ON NUMERICALLY VERIFIABLE EXACTNESS OF MULTIPLIER RELAXATIONS

Carsten W. Scherer [*,1]

[*] *Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands.*

Abstract: Robust semi-definite programming problems with rational dependence on uncertainties are known to have a wide range of applications, in particular in robust control. It is well-established how to systematically construct relaxations on the basis of the full block S-procedure. In general such relaxations are expected to be conservative, but for concrete problem instances they are often observed to be tight. The main purpose of this paper is to investigated in how far recently suggested tests for the exactness of such relaxations are indeed numerically verifiable.

Keywords: Robust semi-definite programming, relaxations, S-procedure, exactness tests, linear matrix inequalities.

1. INTRODUCTION

During the last fifteen years there has been a tremendous activity to identify control problems that can be translated into linear semi-definite programs (SDP's) which are generically formulated as follows: given a vector c and symmetric matrices $F_0, F_1, \ldots, F_n$, minimize $c'x$ over all $x \in \mathbb{R}^n$ which satisfy

$$F_0 + x_1F_1 + \cdots + x_nF_n \prec 0,$$

with $\prec$ denoting negative definiteness. Most classical problems such as H_2- and H_∞-optimal control have been successfully subsumed to such formulations (Boyd *et al.*, 1993; El Ghaoui and Niculescu, 2000).

In robust control the system descriptions are assumed to be affected by either time-invariant, time-varying parametric or dynamic uncertainties. Then the data matrices $F_0(\delta)$, $\ldots$,$F_n(\delta)$ are functions of some real or complex parameter δ that is only know to be contained in some set $\boldsymbol{\delta}$, and the goal is to minimize $c'x$ over all $x \in \mathbb{R}^n$ such that

$$F_0(\delta) + x_1F_1(\delta) + \cdots + x_nF_n(\delta) \prec 0 \text{ for all } \delta \in \boldsymbol{\delta}.$$

If $F_i(\delta)$, $i = 0, 1, \ldots, n$, depend affinely on δ and $\boldsymbol{\delta}$ is the convex hull of (a moderate number of) finitely many generators, the problem is easily reduced to a standard SDP since it suffices to guarantee the validity of the linear matrix inequality (LMI) constraint at the generators of the convex hull. The situation drastically differs in case that the dependence is non-linear in δ. If the dependence is rational without pole at zero - as is often true in control - one can determine matrices A, B, C, D, C_0, D_0 with

$$\left[\begin{array}{cc} C_0 & D_0 \\ \hline C & D \end{array}\right] \left[\begin{array}{c} \Delta(\delta)(I - A\Delta(\delta))^{-1}B \\ I \end{array}\right] = \frac{1}{2}\left[\begin{array}{c} F_0(\delta) \\ \hline F_1(\delta) \\ \vdots \\ F_l(\delta) \end{array}\right]$$

[1] Supported by the Technology Foundation STW, applied science division of NWO and the technology programme of the Ministry of Economic Affairs. A considerably extended version of this paper has been accepted for journal publication (Scherer, 2005).

where $\Delta(\delta)$ is block-diagonal and depends linearly on δ (Doyle *et al.*, 1991; Lambrechts *et al.*, 1994; Zhou *et al.*, 1996). With the abbreviations

$$W(x) = \begin{bmatrix} C_0 & D_0 \\ C & D \\ 0 & I \end{bmatrix}' \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & x \otimes I \\ I & x' \otimes I & 0 \end{bmatrix} \begin{bmatrix} C_0 & D_0 \\ C & D \\ 0 & I \end{bmatrix},$$

$$F(\Delta) = \begin{bmatrix} \Delta(I - A\Delta)^{-1}B \\ I \end{bmatrix},$$

$$L(x, \Delta) = \begin{bmatrix} F(\Delta) \\ I \end{bmatrix}' W(x) \begin{bmatrix} F(\Delta) \\ I \end{bmatrix},$$

$$\boldsymbol{\Delta} = \{\Delta(\delta) : \ \delta \in \boldsymbol{\delta}\},$$

the robust SDP with optimal value γ_{opt} reads as

$$\begin{array}{ll} \text{infimize} & c'x \\ \text{subject to} & L(x, \Delta) \prec 0 \ \text{ for all } \ \Delta \in \boldsymbol{\Delta}. \end{array} \qquad \text{(ROB)}$$

The purpose of this paper is to investigated the following family of relaxations for computing upper bounds on γ_{opt}. Choose linear Hermitian-valued mappings $G(y)$ and $H(y)$ (with variable y living in some finite-dimensional inner product space) such that

$$G(y) \preceq 0 \Rightarrow \begin{bmatrix} \Delta \\ I \end{bmatrix}' H(y) \begin{bmatrix} \Delta \\ I \end{bmatrix} \succeq 0 \ \ \forall \ \Delta \in \boldsymbol{\Delta}. \quad (1)$$

For any such pair G, H consider the standard linear SDP

$$\begin{array}{l} \text{infimize } \ c'x \ \text{ subject to } \ G(y) \prec 0, \\ W(x) + \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}' H(y) \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \prec 0. \end{array} \qquad \text{(REL)}$$

If γ_{rel} denotes its optimal value it is straightforward to prove on the basis of (1) that $\gamma_{\text{opt}} \leq \gamma_{\text{rel}}$. The computational complexity of computing γ_{rel} is determined by the dimension of y and the size of $G(y)$ respectively.

It is stressed that a large variety of relaxations suggested for robust linear algebra (El Ghaoui *et al.*, 1999; Ben-Tal and Nemirovski, 2001), robustness analysis (Packard and Doyle, 1993; Iwasaki and Hara, 1996; Trofino and de Souza, 1999; Iwasaki and Shibata, 2001; Iwasaki and Hara, 2003; Scherer, 2003*a*; Scherer, 2005) or linear-parameter-varying synthesis (Packard, 1994; Scorletti and El Ghaoui, 1998; Helmersson, 1995; Scherer, 2001) are captured within this framework. It is even possible to subsume recently suggested sum-of-squares relaxations which can be shown to be asymptotically exact (Scherer and Hol, 2004).

Although the suggested relaxations are expected to involve conservatism, it is often true that they are actually exact, with the following precise meaning. Suppose $W(x) = W_0 + x_1 W_1 + \cdots + x_n W_n$ with Hermitian $W_0, \ldots, W_n$. If the relaxation is infeasible, standard Lagrange duality (Boyd and Vandenberghe, 2004) allows to construct an infeasibility certificate, a pair $(M, N) \neq 0$ with $\langle W_0, M\rangle \geq 0$ and

$$M \geq 0, \quad N \geq 0, \quad \begin{bmatrix} \langle W_1, M\rangle \\ \vdots \\ \langle W_n, M\rangle \end{bmatrix} + c = 0, \qquad (2)$$

$$H^* \left(\begin{bmatrix} I & 0 \\ A & B \end{bmatrix} M \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}' \right) + G^*(N) = 0, \qquad (3)$$

where G^*, H^* denote the adjoint mappings of G, H and $\langle ., . \rangle$ is the standard inner product for matrices. The following result shows under which conditions and how one can verify whether the original robust LMI problem is infeasible as well (Scherer, 2005).

Theorem 1. (ROB) is not feasible iff there exists some $\Delta_0 \in \boldsymbol{\Delta}$ such that

$$L(x, \Delta_0) \prec 0 \ \text{ is not feasible.} \qquad (4)$$

$\Delta_0 \in \boldsymbol{\Delta}$ satisfies (4) iff there exist an infeasibility certificate (M, N) of (REL) with

$$\begin{bmatrix} I & -\Delta_0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} M = 0. \qquad (5)$$

If (REL) is feasible, it admits dual optimal solutions, matrix pairs (M, N) with $\langle W_0, M\rangle = \gamma_{\text{rel}}$ and (2)-(3). If one can solve the equation (5) for any of these dual optimal solutions it is guaranteed that the relaxation is exact, and that any Δ_0 satisfying (5) is a worst-case uncertainty. Moreover the converse holds true as well (Scherer, 2003*b*).

Theorem 2. If (REL) admits a dual optimal solution (M, N) for which (5) has a solution $\Delta_0 \in \boldsymbol{\Delta}$ then $\gamma_{\text{opt}} = \gamma_{\text{rel}}$ and Δ_0 is a worst-case uncertainty in the sense that

$$\inf \left\{ c'x : \begin{bmatrix} F(\Delta_0) \\ I \end{bmatrix}' W(x) \begin{bmatrix} F(\Delta_0) \\ I \end{bmatrix} \prec 0 \right\} = \gamma_{\text{opt}}. \qquad (6)$$

Conversely if $\gamma_{\text{opt}} = \gamma_{\text{rel}}$ and if there exists some $\Delta_0 \in \boldsymbol{\Delta}$ with (6) then there exist a (REL)-dual optimal solution (M, N) satisfying (5).

The purpose of this paper is to investigate under which conditions the exactness test of Theorem 2 is indeed applicable in practice. It is assumed from now on that $\boldsymbol{\Delta}$ is compact and admits an LMI description. Moreover suppose that (ROB) is feasible and that the (obviously closed and convex) set

$$X = \{x \in \mathbb{R}^n : \ L(x, \Delta) \preceq 0 \ \text{ for all } \ \Delta \in \boldsymbol{\Delta}\}$$

is bounded and hence compact. As a consequence, $c'x$ attains its minimum on X, and the set of minimizers X_{opt} is convex and compact. To avoid trivialities suppose $c \neq 0$. Finally consider a relaxation (REL) which is feasible, and denote by $\mathcal{M}$ the set of all matrices M such that (M, N) is optimal for the relaxation's dual.

2. APPROXIMATE EXACTNESS

In order to verify exactness one computes some element $M \in \mathcal{M}$ and determines

$$\nu_{\rm rel}(M) = \inf_{\Delta \in \boldsymbol{\Delta}} \left\| \left[\, I \;\, -\Delta \,\right] \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} M \right\|. \qquad (7)$$

Since $M \succeq 0$ and since $\boldsymbol{\Delta}$ has an LMI description, the computation of $\nu_{\rm rel}(M)$ can be easily translated into an SDP. If $\nu_{\rm rel}(M)$ vanishes, the relaxation is exact and any minimizer defines a worst-case uncertainty as in Theorem 2. If $\nu_{\rm rel}(M) \leq \nu_0$ for some small ν_0 we can actually infer approximate exactness, as made precise with explicit error bounds in the following result whose proof can be found in Appendix A.

Theorem 3. Suppose that $\|(I - \Delta A)^{-1}\| \leq a$ and $\|F(\Delta)\| \leq b$ for all $\Delta \in \boldsymbol{\Delta}$ and define $r = 2a(1+b)\max\{\|W_0\|, \ldots, \|W_n\|\}$. Moreover assume that $(\Delta_0, \nu_0) \in \boldsymbol{\Delta} \times \mathbb{R}$ satisfies $\left\| \left[\, I - \Delta_0 A \;\, -\Delta_0 B \,\right] M \right\| \leq \nu_0$.

If $\|x\|_1 \leq \xi$ for all (ROB)-feasible $x \in \mathbb{R}^n$ then

$$\gamma_{\rm opt} \leq \gamma_{\rm rel} \leq \gamma_{\rm opt} + \nu_0(1+\xi)r. \qquad (8)$$

If $\|x\|_1 \leq \eta$ for all $x \in \mathbb{R}^n$ with $L(x, \Delta_0) \prec 0$ then

$$\gamma_{\rm rel} - \nu_0(1+\eta)r \leq \inf\{c'x : \; L(x, \Delta_0) \prec 0\} \leq \gamma_{\rm opt}. \qquad (9)$$

Remark. Note that r and ξ are solely determined by (ROB), and (rough) estimates can be obtained by solving (rough) relaxations. In fact Theorem 3 can be viewed as an approximate version of Theorem 2 for small ν_0, and the latter is recovered with $\nu_0 = 0$. Finally observe that $\nu_{\rm rel}(M)$ is continuous in M since $\boldsymbol{\Delta}$ is compact. Hence, if $\nu_{\rm rel}(M)$ vanishes or is small, it is guaranteed that $\nu_{\rm rel}(\tilde{M})$ is small as well for all $\tilde{M}$ close to M, which guarantees stability of the suggested approximate exactness test against small errors in the computation of M. ■

3. VERIFIABLE EXACTNESS

From now on let us fix (existence assumed) a worst-case uncertainty $\Delta_0 \in \boldsymbol{\Delta}$ which satisfies $\gamma_{\rm opt} = \inf\{c'x : \; L(x, \Delta_0) \prec 0\}$. It is then not difficult to check that (5) is equivalent to $\langle \hat{W}, M \rangle = 0$ if defining

$$\hat{W} := \begin{bmatrix} I \\ -F(\Delta_0)' \end{bmatrix} \left[\, I \;\, -F(\Delta_0) \,\right] \succeq 0. \qquad (10)$$

As a major practical difficulty, this condition might not be valid for some computed $M \in \mathcal{M}$, even if the relaxation is exact. This is indeed of relevance since SDP-solvers of different nature return different points in $\mathcal{M}$, and $\mathcal{M}$ is in general not a singleton. This motivates the concept of verifiable exactness which implies that we can check exactness irrespective of which particular element of $\mathcal{M}$ is computed.

Verifiable exactness. (REL) is verifiably exact if $\langle \hat{W}, M \rangle = 0$ for all $M \in \mathcal{M}$.

Remark. If (REL)'s dual is strictly feasible, central-path-following primal-dual interior-point algorithms determine an element $\hat{M} \in \mathcal{M}$ satisfying $\mathrm{im}(M) \subset \mathrm{im}(\hat{M})$ for all $M \in \mathcal{M}$ (Goldfarb and Scheinberg, 1998). Therefore (10) holds for $\hat{M}$ iff it holds for all $M \in \mathcal{M}$. We conclude that we can check exactness in terms of $\hat{M}$ only if the relaxation is verifiable exact in the sense of our definition. ■

Let us first relate verifiable exactness to the following perturbation of (REL) with some fixed $t \in \mathbb{R}$, denoted as $(\mathrm{REL})_t$ with value $\gamma_{\rm rel}(t)$:

$$\text{infimize } \; c'x \; \text{ subject to } \; G(y) \prec 0,$$
$$t\hat{W} + W(x) + \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}' H(y) \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \prec 0.$$

For all t in a neighborhood of $t = 0$, $\gamma_{\rm rel}(t)$ is finite (since finite at $t = 0$ by assumption), nondecreasing (since $\hat{W} \geq 0$), and convex. Therefore its right-derivative $\gamma'_{\rm rel}(0)$ at zero does exist and is non-negative (Bertsekas, 2003). By a rather standard result in convex analysis $\gamma'_{\rm rel}(0)$ equals $\sup\{\langle \hat{W}, M \rangle : M \in \mathcal{M}\}$. This leads to the following alternative characterization of verifiable exactness.

Lemma 4. (REL) is verifiably exact iff $\gamma'_{\rm rel}(0) = 0$.

This reformulation opens the path for a relation to the perturbed version of (ROB), again obtained by just replacing $W(x)$ with $W(x) + t\hat{W}$ for $t \in \mathbb{R}$. Due to the particular structure of $\hat{W}$, $(\mathrm{ROB})_t$ reads explicitly as

$$\gamma_{\rm opt}(t) = \inf\{c'x : \; \forall \Delta \in \boldsymbol{\Delta} : \; L(x, \Delta) +$$
$$+ t[F(\Delta) - F(\Delta_0)]'[F(\Delta) - F(\Delta_0)] \prec 0\}.$$

Similarly as for $\gamma_{\rm rel}(t)$ it is seen that $\gamma_{\rm opt}(t)$ is finite, non-decreasing and convex in a neighborhood of $t = 0$. Therefore its right-derivative $\gamma'_{\rm opt}(0)$ exists and is non-negative.

It is again easy to show that $\gamma_{\rm opt}(t) \leq \gamma_{\rm rel}(t)$. If the relaxation is verifiably exact, $\gamma_{\rm opt}(0) = \gamma_{\rm rel}(0)$ implies $0 \leq \gamma'_{\rm opt}(0) \leq \gamma'_{\rm rel}(0) = 0$ and thus $\gamma'_{\rm opt}(0) = 0$. Hence we have identified $\gamma'_{\rm opt}(0) = 0$ as a necessary condition for (ROB) to admit a verifiable exact relaxation.

Theorem 5. There exists no verifiable exact relaxation if $\gamma'_{\rm opt}(0) > 0$.

It is interesting to observe that $\gamma'_{\rm opt}(0)$ is positive if there exists a worst case uncertainty $\Delta_1 \in \boldsymbol{\Delta}$ different from Δ_0 such that $F(\Delta_1) - F(\Delta_0)$ has no kernel. This is a consequence of the following more refined condition.

Theorem 6. $\gamma'_{\rm opt}(0) > 0$ if there exists some $\Delta_1 \in \boldsymbol{\Delta} \setminus \{\Delta_0\}$ such that

$$\inf\{c'x:\ L(x,\Delta_1) \prec 0\} = \gamma_{\rm opt}, \qquad (11)$$

and such that (11) admits a dual optimal solution Z with $[F(\Delta_1) - F(\Delta_0)]Z \neq 0$.

Proof. Since (ROB) is feasible, (11) is strictly feasible. Consider

$$\gamma(t) = \inf\{c'x:\ L(x,\Delta_1)+ \\ + t[F(\Delta_1) - F(\Delta_0)]'[F(\Delta_1) - F(\Delta_0)] \prec 0\}.$$

Clearly $\gamma(t) \leq \gamma_{\rm opt}(t)$. By hypothesis $\gamma(0) = \gamma_{\rm opt}(0) = \gamma_{\rm opt}$. Therefore $\gamma'(0) \leq \gamma'_{\rm opt}(0)$. Let us prove $\gamma'(0) > 0$. This holds if there exists a dual optimal solution Z for (11) such that $\langle[F(\Delta_1) - F(\Delta_0)]'[F(\Delta_1) - F(\Delta_0)], Z\rangle > 0$ or, since $Z \geq 0$, equivalently $[F(\Delta_1) - F(\Delta_0)]Z \neq 0$. ■

It remains to clarify under which conditions $\gamma'_{\rm opt}(0)$ vanishes such that we have indeed a chance to construct some verifiably exact relaxation. This is the contents of the following result whose proof is found in Appendix B.

Theorem 7. $\gamma'_{\rm opt}(0) = 0$ if there exists some $x_{\rm opt} \in X_{\rm opt}$ such that

$$\ker[L(x_{\rm opt},\Delta)] \subset \ker[F(\Delta) - F(\Delta_0)]$$

for all $\Delta \in \boldsymbol{\Delta} \setminus \{\Delta_0\}$.

As an immediate consequence, $\gamma'_{\rm opt}(0)$ vanishes if there exists some $x_{\rm opt} \in X_{\rm opt}$ such that $L(x_{\rm opt},\Delta) \prec 0$ for all $\Delta \in \boldsymbol{\Delta}$ different from Δ_0. The latter property requires Δ_0 to be unique as a worst-case uncertainty.

To summarize, Theorem 6 formulates a sufficient condition for $\gamma'_{\rm opt}(0) > 0$ which implies that there cannot exist a verifiably exact relaxation. Roughly speaking this property holds if Δ_0 is not unique as a worst-case uncertainty. In contrast Theorem 7 contains a sufficient condition for $\gamma'_{\rm opt}(0) = 0$, which makes it possible to construct verifiably exact relaxations. Roughly, this is true if the worst-case uncertainty Δ_0 is unique. Figure 1 and the subsequent example provide graphical and numerical illustrations. If $\gamma'_{\rm opt}(0) = 0$ it can be shown that the Pólya relaxations of (Scherer, 2003*a*) are verifiably asymptotically exact.

Remark. Without knowing a worst-case uncertainty $\gamma'_{\rm opt}(0)$ cannot be computed in practice. Hence these these insights are mainly of theoretical interest.

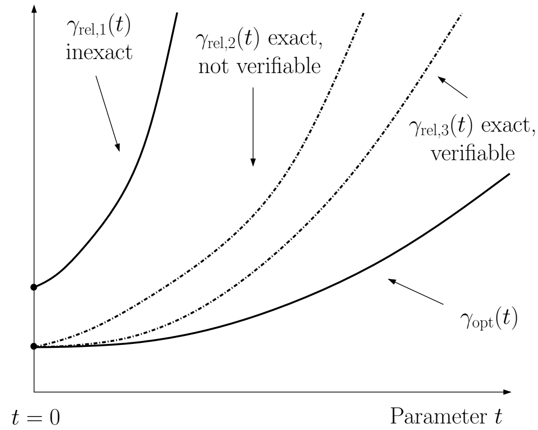

Fig. 1. Illustration of inexact, not verifiably exact and verifiably exact relaxations.

Example. Let us choose a one-parameter example in order to guarantee that the standard relaxations from μ-theory are exact, although they might not be verifiably exact. Consider the problem of minimizing a non-constant rational function $f(\delta)$ over $\delta \in [-1,1]$ (without pole in this interval). Standard realization theory allows to construct a representation $f(\delta) = C\Delta(\delta)(I - A\Delta(\delta))^{-1}B + D$ with $\Delta(\delta) = \delta I$ of minimal size and (A,B) in controllability canonical form. Our problem can be formulated as infimizing x_1 such that $f(\delta) < x_1$ for all $\delta \in [-1,1]$ which is a special version of (ROB). Clearly the unique minimizer of (ROB) is $x_{\rm opt} = \max_{\delta\in[-1,1]} f(\delta)$, and the standard μ-relaxation leads to $\gamma_{\rm rel}(t) = \gamma_{\rm opt}(t)$ and thus $\gamma'_{\rm opt}(0) = \gamma'_{\rm rel}(0)$.

If $f(\delta) = x_{\rm opt}$ for exactly one $\delta_0 \in [-1,1]$, then the relaxation is verifiably exact. Indeed if $\delta \in [-1,1]\setminus\{\delta_0\}$ we infer $f(\delta) - x_{\rm opt} < f(\delta_0) - x_{\rm opt}$ and hence $\ker(f(\delta) - x_{\rm opt}) = \{0\}$; by Theorem 7 we have $\gamma'_{\rm opt}(0) = 0$; therefore $\gamma'_{\rm rel}(0) = 0$ and we can apply Lemma 4. We stress that this fact does not depend on the nature (order) of the zero δ_0 of the function $f(\delta) - x_{\rm opt}$ as one might expect!

Now suppose that $f(\delta_0) = f(\delta_1) = x_{\rm opt}$ for different $\delta_0, \delta_1 \in [-1,1]$. Then neither the standard relaxation nor any relaxation can be verifiably exact. This follows from Theorem 5 since $\gamma'_{\rm opt}(0) > 0$, which is a consequence of Theorem 6 and $\delta_0(I - \delta_0 A)^{-1}B \neq \delta_1(I - \delta_1 A)^{-1}B$. The latter is true since equality implies $\delta_0^j/\det(I - \delta_0 A) = \delta_1^j/\det(I - \delta_0 A)$ for $j = 1,2$ because (A,B) is in controllability canonical form and A has at least dimension two. If $\delta_0 = 0$ we infer $\delta_1 = 0$ from $j = 1$, a contradiction; otherwise we have $\delta_0[\delta_0/\det(I - \delta_0 A)] = \delta_1[\delta_1/\det(I - \delta_1 A)] = \delta_1[\delta_0/\det(I - \delta_0 A)]$ and hence $\delta_0 = \delta_1$, again a contradiction.

Consider the concrete example (Figures 2 and 3)

$$f(\delta) = -1/3 + \frac{\delta^2}{1-\delta^2+\delta^4} - \frac{\delta^4}{1-\delta^2+\delta^4}.$$

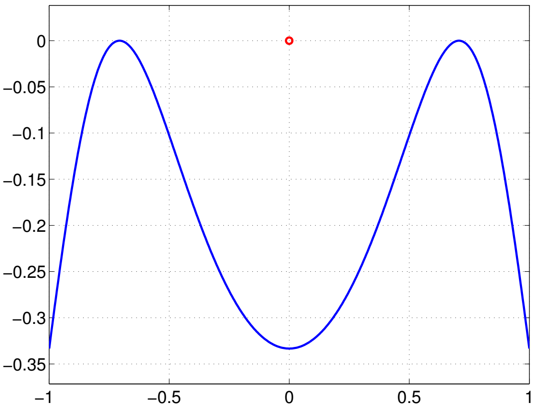

Fig. 2. Two maximizers, computed uncertainty (circle).

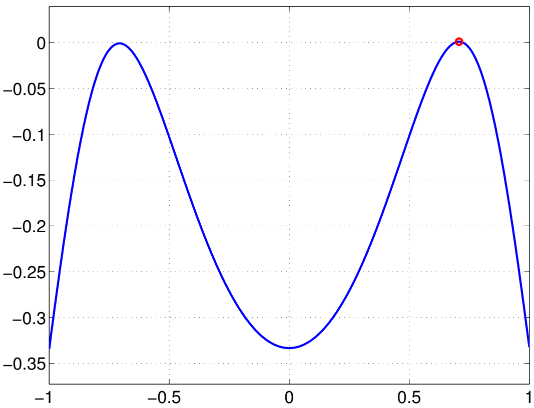

Fig. 3. One maximizer, computed uncertainty (circle).

The standard relaxation computes the correct maximal value, but it is not possible to confirm exactness since the function has two maximizers; the computed M has rank two and (7) admits the minimizer $\delta = 0$. If we slightly perturb the function such that the maximum is attained at one point only, we can indeed confirm exactness of the relaxation and compute the correct maximizer.

4. CONCLUSIONS

For robust semi-definite programming problems with rational dependence on uncertainties we recalled a general framework for formulating upper bound relaxations that encompass a large variety of special versions in the literature. In the main technical contribution we provided explicit necessary and sufficient conditions for being able to numerically verify (approximate) relaxation exactness.

Appendix A. PROOF OF THEOREM 3

Introduce the abbreviations $F_1 = (I \quad -F(\Delta_0))$, $F_2 = (F(\Delta_0)' \quad I)$, $F = \mathrm{col}(F_1, F_2)$. Note that $F_1 F_2' = 0$ and $F'F = \mathrm{diag}(I + F(\Delta_0)F(\Delta_0)', I + F(\Delta_0)'F(\Delta_0))$ which implies $I \preceq F'F \preceq (1+b^2)I$ and hence $\|F\| \leq 1 + b$ as well as $\|F^{-1}\| \leq 1$. Observe that the hypothesis on (Δ_0, ν_0) implies $\|(I - \Delta_0 A)F_1 M\| \leq \nu_0$ and hence $\|F_1 M\| \leq a\nu_0$ which guarantees $\|F_1 M F'\| \leq a(1+b)\nu_0$. Define $\hat{M} = [F_2'(F_2F_2')^{-1}F_2]M[F_2'(F_2F_2')^{-1}F_2]$ to conclude

$$\left\| M - \hat{M} \right\| \leq 2a(1+b)\nu_0.$$

Let us finally introduce $\hat{c}_j = -\langle W_j, \hat{M}\rangle$, $j = 1, \ldots, n$, to define the vector $\hat{c}$. Now suppose that x is (ROB)-feasible. Since $F_2 W(x) F_2' \preceq 0$ and $\hat{M} \succeq 0$, $F_1\hat{M} = 0$, $\langle W_j, \hat{M}\rangle + \hat{c}_j = 0$, $j = 1, \ldots, n$, one shows as in the proof of Theorem 2 (see (Scherer, 2005)) that $\langle \hat{c}, x\rangle \geq \langle W_0, \hat{M}\rangle$. If we recall $\langle W_0, M\rangle = \gamma_{\rm rel}$, $c_j = -\langle W_j, M\rangle$, $j = 1, \ldots, n$, and $\|x\|_1 \leq \xi$, we conclude

$$\langle c, x\rangle = \langle \hat{c}, x\rangle + \sum_{j=1}^{n} \langle W_j, \hat{M} - M\rangle x_j \geq$$

$$\geq \gamma_{\rm rel} + \langle W_0, \hat{M} - M\rangle + \sum_{j=1}^{n} \langle W_j, \hat{M} - M\rangle x_j \geq$$

$$\geq \gamma_{\rm rel} - (1+\xi) \max_{j=0}^{n} \|W_j\| \|\hat{M} - M\| \geq \nu_0(1+\xi)r.$$

Since x was arbitrary (ROB)-feasible we have proved (8). If x only satisfies $L(x, \Delta_0) \prec 0$, the same line of reasoning with $\|x\|_1 \leq \eta$ gives (9). ■

Appendix B. PROOF OF THEOREM 7.

If $\boldsymbol{B}$ denotes the unit ball in $\mathbb{R}^m$, we observe that $\gamma_{\rm opt}(t) = \min\{c'x : \ x \in \mathbb{R}^n, \ g(x,t) \leq 0\}$ where $g(x,t)$ equals

$$\max_{(\Delta, w) \in \boldsymbol{\Delta} \times \boldsymbol{B}} w^T L(x,\Delta) w + t\|[F(\Delta) - F(\Delta_0)]w\|^2.$$

Since $(x,t) \to w^T L(x,\Delta)w + t\|[F(\Delta) - F(\Delta_0)]w\|^2$ is affine, and since the derivative with respect to t is $\|[F(\Delta) - F(\Delta_0)]w\|^2$, we can invoke Danskin's theorem (Bertsekas, 2003, p.245) to infer that $g(.,.)$ is continuous and convex, and that the right-derivative of $g(x,.)$ at $t = 0$ is given by

$$g'(x, 0) = \max_{(\Delta, w) \in M(x)} \|[F(\Delta) - F(\Delta_0)]w\|^2$$

where $M(x) = \{(\Delta, w) \in \boldsymbol{\Delta} \times \boldsymbol{B} : \ w^T L(x,\Delta) w = g(x, 0)\}$. Now note $g(x_{\rm opt}, 0) = 0$ (since c is nonzero and the objective $c'x$ is linear in x) and $L(x_{\rm opt}, \Delta) \preceq 0$ for all $\Delta \in \boldsymbol{\Delta}$. Therefore $M(x_{\rm opt}) = \{(\Delta, w) \in \boldsymbol{\Delta} \times \boldsymbol{B} : \ L(x_{\rm opt}, \Delta)w = 0\}$. At this point we exploit the hypothesis to conclude that $g'(x_{\rm opt}, 0) = 0$.

Since (ROB) was assumed feasible there exists $x_0 \in \mathbb{R}^n$ with $g(x_0, 0) < 0$, and we can hence fix some $t_0 > 0$ and $\delta > 0$ such $g(x_0, t) \leq -\alpha_0$ for all $t \in [0, t_0]$. For these parameters, the set $\{x \in \mathbb{R}^n : \ g(x,t) \leq 0\}$ is hence nonempty, and it is also compact since contained in X. Therefore $\gamma_{\rm opt}(t)$ satisfies $\gamma_{\rm opt}(0) \leq \gamma_{\rm opt}(t) \leq \gamma_{\rm opt}(t_0) < \infty$, the set of optimizers $X_{\rm opt}(t)$ is nonempty and compact, and the set of Lagrange multipliers $\Lambda(t)$ is nonempty. Moreover for all $t \in [0, t_0]$, $\lambda_t \in \Lambda(t)$, $x_t \in X(t)$ we infer with the usual saddle-point property of the Lagrangian (Boyd and Vandenberghe, 2004) that

$$\begin{aligned} -\alpha_0\lambda_t &\geq \lambda_t g(x_0, t) = \\ &= \lambda_t g(x_0, t) + \langle c, x_0\rangle - \langle c, x_0\rangle \geq \\ &\geq \lambda_t g(x_t, t) + \langle c, x_t\rangle - \langle c, x_0\rangle = \\ &= \gamma_{\rm opt}(t) - \langle c, x_0\rangle \geq \gamma_{\rm opt}(0) - \langle c, x_0\rangle. \end{aligned}$$

With $\alpha = (\langle c, x_0\rangle - \gamma_{\rm opt}(0))/\alpha_0 > 0$ this implies $0 \leq \lambda_t \leq \alpha$ for al $\lambda_t \in \Lambda(t)$, $t \in [0, t_0]$.

Now choose an arbitrary $\epsilon > 0$. Since $g'(x_{\rm opt}, 0) = 0$, there exists $\delta \in (0, t_0]$ with $g(x_{\rm opt}, t) \leq g(x_{\rm opt}, 0) + t\epsilon/\alpha$ for all $t \in (0, \delta]$. With $t \in (0, \delta]$, $x_t \in X(t)$, $\lambda_t \in \Lambda(t)$ we exploit again the saddle-point property to obtain

$$\begin{aligned} \gamma_{\rm opt}(t) &= c^T x_t + \lambda_t g(x_t, t) \leq \\ &\leq c^T x_{\rm opt} + \lambda_t g(x_{\rm opt}, t) \leq \\ &\leq c^T x_{\rm opt} + \lambda_t [g(x_{\rm opt}, 0) + t\epsilon/\alpha] \leq \\ &\leq c^T x_{\rm opt} + \lambda_t g(x_{\rm opt}, 0) + t\epsilon \leq \\ &\leq c^T x_{\rm opt} + \lambda_0 g(x_{\rm opt}, 0) + t\epsilon \leq \gamma_{\rm opt}(0) + t\epsilon \end{aligned}$$

and hence $(\gamma_{\rm opt}(t) - \gamma_{\rm opt}(0))/t \leq \epsilon$. This proves $\gamma'_{\rm opt}(0) \leq 0$ and hence $\gamma'_{\rm opt}(0) = 0$. ∎

REFERENCES

Ben-Tal, A. and A. Nemirovski (2001). *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications.* SIAM-MPS Series in Optimizaton. SIAM Publications. Philadelphia.

Bertsekas, D.P. (2003). *Convex analysis and optimization.* Athena Scientific. Belmont, Massachusetts.

Boyd, S. and L. Vandenberghe (2004). *Convex Optimization.* Cambridge University Press. Cambridge.

Boyd, S., V. Balakrishnan, E. Feron and L. El Ghaoui (1993). Control system analysis and synthesis via linear matrix inequalities. In: *Proc. Amer. Contr. Conf.*. San Francisco, CA. pp. 2147–2154.

Doyle, J., A. Packard and K. Zhou (1991). Review of LFT's, LMI's, and μ. In: *Proc. 30th IEEE Conf. Decision and Control.* Brighton, UK. pp. 1227–1232.

El Ghaoui, L. and Niculescu, S.I., Eds.) (2000). *Advances in Linear Matrix Inequality Methods in Control.* SIAM. Philadelphia.

El Ghaoui, L., F. Oustry and H. Lebret (1999). Robust solutions to uncertain semidefinite programs. *SIAM J. Optim.* **9**(1), 33–52.

Goldfarb, D. and K. Scheinberg (1998). Interior point trajectories in semidefinite programming. *SIAM J. Optim.* **8**(4), 871–886.

Helmersson, A. (1995). A finite frequency method for μ-analysis. In: *European Control Conf.*. Vol. 1. Rome, Italy. pp. 171–176.

Iwasaki, T. and G. Shibata (2001). LPV system analysis via quadratic separator for uncertain implicit systems. *IEEE Trans. Aut. Contr.* **46**(8), 1195–1208.

Iwasaki, T. and S. Hara (1996). Well-posedness of feedback systems: insights into exact robustness analysis. In: *Proc. 35th IEEE Conf. Decision and Control.* Kobe, Japan. pp. 1863–1868.

Iwasaki, T. and S. Hara (2003). Generalization of Kalman-Yakubovič-Popov lemma for restricted frequency inequalities. In: *Proc. American Control Conf.*. Denver, Colorado USA.

Lambrechts, P., J.C. Terlouw, S. Bennani and M. Steinbuch (1994). Obtaining lft descriptions of state-space models with parametric uncertainties. In: *Proc. IMACS Symposium on Mathematical Modelling MATHMOD* (I. Koch, Ed.). VIENNA. pp. 261–264.

Packard, A. (1994). Gain scheduling via linear fractional transformations. *Syst. Contr. Letters* **22**, 79–92.

Packard, A. and J. Doyle (1993). The complex structured singular value. *Automatica* **29**, 71–109.

Scherer, C.W. (2001). LPV control and full block multipliers. *Automatica* **37**, 361–375.

Scherer, C.W. (2003*a*). Higher-order relaxations for robust LMI problems with verifications for exactness. In: *Proc. 42nd IEEE Conf. Decision and Control.* Maui, Hawaii. pp. 4652–4657.

Scherer, C.W. (2003*b*). Testing the exactness of convex relaxations for robust LMI problems. In: *Proc. Appl. Math. Mech.*. Vol. 3. Wiley. pp. 52–55.

Scherer, C.W. (2005). Relaxations for robust linear matrix inequality problems with verifications for exactness. *to appear in SIAM Journal on Matrix Analysis and Applications.*

Scherer, C.W. and C.W.J. Hol (2004). Matrix sum-of-squares relaxations for robust semidefinite programs. *submitted for publication.*

Scorletti, G. and L. El Ghaoui (1998). Improved LMI conditions for gain-scheduling and related control problems. *Int. J. of Robust and Nonlinear Control* **8**, 845–877.

Trofino, A. and C.E. de Souza (1999). Biquadratic stability of uncertain linear systems. In: *Proc. 38th IEEE Conf. Decision and Control.* Phoenix, Arizona. pp. 5016–5021.

Zhou, K., J. Doyle and K. Glover (1996). *Robust and Optimal Control.* Prentice Hall. Upper Saddle River, New Jersey.