ANALYSIS OF RANDOM REFERENCE TRACKING IN SYSTEMS WITH SATURATING ACTUATORS

Yongsoon Eun, Pierre T. Kabamba
Semyon M. Meerkov

Dept. of Electrical Engineering and Computer Science,
University of Michigan, Ann Arbor, MI 48109

Abstract: This paper develops a method for analysis of random reference tracking in feedback systems with saturating actuators. The development is motivated by the frequency domain approach to linear systems, where the bandwidth and resonance peak of the sensitivity function are used to predict the quality of step reference tracking. Similarly, based on the so-called saturating random sensitivity function, we introduce tracking quality indicators and show that they can be used to determine both the quality of random reference tracking and the nature of track loss under actuator saturation. *Copyrights© 2005 IFAC*

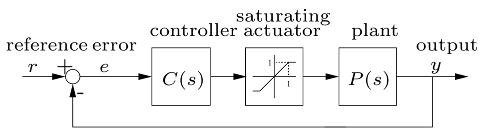
Keywords: Reference tracking, Saturating actuators, Tracking quality indicators

1. INTRODUCTION

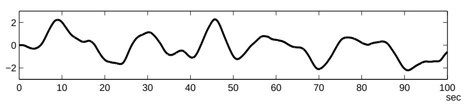
1.1 Motivation

As it is well known, the quality of reference tracking in linear systems is determined by the loop transfer function. In systems with saturating actuators, this is not the case. Indeed, for example, consider the SISO feedback system and the reference signal shown in Figure 1, where the latter is a realization of a colored noise process with power spectral density $S_R(\omega) = \frac{6}{1+(\omega/0.5)^6}$. The quality of tracking for several $C(s)$ and $P(s)$, satisfying $C(s)P(s) = \frac{75}{s(s+10)}$, is illustrated in Figure 2. (In Figure 2(a), the reference and the output signals practically coincide.) Clearly, the nature of tracking errors in each of the three cases is qualitatively different, which supports the above assertion.

The track loss in systems with saturating actuators may occur due to a number of different reasons. These include those that occur in linear systems plus those due to actuator saturation. To illustrate these reasons, consider again the system and the reference signal of Figure 1 and select $C(s)$ and $P(s)$, which result in different patterns of track loss but with the same standard deviation of the tracking error, σ_e. The results are shown in Figure 3 (for $\sigma_e = 0.67$). As one can see, track loss in Figures 3 (a)–(c) is due to static unresponsiveness, dynamic lagging, and oscillatory behavior, respectively. These reasons take place in the purely linear case as well (see (Eun *et al.* 2003)). Track loss in Figures 3 (d)–(g) is due to saturation, namely, amplitude truncation without controller wind-up, amplitude truncation with the controller wind-

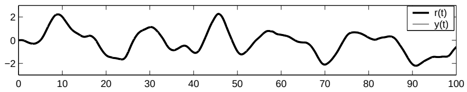


(a) Feedback system with saturating actuator

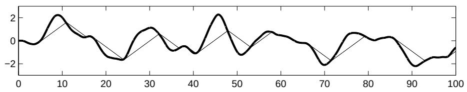


(b) Reference signal $r(t)$

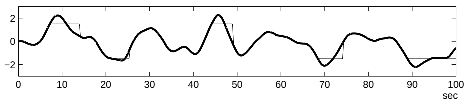
Fig. 1. Feedback control system with saturating actuator and reference signal



(a) $C(s) = 5$ and $P(s) = \frac{15}{s(s+10)}$



(b) $C(s) = 27$ and $P(s) = \frac{25/9}{s(s+10)}$

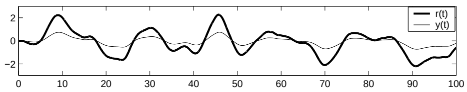


(c) $C(s) = \frac{5}{s}$ and $P(s) = \frac{15}{s+10}$

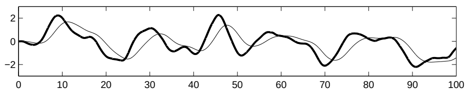
Fig. 2. Tracking of random reference in system of Figure 1(a)

up, nonlinear lagging, and nonlinear oscillations, respectively.

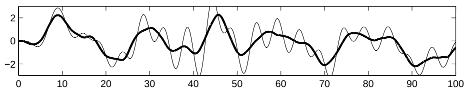
The goals of this paper are to analyze what determines the quality of tracking in systems with saturating actuators and quantify under which conditions one or another type of track loss takes place.



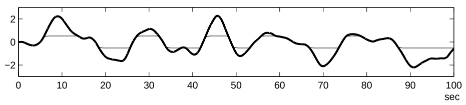
(a) $C(s) = 4/(s + 10)$ and $P(s) = 6.2/(s + 5)$



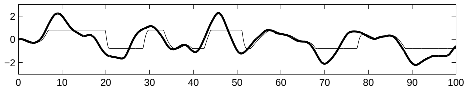
(b) $C(s) = 0.4$ and $P(s) = 0.8125(s+0.1)/s(s+0.02)$



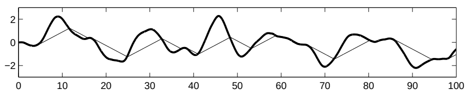
(c) $C(s) = 0.0112(s+30.533)/s$ and $P(s) = 4/s$



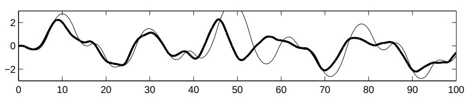
(d) $C(s) = 100$ and $P(s) = 11/(s + 3)(s + 7)$



(e) $C(s) = 2/s$ and $P(s) = 8/(s + 10)$



(f) $C(s) = 25$ and $P(s) = 2/s(s + 10)$



(g) $C(s) = 4.126(s + 1.102)$ and $P(s) = 0.355/s^2$

Fig. 3. Track loss in system of Figure 1(a)

1.2 Approach

In the case of linear systems, the quality of step input tracking is often characterized in the frequency domain by the Sensitivity (S) function, specifically, by its d.c. gain, bandwidth, and resonance peak. Recently, this approach has been extended to tracking random inputs by introducing the notion of random sensitivity (RS) function (Eun *et al.* 2003). In particular, it has been shown, that the d.c. gain, bandwidth, and resonance peak of the RS function characterize the quality of random reference tracking in linear systems in the same manner as the S function characterizes the quality of tracking steps. In the current paper, we extend this approach to systems with saturating actuators. This is accomplished by introducing and analyzing the so-called saturating random sensitivity (SRS) function. Due to actuator nonlinearity, the SRS function depends not only on the frequency but also on the "amplitude" of the signals involved and, therefore, is a function of two independent variables. We provide a method for calculating the SRS using a quasi-linearization technique known as stochastic linearization (Roberts and Spanos 1990). In (Gökçek *et al.* 2001), stochastic linearization has been used for analysis and design of systems with saturating actuators from the point of view of disturbance rejection. In this paper, we use it in the framework of reference tracking.

1.3 Related Literature and Paper Outline

Systems with saturating actuators have been studied for a long time (see recent monographs (Saberi *dt al.* 2000, Hu and Lin 2001, Kapila 2002)). However, just a few publications have been devoted to reference tracking. These include (Yakubovich *et al.* 1999) where tacking domains have been investigated, (Saberi *dt al.* 2000) where asymptotic output tracking has been studied, (Goldfarb and Sirithanapipat 1999) where random reference tracking by a servo with a PD-controller has been analyzed, and (Eun *et al.* 2004a) where the notion of system type has been extended to feedback control with saturating actuators. However, no general methods for analysis of quality of random reference tracking in systems with saturating actuators exist. This paper is intended to contribute to this end.

To accomplish this, Section 2 introduces the SRS and its characteristics: d.c. gain, bandwidth, resonance frequency and resonance peak. In Section 3, we use these characteristics to define dimensionless tracking quality indicators and diagnostic flow charts. Finally, in Section 4 the conclusions are given. Due to space limitations the proofs are not included here and can be found in (Eun *et al.* 2004b).

2. SATURATING RANDOM SENSITIVITY FUNCTION

2.1 Random Reference Signals

Similar to (Eun *et al.* 2003), the class of random reference signals, considered in the work, is defined as the scaled steady state output of the third order Butterworth filter driven by a standard white Gaussian process. The transfer function of this filter is given by

$$F(s;\Omega) = \sqrt{\frac{3}{\Omega}}\left(\frac{\Omega^3}{s^3+2\Omega s^2+2\Omega^2 s+\Omega^3}\right), \quad (1)$$

where the d.c. gain is selected so that, for all 3-dB bandwidths Ω, the standard deviation of the output is 1. Thus, the reference signals considered in this work are given by

$$r(t) = \sigma_r r(t;\Omega), \quad (2)$$

where $r(t;\Omega)$ is the output of (1) and σ_r is the "amplitude" or, more precisely, the standard deviation of $r(t)$.

Clearly, higher order Butterworth filters can be considered instead of (1). However, as it turns out, the results remain quite similar to those obtained using (1) (see also (Eun *et al.* 2003)) and, thus, for the sake of simplicity, we consider band-limited reference signals $r(t)$ defined by (1) and (2).

2.2 System Model

Consider the system shown in Figure 4 with reference signal (1), (2) and $\text{sat}_\alpha(u)$ defined by

$$\text{sat}_\alpha(u) = \begin{cases} \alpha & \text{if } \alpha < u, \\ u & \text{if } -\alpha \le u \le \alpha, \\ -\alpha & \text{if } u < -\alpha. \end{cases} \quad (3)$$

Due to the nonlinearity, exact analysis of this system requires solving the Fokker-Plank equation, which is possible only in a few special cases. Therefore, a simplification is necessary. We use for this purpose the method of stochastic linearization (Roberts and Spanos 1990). According to this method, the saturation function is replaced by a linear function, the slope of which depends on the standard deviation of the signal at the input of the saturation. This method is akin to the method of describing functions and ensures similar accuracy.

Using stochastic linearization, the nonlinear system of Figure 4 can be replaced by the quasi-linear

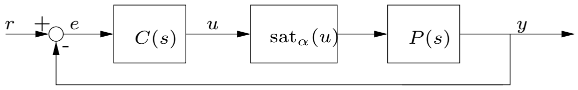


Fig. 4. System with saturating actuator

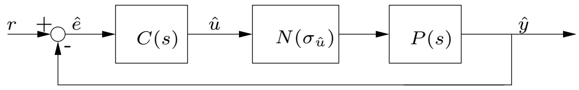


Fig. 5. Stochastically linearized system

system shown in Figure 5, where the equivalent gain $N(\sigma_{\hat{u}})$ is given by (Gökçek *et al.* 2001)

$$N(\sigma_{\hat{u}}) = \text{erf}\left(\frac{\alpha}{\sqrt{2}\sigma_{\hat{u}}}\right), \quad (4)$$

$$\text{erf}(\xi) = \frac{2}{\sqrt{\pi}}\int_0^\xi \exp(-t^2)\, dt. \quad (5)$$

The system of Figure 5 is quasi-linear since N depends on the standard deviation of $\hat{u}$.

The results reported in this paper are obtained using the simplified model of Figure 5. However, the system of Figure 4 is also used – to verify that the results derived are applicable to the original nonlinear system as well.

2.3 Definition and Properties of the Saturating Random Sensitivity Function

If N in Figure 5 were a constant gain equal to 1, the sensitivity and the random sensitivity functions of the closed loop system would be given by (Eun *et al.* 2003)

$$S(s) = \frac{1}{1+P(s)C(s)}, \quad (6)$$

$$RS(\Omega) = \left|\left|\frac{F(s;\Omega)}{1+P(s)C(s)}\right|\right|_2. \quad (7)$$

These functions are extended to the case of the quasi-linear system of Figure 5 by defining the saturating random sensitivity (SRS) as follows:

$$SRS(\Omega,\sigma_r) = \left|\left|\frac{F(s;\Omega)}{1+NP(s)C(s)}\right|\right|_2, \quad (8)$$

$$N = \text{erf}\left(\frac{\alpha}{\sqrt{2}\sigma_r\left|\left|\frac{F(s;\Omega)C(s)}{1+NP(s)C(s)}\right|\right|_2}\right). \quad (9)$$

As one can see, physically $SRS(\Omega,\sigma_r)$ represents the ratio of the standard deviations of the error signal $\hat{e}(t)$ and reference signal $r(t)$, i.e., $SRS(\Omega,\sigma_r) = \sigma_{\hat{e}}/\sigma_r$.

Asymptotic properties of $SRS(\Omega,\sigma_r)$ are as follows:

Theorem 1. Assume that the closed loop system of Figure 5 is asymptotically stable for all $N \in (0,\ 1]$, $P(s)$ is strictly proper and $C(s)$ is proper. Then,

(i) for any $\Omega > 0$,

$$\lim_{\sigma_r\to 0} SRS(\Omega,\sigma_r) = \left|\left|\frac{F(s;\Omega)}{1+P(s)C(s)}\right|\right|_2; \quad (10)$$

(ii) for any $\sigma_r > 0$,

$$\lim_{\Omega\to\infty} SRS(\Omega,\sigma_r) = 1; \quad (11)$$

$$\lim_{\Omega\to 0} SRS(\Omega,\sigma_r) = \left|\frac{1}{1+NP(0)C(0)}\right|, \quad (12)$$

where N satisfies

$$N = \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}\sigma_r\left|\frac{C(0)}{1+NP(0)C(0)}\right|}\right). \quad (13)$$

Clearly, statement (10) implies that for small reference signals, $SRS(\Omega,\sigma_r)$ practically coincides with $RS(\Omega)$. Statement (11) indicates that for large Ω the functions $SRS(\Omega,\sigma_r)$, $RS(\Omega)$ and $S(s)$ are practically identical, and no tracking takes place. Finally, since $N \le 1$, statement (12) shows that for low frequencies $SRS(\Omega,\sigma_r)$ is typically larger than $RS(\Omega)$ and, thus, the presence of saturation impedes tracking.

Figures 6 and 7 illustrate the SRS functions for all systems of Figures 2 and 3, respectively. As it will be shown in Section 3, these functions define the nature of tracking and track loss in the corresponding systems.

2.4 Shape Characteristics

Although a complete description of $SRS(\Omega,\sigma_r)$ requires a two-dimensional surface, a compact (but incomplete) description can be given in terms of characteristics, similar to those used to describe the $S(s)$ and $RS(\Omega)$ functions. Namely, introduce

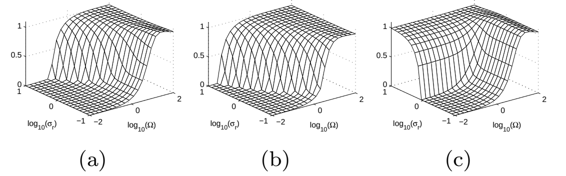


Fig. 6. $SRS(\Omega,\sigma_r)$ for systems of Figure 2.

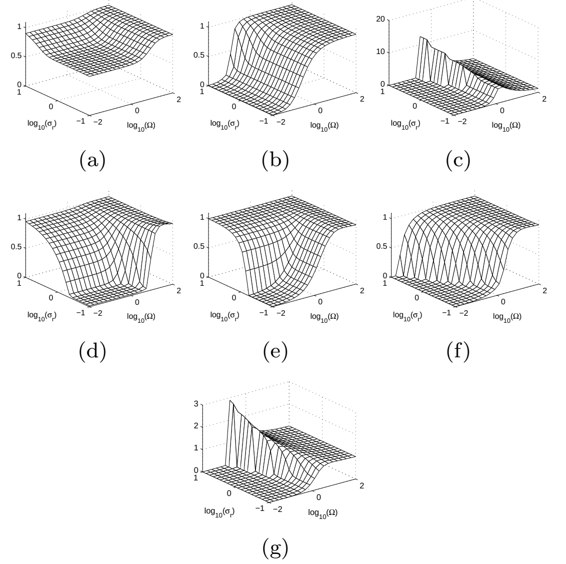


Fig. 7. $SRS(\Omega,\sigma_r)$ for systems of Figure 3.

Table 1. Trackable Domains for systems of Figure 2

	(a)	(b)	(c)
$\|TD\|$	∞	∞	1.5

Table 2. Trackable Domains for systems of Figure 3

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
$\|TD\|$	3.75	∞	∞	0.53	0.4	∞	∞

(i) saturating random d.c. gain:

$$SR_{dc} = \lim_{\Omega\to 0,\ \sigma_r\to 0} SRS(\Omega,\sigma_r), \quad (14)$$

(ii) saturating random bandwidth:

$$SR\Omega_{BW}(\sigma_r) = \min\{\Omega \,|SRS(\Omega,\sigma_r) = 1/\sqrt{2}\}, \quad (15)$$

(iii) saturating random resonance frequency:

$$SR\Omega_r(\sigma_r) = \arg\max_{\Omega>0} SRS(\Omega,\sigma_r), \quad (16)$$

(iv) saturating random resonance peak:

$$SRM_r(\sigma_r) = \sup_{\Omega>0} SRS(\Omega,\sigma_r). \quad (17)$$

For the SRS functions of Figures 7 (a) and (d), SR_{dc} are 0.67 and 0.019, respectively, while for all others it is 0. Clearly, one might expect that tracking of even small and slowly changing signals in the system of Figure 7 (a) is poor, and the track loss is due to static unresponsiveness.

The $SR\Omega_{BW}$ for all systems of Figures 2 and 3 are shown in Figures 8 and 9, respectively. In all cases $SR\Omega_{BW}$ is monotonically decreasing in σ_r, but systems of Figure 2 (c) and Figures 3 (a), (d), (e) result in $SR\Omega_{BW}$ with almost infinite roll-off rate. This phenomenon can be explained using the notion of Trackable Domain (TD) introduced in (Eun *et al.* 2004a). Indeed, it has been shown in (Eun *et al.* 2004a) that the set of step inputs that can be tracked by a system with saturating actuators and its size can be quantified, respectively, as

$$TD = \left\{r_0 \in \mathbb{R} : |r_0| < \left|\frac{1}{C_0} + P_0\right|\alpha\right\}, \quad (18)$$

$$|TD| = \left|\frac{1}{C_0} + P_0\right|\alpha, \quad (19)$$

where r_0 is the size of the step and C_0 and P_0 are d.c. gains of the controller and plant, respectively. Trackable domains for all systems of Figures 2 and 3 are given in Tables 1 and 2, respectively. Clearly, systems of Figure 2 (c) and Figures 3 (a), (d), (e) have finite trackable domains and, therefore, their bandwidth must drop to 0 for σ_r sufficiently large, no matter how small Ω is.

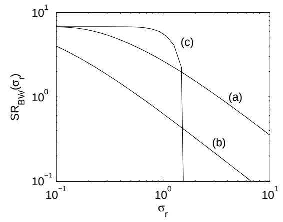


Fig. 8. Saturating random bandwidth for systems of Figure 2.

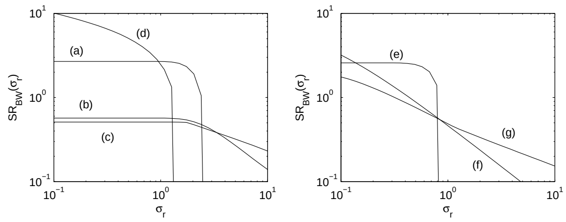


Fig. 9. Saturating random bandwidth for systems of Figure 3.

The $SRS(\Omega, \sigma_r)$ function and its characteristics are used in Section 3 to quantify the nature of random reference tracking and track loss in systems with saturating actuators.

2.5 Accuracy of SRS and its Shape Characteristics

The saturating random sensitivity function provides an estimate, $\sigma_{\hat{e}}$, of the steady state tracking error, σ_e, in the system of Figure 4 as follows:

$$\sigma_{\hat{e}} = \sigma_r \; SRS(\Omega, \sigma_r). \qquad (20)$$

Unfortunately, the question of accuracy of this estimate has no general answer. In a few special cases, where σ_e can be found exactly by solving the corresponding Fokker-Planck equation, this accuracy has been shown to be quite high (well within 10%) (Roberts and Spanos 1990, Gökçek *et al.* 2001). In other cases, the accuracy of this estimate has been evaluated numerically (Gökçek *et al.* 2001). It has been observed that the accuracy is high for small and large σ_r (regardless of Ω) and for large Ω (regardless of σ_r). For some intermediate values of Ω and σ_r, the approximation could be poor. However, since the four shape characteristics, i.e., SR_{dc}, $SR\Omega_{BW}(\sigma_r)$, $SR\Omega_r(\sigma_r)$, $SRM_r(\sigma_r)$, are defined in the domains where the accuracy is typically high, we use them for tracking quality analysis in systems with saturating actuators.

3. TRACKING QUALITY INDICATORS AND DIAGNOSTIC FLOW CHARTS

The tacking quality indicators are introduced as follows:

$$I_0 = \frac{\sigma_r}{|TD|}, \qquad (21)$$

$$I_1 = SR_{dc}, \qquad (22)$$

$$I_2 = \frac{\Omega}{SR\Omega_{BW}(\sigma_r)}, \qquad (23)$$

$$I_3 = \min\left(\frac{\Omega}{SR\Omega_r(\sigma_r)},\ SRM_r(\sigma_r) - 1\right). \qquad (24)$$

Clearly, I_0 quantifies the "size" of the reference signal vis-a-vis the trackable domain; large I_0 implies that amplitude truncation must take place. Indicator I_1 quantifies the level of static responsiveness; large I_1 implies that responsiveness, even to small and slow signals, is poor. Indicator I_2 quantifies the bandwidth of the reference signal in units of the closed loop bandwidth; large I_2 implies that dynamic lagging must take place. Finally, I_3 characterizes oscillatory properties of the response; large I_3 implies that oscillations must be present.

Although indicators I_1–I_3 are proper extensions of the corresponding tracking quality indicators for linear systems (Eun *et al.* 2003), they may be large due to either linear or nonlinear part of the system. The two cases can be discriminated by the value of the equivalent gain, N, defined by (9). Specifically if N is close to 1, the phenomenon is caused by the linear part of the system, otherwise, it is due to saturation.

Based on the above discussion, the nature of tracking quality and reasons for track loss can be diagnosed using the flow charts shown in Figure 10. Each of them includes a qualitative term "large". Based on our experience, an indicator can be viewed as large if

$$I_0 > 0.4,\ I_1 > 0.1,\ I_2 > 0.4,\ I_3 > 0.2. \qquad (25)$$

Consider, for example, the system of Figure 4 with $C(s) = 5/s$, $P(s) = 15/(s+10)$ and $r(t) = 1.5\, r(t; 10)$. The tracking quality indicators for this system and reference signal are:

$$I_0 = 1, I_1 = 0, I_2 = 4.705, I_3 = 0.078, \qquad (26)$$

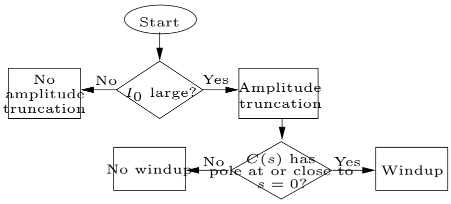
while $N = 0.47$. Thus, using Figure 10 (a) we determine that tracking is poor due to the amplitude truncation with wind-up . Using Figure 10 (b), we conclude that there is no loss of tracking due to unresponsiveness. Using Figure 10 (c), we expect lagging due to saturation (i.e., nonlinear lagging). These conclusions are supported by the traces of $y(t)$ (obtained by simulating the system of Figure 4) shown in Figure 11.

Table 3 presents the tracking quality indicators and the conclusions as to the nature of tracking and track loss for all systems considered in Section 1.

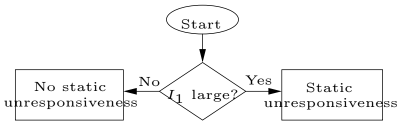
Remark The diagnostics approach, described above, leads to qualitatively correct results in the majority of cases analyzed. However, it is not always the case. Typically, this approach fails when $C(s)$ and $P(s)$ are such that the usual sensitivity function, $S(s)$, does not predict the step response well. An

Table 3. Diagnosed quality of tracking in systems of Figures 2 and 3.

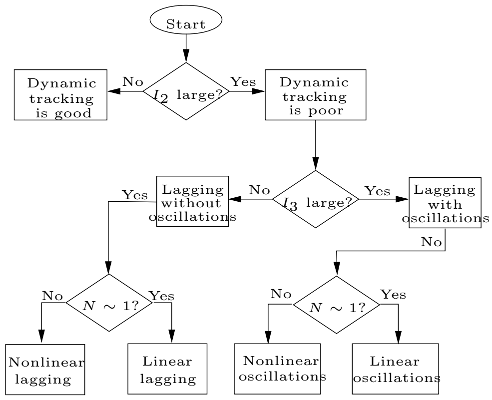
Fig.	I_0	I_1	I_2	I_3	N	$C(s)$ pole at $s = 0$?	Track qual. & track loss
2(a)	0	0	0.187	0.017	1	No	good
2(b)	0	0	0.791	0.003	0.049	No	nonlin. lag.
2(c)	0.667	0	0.089	0.018	0.718	Yes	ampl. trunc. with windup
3(a)	0.267	0.668	0.187	0.002	1	No	static unresponsiveness
3(b)	0	0	0.807	0	1	No	linear lag.
3(c)	0	0	0.977	0.756	1	Yes	linear osc.
3(d)	1.873	0.019	0.200	0.001	0.014	No	ampl. trunc. without windup
3(e)	1.25	0.0	∞	0.02	0.106	Yes	ampl. trunc. without windup and nonlin. lag.
3(f)	0	0	1.080	0.002	0.043	No	nonlin. lag.
3(g)	0	0	1.025	0.406	0.182	No	nonlin. osc.



(a) Diagnostics based on I_0



(b) Diagnostics based on I_1



(c) Diagnostics based on I_2 and I_3

Fig. 10. Diagnostic flow charts for analysis of tracking quality in systems with saturating actuators

example of this type, where neither linear nor saturating cases are well characterized by their sensitivity functions, can be found in (Eun *et al.* 2004b).

4. CONCLUSIONS

This paper provides a simple method for analysis of random reference tracking in systems with saturating actuators. The method mimics the classical frequency domain approach to step reference tracking in linear systems. Indeed, it is based on the indicators, which are similar to bandwidth and

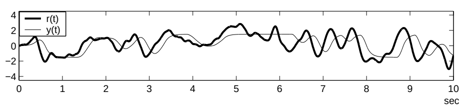


Fig. 11. Tracking $r(t) = 1.5\ r(t; 10)$ in system of Figure 4 with $C(s) = 5/s$, $P(s) = 15/(s+10)$ and $\alpha = 1$

resonance peak, used in the linear case, and which allow one to predict the quality of random reference tracking and nature of track loss in systems with saturating actuators.

The method developed in this paper offers control system designers a quick and easy way to predict system performance without resorting to lengthy and expensive numerical simulations. In addition, it illuminates reasons for track loss, which might be useful for developing improvement measures.

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