DIRECT IDENTIFICATION OF CONTINUOUS-TIME ERRORS-IN-VARIABLES MODELS [1]

Kaushik Mahata [*] and **Hugues Garnier** [**]

[*] *Centre for Complex Dynamic Systems and Control, University of Newcastle, Callaghan, NSW 2308, Australia.*
Email: `Kaushik.Mahata@newcastle.edu.au`
[**] *Centre de Recherche en Automatique de Nancy, CRAN - UMR 7039 CNRS-UHP-INPL 7039, Université Henri Poincaré, Nancy 1, BP 239, 54506 Vandoeuvre-les-Nancy Cedex, France.*
Email: `hugues.garnier@cran.uhp-nancy.fr`

Abstract: A novel direct approach for identifying continuous-time linear dynamic errors-in-variables models is presented in this paper. The effects of the noise on the state-variable filter outputs are analyzed. Subsequently, a search-free algorithm to obtain consistent continuous-time parameter estimates in the errors-in-variables framework is derived. The performances of the proposed algorithm are illustrated with some numerical simulation examples.

Keywords: Errors-in-variables, system identification, continuous-time models, linear estimation

1. INTRODUCTION

Consider a single-input, single-output, linear, time-invariant, continuous-time system having noise-free input $u_0(t)$ and output $y_0(t)$ linked by:

$$\sum_{j=0}^{n} a_j y_0^{(j)}(t) = \sum_{j=0}^{m} b_j u_0^{(j)}(t) \tag{1}$$

where $x^{(j)}(t) := \frac{\mathrm{d}^j x(t)}{\mathrm{d}t^j}$. The system is assumed to be proper, *i.e.* $m \leq n$. Without any loss of generality we assume $a_0 = 1$. It is assumed that the input and output signals are sampled at time-instants $\{t_k\}_{k=1}^N$, not necessarily uniformly spaced. The sampled signals are denoted by $\{u_0(t_k); y_0(t_k)\}$. It is further assumed that the measurements are noise corrupted. The observed sampled data $\{u(t_k)\}_{k=1}^N$ and $\{y(t_k)\}_{k=1}^N$ are given by

$$u(t_k) = u_0(t_k) + \tilde{u}(t_k), \qquad y(t_k) = y_0(t_k) + \tilde{y}(t_k),$$

where $\{\tilde{u}(t_k)\}_{k=1}^N$ and $\{\tilde{y}(t_k)\}_{k=1}^N$ are zero mean white noise sequences with variances σ_u and σ_y, respectively. In this paper we are concerned with the problem of identifying the continuous-time parameters $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=0}^m$ from the noise corrupted observations of the input-output sampled data $\{u(t_k)\}_{k=1}^N$ and $\{y(t_k)\}_{k=1}^N$.

The model under consideration is often referred to as the continuous-time errors-in-variables (EIV) model. Many methods have been proposed to solve the related problem in discrete-time. The popular approaches can be classified in a few broad categories (Söderström *et al.*, 2002); namely, the bias compensating least squares methods (Beghelli *et al.*, 1990; Zheng, 1998), prediction error and maximum likelihood methods (Söderström, 1981), instrumental variable based approaches (Stoica *et al.*, 1995; Söderström and Mahata, 2002) and frequency-domain methods based on non-parametric noise modeling (Schoukens *et al.*, 1997). Unless we impose certain

[1] This work was supported by the Australian Research Council.

assumptions on the signal and noise models, it is well-known that the *general* EIV model is not uniquely identifiable from second order statistics (Anderson and Deistler, 1984). This motivates the approaches based on higher order statistics (Tugnait and Ye, 1995).

To our best knowledge, the case of continuous-time EIV model identification has not received appropriate attention so far. A first attempt to solve the EIV filtering problem for continuous-time models has been recently proposed in (Markovski *et al.*, 2002). An alternative approach for transfer function model is derived in this paper. The traditional state-variable filter (SVF) method is used to handle the time-derivative measurement problem (see for example (Garnier and Young, 2004) and the references therein). In this work, we first analyze the effect of additive noise on the SVF outputs. As a next step we use this novel characterization to develop a consistent estimator for the continuous-time EIV identification problem introduced above.

2. CONTINUOUS-TIME MODEL IDENTIFICATION

In this section we present a brief review of direct least squares-based identification of continuous-time models (Garnier *et al.*, 2003; Garnier and Young, 2004). It is well-known that in presence of additive noise in either or both of the input and output measurements, the conventional least squares method gives biased estimates. If the input data are noise-free, one can however obtain consistent estimates by using an instrumental variable estimator even if the output measurements are noise corrupted. Unfortunately, in presence of additive noise in the input measurements the instrumental variable based methods fail to achieve parameter consistency. Since the algorithm proposed in this paper is based on least squares normal equations, it is more appropriate to introduce the least squares estimation method.

A crucial step in direct continuous-time model identification is the appropriate reconstruction of the time-derivatives from the sampled data. It is well-known that the computation of derivatives from sampled data is an ill posed problem. The traditional SVF approach mitigates this problem by passing both input/output signals through an all-pole filter of minimum order n. To explain the idea, consider that both the noise-free input and output data are prefiltered as follows:

$$\sum_{i=0}^{n} f_i y_f^{(i)}(t) = y_0(t), \quad \sum_{i=0}^{n} f_i u_f^{(i)}(t) = u_0(t). \quad (2)$$

Without any loss of generality we shall assume $f_0 = 1$. Then it is readily verified that the filtered signals $y_f(t)$ and $u_f(t)$ satisfy the differential equation

$$\sum_{i=0}^{n} a_i y_f^{(i)}(t) = \sum_{i=0}^{m} b_i u_f^{(i)}(t).$$

The equation above can be written alternatively as

$$\boldsymbol{y}_0'(t)\boldsymbol{a} = \boldsymbol{u}_0'(t)\boldsymbol{b}, \quad (3)$$

where we have introduced regressors

$$\begin{aligned} \boldsymbol{y}_0'(t) &= [\, y_f^{(n)}(t) \ \cdots \ y_f^{(0)}(t)\,], \\ \boldsymbol{u}_0'(t) &= [\, u_f^{(m)}(t) \ \cdots \ u_f^{(0)}(t)\,], \end{aligned} \quad (4)$$

and parameter vectors

$$\boldsymbol{a}' = [\, a_n \ \cdots \ a_0 \,], \qquad \boldsymbol{b}' = [\, b_m \ \cdots \ b_0 \,].$$

Let us denote the Laplace transform of $\boldsymbol{y}_0(t)$, $y_0(t)$ etc as

$$\check{\boldsymbol{y}}_0(s) = \mathcal{L}\{\boldsymbol{y}_0(t)\}, \qquad \check{y}_0(s) = \mathcal{L}\{y_0(t)\},$$

etc. Also denote the denominator of the all pole prefilter [in (2)] transfer function as

$$F(s) = \sum_{i=0}^{n} f_i s^i.$$

Then we can verify from (2) and (4) that (neglecting the effect of the initial conditions)

$$\check{\boldsymbol{y}}_0(s) = \mathbf{F}_n(s)\check{y}_0(s), \quad \check{\boldsymbol{u}}_0(s) = \mathbf{F}_m(s)\check{u}_0(s), \quad (5)$$

where

$$\mathbf{F}_i(s) = \frac{1}{F(s)} \left[\, s^i \ \cdots \ s \ 1 \,\right]', \ \ i \le n.$$

Note that the transfer functions $\mathbf{F}_i(s)$ in the above equation are causally implementable. As we shall describe later in more detail, there exists reliable numerical methods to implement the vector valued filter $\mathbf{F}_i(s)$ from sampled data assuming suitable inter-sample behaviour. We also point out that it is often preferred to choose the coefficients $\{f_i\}_{i=0}^{n}$ such that $F(s)$ has n multiple zeros. The bandwidth of $F(s)$ is chosen to match the bandwidth of interest.

Assume that it is possible the compute $\{\boldsymbol{y}_0(t_k)\}_{k=1}^{N}$ and $\{\boldsymbol{u}_0(t_k)\}_{k=1}^{N}$ by using a suitable numerical technique. Then we can form the matrices

$$\begin{aligned} \mathbf{Y}_{0N} &= \sum_{k=1}^{N} \boldsymbol{y}_0(t_k)\boldsymbol{y}_0'(t_k), \\ \mathbf{V}_{0N} &= \sum_{k=1}^{N} \boldsymbol{y}_0(t_k)\boldsymbol{u}_0'(t_k), \\ \mathbf{U}_{0N} &= \sum_{k=1}^{N} \boldsymbol{u}_0(t_k)\boldsymbol{u}_0'(t_k). \end{aligned}$$

Then from (3) we have

$$\begin{bmatrix} \mathbf{Y}_{0N} & \mathbf{V}_{0N} \\ \mathbf{V}_{0N}' & \mathbf{U}_{0N} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \mathbf{0}_{(m+n+2)\times 1} \quad (6)$$

Therefore, estimates of $\boldsymbol{a}$ and $\boldsymbol{b}$ are obtained by solving (6). We point out that due to numerical errors introduced in the digital simulation of the continuous-time state-variable filtering, (6) does not hold exactly[1]. Therefore, it is required to solve (6) in a least squares or total least squares sense.

3. IMPLEMENTATION OF SVF

A crucial step in direct continuous-time model identification lies in the implementation of the SVF. This step involves accurate computation of the regressor vectors $\boldsymbol{y}_0(t_k)$ and $\boldsymbol{u}_0(t_k)$ at the sampling instants $\{t_k\}_{k=1}^N$ from the sampled data $\{u_0(t_k)\}_{k=1}^N$ and $\{y_0(t_k)\}_{k=1}^N$. The numerical technique used here is the standard discretization of the underlying continuous-time state space model of the filter $\mathbf{F}_n(s)$. Note that the discrete-time representation of the continuous-time filter $\mathbf{F}_n(s)$ depends also on the assumptions made on the input inter-sample behaviour. The popular assumptions considered here are zero order hold (ZOH) and first order hold (FOH). In the ZOH assumption the sampled signal is assumed to remain constant between the sampling instants, while for the FOH assumption the input signal is assumed to vary linearly between sampling instants. In an EIV environment, it is necessary to know the details of the discrete-time implementation of the SVF. This is because the observed data being noise corrupted, we also pass the measurement noise along with the true signal through the SVF. Therefore, at a later stage it is necessary to characterize the statistical properties of the SVF output. Hence in this section we give a brief description of the discrete-time state space model employed to implement the SVF. Although our discussion here is limited to ZOH and FOH assumptions[2], the following theory can be extended to higher order hold assumptions.

Consider that an arbitrary signal $w(t)$ is the filter input $\mathbf{F}_n(s)$ and denote the corresponding $(n+1)\times 1$ vector valued output by $\boldsymbol{w}(t)$. Then from the standard results on controllable canonical form representation of continuous-time systems, the filtering operation can be expressed in the following state space form:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \mathbf{A}\boldsymbol{x}(t) + \boldsymbol{e}w(t),\\ v(t) &= \boldsymbol{f}'\boldsymbol{x}(t) + w(t),\end{aligned}$$

where $\boldsymbol{e}$ is the first column of the n dimensional identity matrix and

$$\mathbf{A} = \begin{bmatrix} -f_1 & \cdots & -f_{n-1} & -f_n \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \ \boldsymbol{f} = \begin{bmatrix} -f_1 \\ -f_2 \\ \vdots \\ -f_n \end{bmatrix}.$$

Consequently, the output vector $\boldsymbol{w}(t)$ is given by

$$\boldsymbol{w}(t) = [\, v(t) \ \boldsymbol{x}'(t)\,]'. \tag{7}$$

In order to compute the output $\boldsymbol{w}(t_k)$ at sampling instants $\{t_k\}_{k=1}^N$ from the input samples $\{w(t_k)\}_{k=1}^N$, we can discretize the above state space representation. Introduce

$$h_k := t_{k+1} - t_k. \tag{8}$$

At this point we shall assume that $\mathbf{A}$ is invertible, *i.e.* the SVF is asymptotically stable. This is, in fact, a natural assumption. Then for ZOH assumption one can show that the SVF can be implemented using the sampled data in the following state space form (Sinha and Rao, 1991):

$$\boldsymbol{x}(t_{k+1}) = \mathrm{e}^{\mathbf{A}h_k}\boldsymbol{x}(t_k) + [\mathrm{e}^{\mathbf{A}h_k} - \mathbf{I}]\mathbf{A}^{-1}\boldsymbol{e}w(t_k).$$

Under FOH assumption, one can show that the state vector $x(t_k)$ satisfies (Sinha and Rao, 1991)

$$\boldsymbol{x}(t_{k+1}) = \mathrm{e}^{\mathbf{A}h_k}\boldsymbol{x}(t_k) + \boldsymbol{\beta}_{1k}w(t_{k+1}) + \boldsymbol{\beta}_{0k}w(t_k),$$

where $\boldsymbol{\beta}_{0k}$ and $\boldsymbol{\beta}_{1k}$ are defined as

$$\begin{aligned}\boldsymbol{\beta}_{0k} &= \{\mathrm{e}^{\mathbf{A}h_k}\mathbf{A}^{-1} - (1/h_k)(\mathrm{e}^{\mathbf{A}h_k} - \mathbf{I})\mathbf{A}^{-2}\}\boldsymbol{e},\\ \boldsymbol{\beta}_{1k} &= \{(1/h_k)(\mathrm{e}^{\mathbf{A}h_k} - \mathbf{I})\mathbf{A}^{-2} - \mathbf{A}^{-1}\}\boldsymbol{e}.\end{aligned}$$

Once $\boldsymbol{x}(t_k)$ is computed, the computation of $v(t_k)$ is straightforward in both cases (ZOH and FOH):

$$v(t_k) = \boldsymbol{f}'x(t_k) + w(t_k).$$

Therefore, the state space implementation of the SVF output takes the following form:

$$\begin{aligned}\boldsymbol{x}_\star(t_{k+1}) &= \boldsymbol{\Psi}_k\boldsymbol{x}_\star(t_k) + \boldsymbol{\xi}_k w_\star(t_k),\\ v(t_k) &= \boldsymbol{\gamma}'\boldsymbol{x}_\star(t_k) + \delta w_\star(t_k),\end{aligned} \tag{9}$$

where for ZOH assumption we have

$$\boldsymbol{x}_\star(t_k) = \boldsymbol{x}(t_k), \ \ w_\star(t_k) = w(t_k) \tag{10}$$

and

$$\begin{aligned}\boldsymbol{\Psi}_k &= \mathrm{e}^{\mathbf{A}h_k}, & \boldsymbol{\gamma} &= \boldsymbol{f},\\ \boldsymbol{\xi}_k &= [\mathrm{e}^{\mathbf{A}h_k} - \mathbf{I}]\mathbf{A}^{-1}\boldsymbol{e}, & \delta &= 1.\end{aligned} \tag{11}$$

For the FOH assumption we have an augmented state vector and time delayed input signal:

$$\boldsymbol{x}_\star(t_k) = [\,\boldsymbol{x}'(t_k) \ w(t_k)\,]', \ \ w_\star(t_k) = w(t_{k+1}), \tag{12}$$

while the matrices in the state space form are given by

$$\begin{aligned}\boldsymbol{\Psi}_k &= \begin{bmatrix} \mathrm{e}^{\mathbf{A}h_k} & \boldsymbol{\beta}_{0k} \\ 0 & 0 \end{bmatrix}, & \boldsymbol{\xi}_k &= \begin{bmatrix} \boldsymbol{\beta}_{1k} \\ 1 \end{bmatrix},\\ \boldsymbol{\gamma} &= [\,\boldsymbol{f}' \ 1\,]', & \delta &= 0.\end{aligned} \tag{13}$$

[1] We assume here that the simulation errors can be neglected in comparison with the estimation errors introduced by the presence of additive noise on the input-output data. Therefore, we assume the numerical simulation errors to be negligible, and concentrate in the sequel on the effect of the additive noise on the SVF outputs.

[2] In most applications ZOH and FOH assumptions are sufficient to get estimates of desired accuracy.

4. NOISE EFFECTS ON SVF

One of the main step involved in the development of an identification approach for continuous-time dynamic EIV models is to analyze the statistical behaviour of the SVF output. Note that in presence of additive measurement noise in the observed data, the SVF output will also be noise contaminated. In this section we give a second-order statistical analysis of the SVF output vector when a zero-mean white noise of unit variance contaminates the SVF input. In that goal, consider that the SVF input $w(t_k)$ at sampling instant t_k has two parts:

$$w(t_k) = w_0(t_k) + \tilde{w}(t_k),$$

where $w_0(t_k)$ is deterministic, and $\tilde{w}(t_k)$ is the noise contribution. Furthermore, assume that $\{\tilde{w}(t_k)\}_{k=1}^{N}$ constitutes a zero-mean white noise sequence of unit variance. Using a similar notation as above we shall denote the noise-free part of the SVF output $\boldsymbol{w}(t_k)$ by $\boldsymbol{w}_0(t_k)$, while the noise contribution in $\boldsymbol{w}(t_k)$ will be denoted by $\tilde{\boldsymbol{w}}(t_k)$. Our main objective here is to study the asymptotic ($N \to \infty$) second-order statistical properties of the matrix

$$\mathbf{W}_N = \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{w}(t_k)\boldsymbol{w}'(t_k).$$

We emphasize that the estimation algorithms to be developed later employ matrices having similar form as $\mathbf{W}_N$. This is the reason why we analyze the asymptotic behaviour of $\mathbf{W}_N$. Now from (7), (10) and (12) we see that $\boldsymbol{w}(t_k)$ is a linear function of $\boldsymbol{x}_\star(t_k)$, and the associated mapping is time-invariant. Therefore, we shall characterize the asymptotic properties of

$$\mathbf{X}_N = \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{x}_\star(t_k)\boldsymbol{x}'_\star(t_k).$$

The following result plays a key role in the development of the algorithms for continuous-time EIV models.

Proposition 1. Let the sampling intervals h_k have a positive lower bound such that $h_k \geq h > 0$. Assume also that the sequence $\tilde{w}(t_k)$ has bounded fourth-order moments, and the signal $w_0(t)$ is bounded for all t. Then

$$\lim_{N\to\infty} \mathbf{X}_N - \mathcal{E}\,\mathbf{X}_N = 0.$$

with probability one, where $\mathcal{E}$ denotes the mathematical expectation operator.

Proof: Due to space limitations the proof of the proposition will be omitted here. However, we shall give an outline. At the first step it is shown that there exists a sequence of positive numbers r_k such that $r_k \to 0$ as $k \to \infty$ and the elements of the matrix $|\mathcal{E}\,\boldsymbol{w}(t_k)\boldsymbol{w}'(t_l)|$ are bounded above by r_{k-l} for $k \geq l$. At this point we use the fact that the eigenvalues of the matrix $\boldsymbol{\Psi}_k$ are strictly inside the unit disc [3]. It is also required to use the assumption $h_k \geq h > 0$. Subsequently, standard ergodicity results (Ljung, 1999) for stationary stochastic processes can be extended in this case to derive the final result. ■

Note that for uniform sampling the discrete-time system in (13) is stable and time-invariant, and in that case the above proposition is well-known, see (Ljung, 1999), for example.

Next, we shall examine the quantity $\mathcal{E}\,\mathbf{X}_N$. Denote the noise-free part of $\boldsymbol{x}_\star(t_k)$ by $\boldsymbol{x}_{\star 0}(t_k)$, and the noise contribution by $\tilde{\boldsymbol{x}}_\star(t_k)$. Then it is readily verified that

$$\mathcal{E}\,\mathbf{X}_N = \mathbf{X}_{0N} + \widetilde{\mathbf{X}}_N, \tag{14}$$

where

$$\begin{aligned} \mathbf{X}_{0N} &= \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{x}_{\star 0}(t_k)\boldsymbol{x}'_{\star 0}(t_k), \\ \widetilde{\mathbf{X}}_N &= \frac{1}{N}\sum_{k=1}^{N} \mathbf{E}_k, \quad \mathbf{E}_k = \mathcal{E}\,\{\tilde{\boldsymbol{x}}_\star(t_k)\tilde{\boldsymbol{x}}'_\star(t_k)\}. \end{aligned} \tag{15}$$

Next we examine $\widetilde{\mathbf{X}}_N$. For that purpose let us introduce a few more notations. Define the doubly indexed sequence

$$\boldsymbol{\Phi}(k,l) = \begin{cases} \mathbf{I}, & l = k+1 \\ \boldsymbol{\Phi}(k,l+1)\boldsymbol{\Psi}_l, & l \leq k \leq 2 \end{cases} \tag{16}$$

Also note that

$$\boldsymbol{\Phi}(k,l) = \boldsymbol{\Psi}_k \boldsymbol{\Phi}(k-1,l). \tag{17}$$

Then using (13) recursively we can derive that

$$\boldsymbol{x}_\star(t_k) = \sum_{l=1}^{k-1} \boldsymbol{\Phi}(k-1,l+1)\boldsymbol{\xi}_l w_\star(t_l).$$

Therefore, using the whiteness assumption on the measurement noise sequence $\{\tilde{w}(t_k)\}_{k=1}^{N}$, it can be verified after a few steps of straightforward calculations that

$$\mathbf{E}_k = \sum_{l=1}^{N-1} \boldsymbol{\Phi}(k-1,l+1)\boldsymbol{\xi}_l\boldsymbol{\xi}'_l\boldsymbol{\Phi}'(k-1,l+1).$$

In the estimation algorithms to be described later it is required to compute $\widetilde{\mathbf{X}}_N$. The computation might appear to be computationally demanding. However, $\widetilde{\mathbf{X}}_N$ can be computed efficiently by a recursion. Using (16) and (17) in the last equation we can check that $\mathbf{E}_k$ is given recursively as the solution to the Lyapunov equation (for a time varying system)

$$\mathbf{E}_{k+1} = \begin{cases} \boldsymbol{\Psi}_k\mathbf{E}_k\boldsymbol{\Psi}'_k + \boldsymbol{\xi}_k\boldsymbol{\xi}'_k, & k > 1, \\ \boldsymbol{\xi}_1\boldsymbol{\xi}'_1, & k = 1. \end{cases} \tag{18}$$

[3] This follows from (8), (11) and (13) because $F(s)$ has all its roots in the left half plane.

Now from (15) we see that $\widetilde{\mathbf{X}}_k$ satisfies the recursion

$$\widetilde{\mathbf{X}}_{k+1} = \frac{k}{k+1}\widetilde{\mathbf{X}}_k + \frac{1}{k+1}\mathbf{E}_{k+1}. \qquad (19)$$

We note by passing that for uniform sampling, the linear system (13) is time-invariant. Then for large N, the matrix $\mathbf{E}_N$ converges to the solution to the associated steady state Lyapunov equation. It is also easy to verify that

$$\lim_{N\to\infty} \widetilde{\mathbf{X}}_N - \mathbf{E}_N = 0$$

for the time-invariant case. Therefore, $\widetilde{\mathbf{X}}_N$ also satisfies the same Lyapunov equation.

We conclude this section by evaluating $\mathbf{W}_N$ from $\mathbf{X}_N$. From (14) and (15) we see that for sufficiently large N, we can approximate $\mathbf{W}_N$ as

$$\mathbf{W}_N = \mathbf{W}_{0N} + \widetilde{\mathbf{W}}_N, \qquad (20)$$

where

$$\mathbf{W}_{0N} = \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{w}_0(t_k)\boldsymbol{w}_0(t_k), \qquad (21)$$

$$\widetilde{\mathbf{W}}_N = \begin{bmatrix} \boldsymbol{\gamma}'\widetilde{\mathbf{X}}_N\boldsymbol{\gamma} + \delta^2 & \boldsymbol{\gamma}'\widetilde{\mathbf{X}}_N(1:n,:) \\ \widetilde{\mathbf{X}}_N(:,1:n) & \widetilde{\mathbf{X}}_N(1:n,1:n) \end{bmatrix}. (22)$$

Note that in the last equality we have used (7), the state space descriptions in (9)-(13), and common Matlab notation. We emphasize that (18) holds for both ZOH and FOH assumptions.

5. ERRORS-IN-VARIABLES PROBLEM

When we deal with an EIV problem using a direct approach, we have noise corrupted regressors. Assume that we pass the input-output observations $u(t)$ and $y(t)$ through the filters $\mathbf{F}_m(s)$ and $\mathbf{F}_n(s)$, respectively as in (5) and denote the corresponding filter outputs as $\boldsymbol{u}(t_k)$ and $\boldsymbol{y}(t_k)$. Using these regressors we form the matrices

$$\mathbf{Y}_N = \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{y}(t_k)\boldsymbol{y}'(t_k), \ \mathbf{U}_N = \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{u}(t_k)\boldsymbol{u}'(t_k).$$

Then from the discussion in the previous section we have

$$\mathbf{Y}_N = \mathbf{Y}_{0N} + \sigma_y\widetilde{\mathbf{Y}}_N, \ \mathbf{U}_N = \mathbf{U}_{0N} + \sigma_u\widetilde{\mathbf{U}}_N,$$

where

$$\mathbf{Y}_{0N} = \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{y}_0(t_k)\boldsymbol{y}_0'(t_k),$$
$$\mathbf{U}_{0N} = \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{u}_0(t_k)\boldsymbol{u}_0'(t_k).$$

are the contribution from the noise-free part of the observed data, while $\widetilde{\mathbf{U}}_N$ and $\widetilde{\mathbf{Y}}_N$ are the asymptotic contribution from the measurement noise. Note that the noise sequences are not necessarily unit-variance. Hence we have the scaling σ_y and σ_u. The matrices $\widetilde{\mathbf{U}}_N$ and $\widetilde{\mathbf{Y}}_N$ can be computed using the algorithm described in the previous section. We also form the matrix

$$\mathbf{V}_N = \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{y}(t_k)\boldsymbol{u}'(t_k),$$

for which a similar result as proposition 1 can be proved where we can show for large N that

$$\mathbf{V}_N \to \frac{1}{N}\sum_{k=1}^{N} \boldsymbol{y}_0(t_k)\boldsymbol{u}_0'(t_k)$$

with probability one. Therefore a calculation similar to that of (6) gives

$$\begin{bmatrix} \mathbf{Y}_N & \mathbf{V}_N \\ \mathbf{V}_N' & \mathbf{U}_N \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} \sigma_y\widetilde{\mathbf{Y}}_N\boldsymbol{a} \\ \sigma_u\widetilde{\mathbf{U}}_N\boldsymbol{b} \end{bmatrix}. \qquad (23)$$

In the last equation we have $n+m+3$ unknowns to solve from $n+m+2$ equations. Therefore, we need to find additional equations in order to solve the unknowns uniquely. The approach we adopt here to circumvent this problem is to use two different prefilters. This means that we obtain two sets of regressors $\{\boldsymbol{y}^{(i)}(t_k), \boldsymbol{u}^{(i)}(t_k)\}_{i=1}^2$ using two different prefilter pairs $\{\mathbf{F}_n^{(i)}(s), \mathbf{F}_m^{(i)}(s)\}_{i=1}^2$. Consequently we get two systems of equations like (23):

$$\begin{bmatrix} \mathbf{Y}_N^{(i)} & \mathbf{V}_N^{(i)} \\ \mathbf{V}_N^{(i)'} & \mathbf{U}_N^{(i)} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} \sigma_y\widetilde{\mathbf{Y}}_N^{(i)}\boldsymbol{a} \\ \sigma_u\widetilde{\mathbf{U}}_N^{(i)}\boldsymbol{b} \end{bmatrix}, \ i = 1,2; \qquad (24)$$

where $\mathbf{Y}_N^{(i)}$ etc, denote the matrices obtained from $\boldsymbol{y}^{(i)}(t_k)$ etc. It is now straightforward to eliminate σ_y and σ_u from the above systems of equations. We have

$$\begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = 0 \qquad (25)$$

where

$$\begin{aligned} \mathbf{Z}_{11} &= \widetilde{\mathbf{Y}}_N^{(1)-1}\mathbf{Y}_N^{(1)} - \widetilde{\mathbf{Y}}_N^{(2)-1}\mathbf{Y}_N^{(2)} \\ \mathbf{Z}_{12} &= \widetilde{\mathbf{Y}}_N^{(1)-1}\mathbf{V}_N^{(1)} - \widetilde{\mathbf{Y}}_N^{(2)-1}\mathbf{V}_N^{(2)} \\ \mathbf{Z}_{21} &= \widetilde{\mathbf{U}}_N^{(1)-1}\mathbf{V}_N^{(1)'} - \widetilde{\mathbf{U}}_N^{(2)-1}\mathbf{V}_N^{(2)'} \\ \mathbf{Z}_{22} &= \widetilde{\mathbf{U}}_N^{(1)-1}\mathbf{U}_N^{(1)} - \widetilde{\mathbf{U}}_N^{(2)-1}\mathbf{U}_N^{(2)} \end{aligned}$$

Therefore a least squares or total least squares solution of (25) leads to the estimates of $\boldsymbol{a}$ and $\boldsymbol{b}$.

6. ILLUSTRATIVE EXAMPLE

In the simulations we consider the system (Chou *et al.*, 1999)

$$y_0^{(2)}(t) + 2y_0^{(1)}(t) + y_0(t) = u_0^{(1)}(t) - u_0(t).$$

The input $u_0(t)$ is chosen as a white binary ± 1 signal obtained as $sign(randn(N,1))$. Note that in the case of the chosen piece-wise constant excitation signal, the system response can be

	a_1	a_2	b_1	b_0
True value	2	1	1	−1
Mean	2.01	0.97	0.96	−1.01
MSE	0.028	0.019	0.113	0.024

Table 1. Simulation results for SNR = 5 dB.

	a_1	a_2	b_1	b_0
True value	2	1	1	−1
Mean	1.99	0.99	0.98	−0.99
MSE	0.004	0.004	0.025	0.005

Table 2. Simulation results for SNR = 10 dB.

calculated *exactly* at the sampling instances via appropriate ZOH discretization of the continuous-time system. The variance σ_y of the additive noise at the output is 0.0886. The input noise variance σ_u is 0.3164. With this choice the signal-to-noise ratio (SNR) at both input and output is 5 dB. The signals are sampled uniformly with a sampling interval 0.01 sec. The number of samples is $N = 1785$, *i.e.* the observation time is 17.85 sec. The denominators of the two prefilters for reconstructing the derivatives are chosen as

$$F^{(1)}(s) = s^2 + 2s + 1, \qquad F^{(2)}(s) = s^2 + 2s + 5.$$

In the identification process, we used ZOH assumption for the input signal and FOH assumption for the output signal. The estimation results are shown in Table 1. They are based on 100 independent Monte Carlo simulations. In the table we have shown the mean values of the estimates obtained from 100 runs along with the mean square error (MSE). Results for 10 dB input-output SNR are shown in Table 2.

7. CONCLUSIONS

In this paper we have addressed the problem of identifying continuous-time dynamic EIV models using a direct identification approach. In that goal, we have presented a new framework for analyzing the noise effect on the state-variable filtering. The results therefrom are used to develop an algorithm for identifying continuous-time dynamic EIV models. The new algorithm employs two state-variable filters. It does not require any numerical search and is computationally efficient. The proposed method can be applied to non-uniformly sampled data and extended for identifying discrete-time EIV models.

8. REFERENCES

Anderson, B. D. O. and M. Deistler (1984). Identifiability of dynamic errors-in-variables models. *J. Time Series Analysis* **5**, 1–13.

Beghelli, S., R.P. Guidorzi and U. Soverini (1990). The Frisch scheme in dynamic system identification. *Automatica* **26**, 171–176.

Chou, C.T., M. Verhaegen and R. Johansson (1999). Continuous-time identification of SISO systems using Laguerre functions. *IEEE Transactions on Signal Processing* **47**(2), 349–362.

Garnier, H. and P.C. Young (2004). Time-domain approaches for continuous-time model identification from sampled data. In: *Invited tutorial paper for the American Control Conference (ACC'2004).* Boston, MA (USA).

Garnier, H., M. Mensler and A. Richard (2003). Continuous-time model identification from sampled data. implementation issues and performance evaluation. *International Journal of Control* **76:13**, 1337–1357.

Ljung, L. (1999). *System Identification - Theory for the User, 2nd edition.* Prentice Hall. Upper Saddle River, NJ, USA.

Markovski, I., J.C. Willems and B. De Moor (2002). Continuous-time errors-in-variables filtering. In: *41st IEEE Conference on Decision and Control (CDC'02).* Las-Vegas, Nevada (USA). pp. 2576–2581.

Schoukens, J., R. Pintelon, G. Vandersteen and P. Guillaume (1997). Frequency-domain system identification using non-parametric noise models estimated from a small number of data sets. *Automatica* **33:6**, 1073–1086.

Sinha, N. K. and Rao, G.P., Eds.) (1991). *Identification of continuous-time systems: Methodology and com-puter implementation.* Kluwer Academic Publishers. Dordrecht.

Söderström, T (1981). Identification of stochastic linear systems in presence of input noise. *Automatica* **17**, 713–725.

Söderström, T. and K. Mahata (2002). On instrumental variable and total least squares approaches for identification of noisy systems. *International Journal of Control* **75**(6), 381–389.

Söderström, T., U. Soverini and K. Mahata (2002). Perspectives on errors-in-variables estimation for dynamic systems. *Signal Processing* **82**, 1139–1154.

Stoica, P, M Cedervall and A Eriksson (1995). Combined instrumental variable and subspace fitting approach to parameter estimation of noisy input-output systems. *IEEE Transactions on Signal Processing* **43**, 2386–2397.

Tugnait, J. K. and Y. Ye (1995). Stochastic system identification with noisy input-output measurement using polyspectra. *IEEE Transactions on Automatic Control* **AC-40**, 670–683.

Zheng, W. X. (1998). Transfer function estimation form noisy input and output data. *International Journal of Adaptive Control and Signal Processing* **12**, 365–380.