THE CONTROLLABILITY TEST FOR BEHAVIORS REVISITED

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Abstract: Let $\mathfrak{B} = \{w \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid R(\frac{d}{dt})w = 0\}$. It is well-know that $\mathfrak{B}$ is controllable if and only if $R(\lambda)$ has the same rank for all complex λ. We want to re-examen the proof of this fundamental result. Denote by $\mathfrak{B}^\infty$ the behavior $\mathfrak{B}$ intersected with set of smooth functions. For $\mathfrak{B}^\infty$ the proof is easy and one would expect that the fact that $\mathfrak{B}^\infty$ is dense in $\mathfrak{B}$ would provide a quick and easy proof for the controllability test for $\mathfrak{B}$. This, unfortunately, is not true. For the smooth case the proof uses a differential transformation of the behavior. This transformation corresponds to a right unimodular transformation $V(\xi)$ of $R(\xi)$ yielding its Smith form. Generally, such a transformation cannot be extended to $\mathfrak{B}$ since $\mathfrak{B}$ contains non-smooth trajectories. In this contribution we argue that, despite this fact, the unimodular matrix $V(\xi)$ defines an *injection* from $\mathfrak{B}$ into the behavior defined by the Smith form. With this observation, that is interesting in its own right, the controllability test can be proved relatively easy.

Keywords: Controllability, behavior, Hautus test

1. INTRODUCTION

The behavioral approach to systems theory provides a parameter free description of systems without prior distinction between inputs and outputs. The central object of study is the set of possible trajectories, called the behavior of the system. Within this framework all system theoretic notions like controllability, observability, stability, state property, etc. are defined and expressed in terms of the behavior rather than in terms of a specific parametric representation of the behavior. Of course, parametric representations such as differential and difference equations are important and indispensable means in the description of behaviors. Naturally, the link between a notion like, for example, controllability of a behavior and computable properties of a representation are extremely important. Without efficiently computable tests the theory would not be very powerful.

In (Polderman and Willems, 1998) the elementary theory of behaviors represented by linear constant coefficient differential equations is described. Of course, when working with differential equations one has to specify the function space to which the solutions belong. For several reasons the set of infinitely differential functions is too restrictive. After all, one wants, for example, to be able to describe switching on a voltage source. A much more flexible, and mathematically convenient choice is the set of locally integrable functions. Solutions of differential equations then are solutions in the sense of distributions and are referred to as weak solutions. A fundamental property of the set of weak solutions is that the set of infinitely differentiable solutions forms a dense subset. This is not only a fundamental but also very useful property. For, it enables a technique of proof that is based on the analysis of the smooth behavior and a denseness argument to carry over properties from the smooth behavior to the behavior of weak solutions.

However, the choice of function space together with the notion of weak solution also has some serious shortcomings. For instance, locally integrable functions are not individual objects, rather they are equivalence classes of functions and consequently weak solutions are unique modulo differences on sets of zero measure. In applications this may be unfortunate. A more serious drawback is the following. To study structural properties of behaviors defined by equations of the type $R(\frac{d}{dt})w = 0$, where $R(\xi)$ is a polynomial matrix, it is often handy to transform the matrix $R(\xi)$ into a more convenient form. An important example is the Smith form that is obtained as $U(\xi)R(\xi)V(\xi)$ for appropriate choice of unimodular matrices $U(\xi), V(\xi)$. The matrix $U(\xi)$ transforms the differential equations but not the behavior. The matrix $V(\xi)$, however, transforms both the equations and the behavior. If the attention is restricted to the smooth behavior then $V(\xi)$ defines an isomorphism of vector spaces between the original and the transformed behavior. Unfortunately, this isomorphism can, in general, not be extended to the weak behavior. This is somewhat annoying in its own right but it also blocks an easy and elementary treatment of controllability. More specifically, Theorem 5.2.10 of (Polderman and Willems, 1998) states that the behavior defined by $R(\frac{d}{dt})w = 0$ is controllable if and only if the rank of the complex matrix $R(\lambda)$ does not depend on $\lambda \in \mathbb{C}$. Of course, a quick line of proof would be to take the matrix $R(\xi)$, transform it into Smith form, prove the result for the Smith form, and transform back. The idea, of course, being that a structural property like controllability is invariant under isomorphisms. The latter is true, but as remarked, the transformation that brings $R(\xi)$ into its Smith form does not induce an isomorphism and therefore this line of proof breaks down. As a result of this observation, the proof provided in (Polderman and Willems, 1998) follows a totally different avenue and is rather technical and more importantly does not provide much insight.

Of course controllability has been a fundamental concept in systems theory since its introduction by R.E. Kalman, (Kalman, 1960). Controllability (and observability) have been studied in numerous texts, for example in (Brockett, 1970) and (Kailath, 1980). The natural generalizations of controllability of state space systems to general behaviors was first presented in (Willems, 1989). The behavioral test for controllability is very similar to the Hautus for controllability. The latter appeared in (Hautus, 1969) and is sometimes referred to as the PBH (Popov–Belevich–Hautus) test, since it was independently derived also in (Popov, 1969) and (Belevich, 1968). More historical details about these notions may be found in (Kailath, 1980; Kalman, 1968).

In this note we revisit the controllability test, (Polderman and Willems, 1998, Theorem 5.2.10), in the perspective of right unimodular transformations. We re-examine the question to what extent unimodular transformations fail to generate an isomorphism between behaviors. It is found that in the case of a right unimodular transformation that transforms a matrix in its Smith form, the transformation generates an *injective* homomorphism from the original behavior into the transformed behavior. This result enables us to derive a new proof of the controllability test. Admitted, this analysis is also somewhat long and also requires some technical steps. However, it provides much more insight, it is more logical, and yields some byproducts that are of general interest.

2. THE MAIN RESULT

In the sequel $R(\xi)$ is a matrix of polynomials with coefficients in $\mathbb{R}$ consisting of g rows and q columns. We assume that $R(\xi)$ is of full row rank. We also use integers p and m defined as $p := g$ and $m := q - g$.

The space of locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^q$ is denoted by $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$. Furthermore we (ab)use the following notation. $\mathcal{C}^{-1}(\mathbb{R}, \mathbb{R}^q) := \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$, $\mathcal{C}^0(\mathbb{R}, \mathbb{R}^q)$ denote the functions from $\mathbb{R}$ to $\mathbb{R}^q$ that are absolutely continuous, and $\mathcal{C}^k(\mathbb{R}, \mathbb{R}^q)$, $k \geq 1$ denote the functions that are k times continuously differentiable. In the sequel we mean by solution of a differential equation, solution in the sense of distributions. We refer to such solutions as *weak* solutions, (Polderman and Willems, 1998, Chapter 2). With this notation we have the following lemma.

LEMMA 2.1. *(Polderman and Willems, 1998, Exercise 3.31) Let* $p(\xi), q(\xi) \in \mathbb{R}[\xi]$. *Let* $r = \deg p(\xi) - \deg q(\xi) \geq 0$. *Furthermore let*

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u \quad \textit{weakly.} \tag{1}$$

If $u \in \mathcal{C}^k(\mathbb{R}, \mathbb{R})$ *for some* $k \geq -1$, *then* $y \in \mathcal{C}^{k+r}(\mathbb{R}, \mathbb{R})$

EXAMPLE 2.2. Consider the behavior defined as

$$\mathfrak{B} = \left\{ w \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^2) \mid \frac{d}{dt}w_1 + w_2 = 0 \right\}. \tag{2}$$

The corresponding polynomial matrix is $R(\xi) = \begin{bmatrix} \xi & 1 \end{bmatrix}$. $R(\xi)$ is transformed into Smith form:

$$[\xi\ 1]\underbrace{\begin{bmatrix}0 & 1\\ 1 & -\xi\end{bmatrix}}_{V(\xi)} = \underbrace{[1\ 0]}_{\tilde{R}(\xi)}. \qquad (3)$$

The behavior defined by the Smith form is, of course, given by

$$\tilde{\mathfrak{B}} = \left\{\ell \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2) \mid \ell_1 = 0\right\}. \qquad (4)$$

The equation $V^{-1}(\frac{d}{dt})w = \ell$ reads as

$$\begin{aligned}\frac{d}{dt}w_1 + w_2 &= \ell_1\\ w_1 &= \ell_2.\end{aligned} \qquad (5)$$

It is easy to see that for each $w \in \mathfrak{B}$ (5) has a unique solution, namely $\ell_1 = 0$, $\ell_2 = w_1$. Now, let $w \in \mathfrak{B}$, then, by Lemma 2.1 $w_1 \in \mathcal{C}^0(\mathbb{R},\mathbb{R})$. It follows that $\ell_2 \in \mathcal{C}^0(\mathbb{R},\mathbb{R})$. Vice versa, it follows that for a given $\ell \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2)$ (5) has a solution $w \in \mathfrak{B}$ if and only if $\ell_1 = 0$ and $\ell_2 \in \mathcal{C}^0(\mathbb{R},\mathbb{R})$.

The conclusion is that for each $w \in \mathfrak{B}$ there is exactly one $\ell \in \tilde{\mathfrak{B}}$ such that $w = V(\frac{d}{dt})\ell$, but that the image $V(\frac{d}{dt})\mathfrak{B} \subsetneq \tilde{\mathfrak{B}}$. Apparently the differential transformation $V^{-1}(\frac{d}{dt})$ defines an non-surjective injection from $\mathfrak{B}$ into $\tilde{\mathfrak{B}}$.

For future reference we recall the notion of row reduced matrix, (Kailath, 1980).

Definition 2.3. Let $R(\xi) \in \mathbb{R}^{g\times q}[\xi]$ and denote the rows of $R(\xi)$ by $r_i(\xi)$, $i = 1,\ldots,g$. The *row degrees* $d_1,\ldots,d_g$ are defined as $d_i = \max_{j=1,\ldots,q}\deg r_{ij}(\xi)$. Define the diagonal matrix $D(\xi) = \mathrm{diag}(x^{d_1},\ldots,x^{d_g})$ and write $R(\xi) = D(\xi)R_0 + R_1(\xi)$ with $D(\xi)^{-1}R_1(\xi)$ strictly proper, meaning that in every entry of $D(\xi)^{-1}R_1(\xi)$ the degree of the denominator exceeds the degree of the numerator. Then, $R(\xi)$ is said to be *row reduced* if R_0 is of full row rank as a matrix in $\mathbb{R}^{g\times q}$. The matrix R_0 is called the *leading row coefficient matrix.*

An important result is that every polynomial matrix of full row rank may be transformed to a row reduced matrix by pre-multiplication with an appropriate unimodular matrix.

The next theorem generalizes Example 2.2.

Theorem 2.4. *Let $R(\xi) \in \mathbb{R}^{g\times q}[\xi]$ of full row rank. Assume that* $\mathrm{rank}\,R(\lambda) = g$ *for all* $\lambda \in \mathbb{C}$. *Then there exist unimodular matrices $U(\xi), V(\xi)$ such that*

$$U(\xi)R(\xi)V(\xi) = \begin{bmatrix}I & 0\end{bmatrix}, \qquad (6)$$

and for every weak solution w of $R(\frac{d}{dt})w = 0$ there exists exactly one $\ell \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^q)$ such that

$$w = V(\frac{d}{dt})\ell \quad \textit{weakly.} \qquad (7)$$

Proof Assume, without loss of generality (Polderman and Willems, 1998, Theorem 3.3.22), that

$$R(\xi) = \begin{bmatrix}P(\xi) & -Q(\xi)\end{bmatrix} \quad P(\xi)^{-1}Q(\xi) \text{ proper}. \qquad (8)$$

Let $U(\xi)$ and $W(\xi)$ be unimodular matrices of appropriate dimensions such that

$$U(\xi)\begin{bmatrix}P(\xi) & -Q(\xi)\end{bmatrix}\begin{bmatrix}W_{11}(\xi) & W_{12}(\xi)\\ W_{21}(\xi) & W_{22}(\xi)\end{bmatrix} = \begin{bmatrix}I & 0\end{bmatrix}. \qquad (9)$$

Again without loss of generality and for ease of notation we assume that $U(\xi) = I$. From (9) and the unimodularity of $W(\xi)$ it follows that $\det W_{22}(\xi) \neq 0$. Therefore there exists a unimodular matrix $W_2'(\xi)$ such that $W_{22}(\xi)W_2'(\xi)$ is row reduced. Define

$$\begin{aligned}V(\xi) &= \begin{bmatrix}V_{11}(\xi) & V_{12}(\xi)\\ V_{21}(\xi) & V_{22}(\xi)\end{bmatrix}\\ &= \begin{bmatrix}W_{11}(\xi) & W_{12}(\xi)\\ W_{21}(\xi) & W_{22}(\xi)\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & W_2'(\xi)\end{bmatrix}.\end{aligned} \qquad (10)$$

Then also:

$$\begin{bmatrix}P(\xi) & -Q(\xi)\end{bmatrix}\begin{bmatrix}V_{11}(\xi) & V_{12}(\xi)\\ V_{21}(\xi) & V_{22}(\xi)\end{bmatrix} = \begin{bmatrix}I & 0\end{bmatrix}. \qquad (11)$$

Next, rewrite $R(\frac{d}{dt})w = 0, w = V(\frac{d}{dt})\ell$ as:

$$\underbrace{\begin{bmatrix}P(\frac{d}{dt}) & 0 & 0\\ -I & V_{11}(\frac{d}{dt}) & V_{12}(\frac{d}{dt})\\ 0 & V_{21}(\frac{d}{dt}) & V_{22}(\frac{d}{dt})\end{bmatrix}}_{\tilde{P}(\frac{d}{dt})}\begin{bmatrix}y\\ \ell_1\\ \ell_2\end{bmatrix} = \underbrace{\begin{bmatrix}Q(\frac{d}{dt})\\ 0\\ I\end{bmatrix}}_{\tilde{Q}(\frac{d}{dt})}u. \qquad (12)$$

It is readily verified that

$$\tilde{P}(\xi)^{-1}\tilde{Q}(\xi) = \begin{bmatrix}P(\xi)^{-1}Q(\xi)\\ 0\\ V_{22}(\xi)^{-1}\end{bmatrix}. \qquad (13)$$

This rational matrix is proper because $P(\xi)^{-1}Q(\xi)$ is proper and $V_{22}(\xi)$ is row reduced. Therefore for every $u \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^m)$ there exists (y,ℓ) such that

$$\tilde{P}(\frac{d}{dt})\begin{bmatrix}y\\ \ell\end{bmatrix} = \tilde{Q}(\frac{d}{dt})u. \qquad (14)$$

Let u, y be any weak solution of $P(\frac{d}{dt})y = Q(\frac{d}{dt})u$ and let $\tilde{y}, \tilde{\ell}$ be such that

$$\tilde{P}(\frac{d}{dt})\begin{bmatrix}\tilde{y}\\ \tilde{\ell}\end{bmatrix} = \tilde{Q}(\frac{d}{dt})u. \qquad (15)$$

Define $y' = y - \tilde{y}$, then, of course, $P(\frac{d}{dt})y' = 0$ and it follows that $y' \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R}^p)$, (Polderman and Willems, 1998, Theorem 3.2.15). As a consequence there exists ℓ' such that

$$\tilde{P}(\frac{d}{dt})\begin{bmatrix}y'\\ \ell'\end{bmatrix} = 0. \qquad (16)$$

Finally, define $\ell = \ell' + \tilde{\ell}$ then

$$\tilde{P}(\frac{d}{dt})\begin{bmatrix}y\\ \ell\end{bmatrix} = \tilde{Q}(\frac{d}{dt})u. \qquad (17)$$

For given u, y the uniqueness of ℓ is obvious. This completes the proof. □

Apparently $V(\xi)$ induces an injective linear map from the behavior $\mathfrak{B}$ defined by $R(\frac{d}{dt})w = 0$ into the behavior defined by $\begin{bmatrix} I & 0 \end{bmatrix} \ell = 0$. The next lemma shows that the image $V^{-1}(\frac{d}{dt})\mathfrak{B}$ is characterized by a smoothness condition on each of the components of ℓ.

Lemma 2.5. *Let $W(\xi) \in \mathbb{R}^{m\times m}[\xi]$ be row reduced. Define*

$$\tilde{\mathfrak{B}} = \left\{ \ell \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^m) \;| W(\frac{d}{dt})\ell = u \quad u \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^m) \right\}. \tag{18}$$

Denote the row degrees of $W(\xi)$ by $d_1, \ldots, d_m$ respectively. Then there exists a permutation $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, m\}$ such that $\ell \in \tilde{\mathfrak{B}}$ if and only if $\ell_{i_j} \in \mathcal{C}^{d_j - 1}(\mathbb{R}, \mathbb{R})$.

Proof Let $\ell \in \tilde{\mathfrak{B}}$. Let i_j be the index of the highest degree entry in the jth row of $W(\xi)$. Since $W(\xi)$ is row reduced the indices i_j may be chosen such that $\{i_1, \ldots, i_m\} = \{1, \ldots, m\}$. The variable ℓ_{i_j} satisfies

$$W_{i_j i_j}(\frac{d}{dt})\ell_{i_j} = u_j - \sum_{k\neq j} W_{jk}(\frac{d}{dt})\ell_k \tag{19}$$

Since $\deg W_{i_j i_j}(\xi) = d_j$ and since $\deg W_{jk}(\xi) \leq \deg W_{i_j i_j}(\xi)$ for all $k \neq i_j$, it follows from Lemma 2.1 that $\ell_{i_j} \in \mathcal{C}^{d_i - 1}(\mathbb{R}, \mathbb{R})$.

Conversely, if for $j = 1, \ldots, j$ there holds that $\ell_{i_j} \in \mathcal{C}^{d_j-1}(\mathbb{R}, \mathbb{R})$, then there exists a $u \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^m)$ such that $W(\frac{d}{dt})\ell = u$. This proves the claim. □

Next we apply the above results to controllability analysis. Recall, (Polderman and Willems, 1998, Definition 5.2.2) that a behavior $\mathfrak{B}$ is *controllable* if for any two trajectories $w_1, w_2 \in \mathfrak{B}$ there exist a $t_0 \geq 0$ and a trajectory $w \in \mathfrak{B}$ with the property

$$w(t) = \begin{cases} w_1(t) & t \leq 0, \\ w_2(t) & t \geq t_0. \end{cases} \tag{20}$$

Lemma 2.6. *The behavior $\tilde{\mathfrak{B}}$ defined in (18) is controllable.*

Proof By Lemma 2.5 it suffices to check the controllability component by component. Each component is characterized by the level of smoothness. Let $\ell_{i,1}, \ell + \ell_{i,2} \in \mathcal{C}^{d_i}(\mathbb{R}, \mathbb{R})[\xi]$ and $t_0 > 0$. Then there exists $\ell_i \in \mathcal{C}^{d_i}(\mathbb{R}, \mathbb{R})$ such that $\ell_i(t) = \ell_{i,1}(t)$ for $t \leq 0$ and $\ell_i(t) = \ell_{i,2}(t)$ for $t \geq t_0$. □

We are now ready to present the new proof of the controllability test.

Theorem 2.7. *Let $R(\xi) \in \mathbb{R}^{g\times q}[\xi]$ of full row rank. Assume that* $\operatorname{rank} R(\lambda) = g$ *for all $\lambda \in \mathbb{C}$. Then $\mathfrak{B}$ is controllable.*

Proof According to Theorem 2.4 there exist unimodular matrices $U(\xi), V(\xi)$ such that $U(\xi)R(\xi)V(\xi) = \begin{bmatrix} I & 0 \end{bmatrix}$, and for each $w \in \mathfrak{B}$ there exists exactly one $\ell \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q)$ such that $V(\frac{d}{dt})w = \ell$. Define

$$\tilde{\mathfrak{B}} = \left\{ \ell \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q) \mid \exists w \in \mathfrak{B} \quad V(\frac{d}{dt})w = \ell \right\}. \tag{21}$$

In the notation of Theorem 2.4 $\tilde{\mathfrak{B}}$ is given by

$$\tilde{\mathfrak{B}} = \left\{ \ell = (\ell_1, \ell_2) \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{p+m}) \mid \ell_1 = 0, V_{22}(\frac{d}{dt})\ell_2 \in \mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^m) \right\}, \tag{22}$$

with $V_{22}(\xi)$ row-reduced. According to Lemma 2.6 $\tilde{\mathfrak{B}}$ is controllable. Let $w_1, w_2 \in \mathfrak{B}$ and $t_0 > 0$. Let $\ell_i \in \tilde{\mathfrak{B}}$ such that $w_i = V(\frac{d}{dt})\ell_i$, $i = 1, 2$. Since $\tilde{\mathfrak{B}}$ is controllable there exists $\ell \in \tilde{\mathfrak{B}}$ such that $\ell(t) = \ell_1(t)$ for $t \leq 0$ and $\ell(t) = \ell_2(t)$ for $t \geq t_0$. Since $\ell \in \tilde{\mathfrak{B}}$ there exists $w \in \mathfrak{B}$ such that $w = V(\frac{d}{dt})\ell$. Obviously $w(t) = w_1(t)$ for $t \leq 0$ and $w(t) = w_2(t)$ for $t \geq t_0$. This proves that $\mathfrak{B}$ is controllable. □

3. CONCLUSIONS

In this note we have analyzed the relation between a behavior defined by $R(\frac{d}{dt})w = 0$ and the behavior defined by the Smith form of $R(\xi)$. It was found that there exists an injective homomorphism of the original behavior into the Smith-behavior. The image of that injection is a dense sub behavior of the Smith behavior and is derived from the Smith behavior by adding appropriate smoothness conditions on the free components. These results were subsequently used to present a new proof of the controllability test.

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