comparing to classical implicit FDM the approach can produce results with higher estimation quality.

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