# RHYTHMIC STABILIZATION OF PERIODIC ORBITS IN A WEDGE


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Abstract: The paper addresses the rhythmic stabilization of periodic orbits in a wedge billiard with actuated edges. The output feedback strategy, based on the sole measurement of impact times, results from the combination of a stabilizing state feedback control law and a nonlinear deadbeat state estimator. It is shown that the robustness of both the control law and the observer leads to a simple rhythmic controller achieving a large basin of attraction.




Keywords: Rhythmic feedback control, juggling, billiard, discrete-time


## 1. INTRODUCTION

This paper is concerned with the stabilization of periodic orbits in a "wedge billiard" (or "planar juggler") illustrated in Figure 1. A point mass

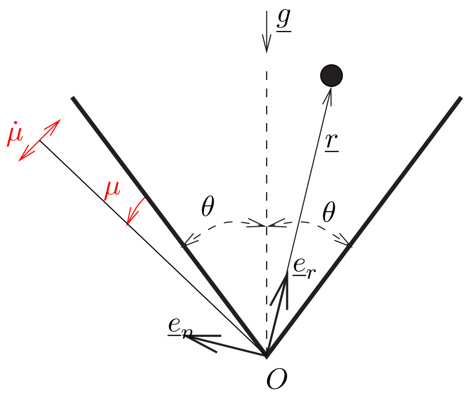


Fig. 1. The wedge billiard

(ball) moves in the plane under the action of a constant gravitational field. The ball undergoes elastic collisions with two intersecting edges, an idealization of the juggler's two arms. In the absence of control, the two edges form a fixed angle $\theta$ with the direction of gravity. Depending on the angle $\theta$, this conservative system exhibits a variety of dynamical phenomena, including an abundance of unstable periodic orbits. Rotational actuation of the edges around their fixed intersection point is used to stabilize one particular orbit of the uncontrolled system.

The wedge billiard stabilization provides a simple benchmark for investigating rhythmic tasks control such as human and animal locomotion. The study of such mechanisms is rendered difficult by the intermittent and underactuated nature of the control (see Brogliato (1999), Menini and Tornambé (2003)).

In previous papers (Sepulchre and Gerard (2003), Gerard and Sepulchre (2004)), stabilizing control strategies have been designed for configurations of the wedge billiard system differing in the value of the wedge angle $\theta$. The stabilization of periodic orbits through impact control is rephrased as the fixed point discrete-time stabilization of the corresponding Poincaré map.

Exploiting the open-loop dynamics, discrete-time feedback laws were proposed that prescribe the

[1] Research Fellow of the Belgian National Fund for Scientific Research


[2] This paper presents research partially supported by the Belgian Programme on Inter-university Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture.

values of the control variables (edge angular position and velocity) at impact times based on the state of the ball at previous impact times.

In a practical implementation, these discrete-time control laws are converted into a continuous-time actuation of the edges such as to produce the right discrete control values at impact times (see Section 6).

The objective of the present paper is to show that the discrete state of the system can be reconstructed with the sole measurement of impact times. The resulting "rhythmic" feedback control laws emphasize the timing nature of intermittent control and show that large basins of attraction can be achieved with minimal feedback information. They complement recent results (Ronsse and Sepulchre (2004)) that demonstrated that unstable periodic orbits of the elastic wedge can actually be (locally) stabilized by sensorless (open-loop) sinusoïdal actuation of the wedge.

The paper is organized as follows. In Section 2 the dynamical model of the wedge billiard is briefly reviewed. The square wedge configuration properties are summarized in Section 3 and a stabilizing state feedback control method is designed in Section 4. Stabilization using impact times as an output is studied in Section 5. The mirror law implementation of the proposed output feedback control law is briefly discussed in Section 6. Simulations results are presented in Section 7.

## 2. CONTROLLED WEDGE BILLIARD

This section summarizes the model presented in Sepulchre and Gerard (2003).

Periodic orbits of the four-dimensional wedge billiard dynamics will be studied via the three-dimensional discrete (Poincaré) map relating the state from one impact to the next one. The discrete-state vector, noted $x[k]$, will consist of continuous-time variables $x(t)$ evaluated at impact time $t[k]$. Because the continuous-time variables can be discontinuous at impact times, we use the notation $x^-(t[k])$ for pre-impact values and $x^+(t[k])$ for post-impact values. As a convention, the discrete-time state will denote post-impact values, that is $x[k] = x^+(t[k])$.
Let $(\underline{e}_r, \underline{e}_n)$ an orthonormal frame attached to the fixed point $O$ with $\underline{e}_r$ aligned with the impacted edge. Let $\underline{r}$ denote the position of the ball (unit mass point) and $\underline{v} = v_r\underline{e}_r + v_n\underline{e}_n$ its velocity. The total energy of the ball is

$$E = \frac{1}{2}\left(v_r^2 + v_n^2\right) - < \underline{r}, \underline{g} > \tag{1}$$

Following Lehtihet et al. (1986), we use the state variables $V_r = \frac{v_r}{\cos\theta}$, $V_n = \frac{v_n}{\sin\theta}$ and $E$, the discrete state vector being

$$x[k] = \left( V_r^+(t[k]) \; V_n^+(t[k]) \; E^+(t[k]) \right)^T$$

In the absence of control, each edge forms an angle $\theta$ with the vertical, i.e. the direction of the constant gravitational field $g$. The discrete control vector $u[k]$ consists of the angular deviation $\mu(t[k])$ of the impacted edge at impact time $t[k]$ and its angular velocity $\dot{\mu}(t[k])$. It is assumed that the edge is not affected by the impacts, i.e. $\dot{\mu}^-(t[k]) = \dot{\mu}^+(t[k])$. The discrete wedge-billiard map is the composition of a (parabolic) flight map and an impact rule. The flight map integrates the continuous-time equation of motion between two successive impact times while the impact map expresses post-impact variables as a (static) map of pre-impact variables and control.

We first review the derivation of the uncontrolled billiard map (Lehtihet et al. (1986)). The flight map is then entirely determined by the wedge geometry, that is by the parameter $\alpha = \tan\theta$. The flight map takes the analytical form $\mathcal{F}_1$

$$\begin{aligned} V_n^-(t[k+1]) &= -V_n[k] \\ V_r^-(t[k+1]) &= V_r[k] - 2|V_n[k]| \\ (E^-(t[k+1]) &= E[k]) \end{aligned}$$

when the impacts $k$ and $k+1$ occur on the same edge, and the analytical form $\mathcal{F}_2$

$$\begin{aligned} &V_n^-(t[k+1])^2 = 4E[k] + \frac{2(1-\alpha^2)}{(1+\alpha^2)^2}\left(|V_n[k]| - V_r[k]\right)^2 - V_n^2[k] \\ &V_r^-(t[k+1]) = |V_n[k]| - V_r[k] - |V_n^-(t[k+1])| \qquad (2) \\ &(E^-(t[k+1]) = E[k]) \end{aligned}$$

when the impacts $k$ and $k+1$ occur on two different edges. The map $\mathcal{F}_1$ is applied as long as the condition

$$2E[k] - V_n^2[k]\sin^2\theta - (V_r[k] - 2|V_n[k]|)^2\cos^2\theta \geq 0$$

is fulfilled. Otherwise, the map $\mathcal{F}_2$ is applied. This condition restricts the ball to impact above the intersection of the edges.
The impact rule $\mathcal{I}$ adopted in this paper simply assumes that the tangential velocity is conserved and that the normal velocity is reversed :

$$V_r^+(t[k]) = V_r^-(t[k]), \; V_n^+(t[k]) = -V_n^-(t[k]) \tag{3}$$

Collisions are thus perfectly elastic (leaving the energy conserved in the absence of control). The uncontrolled wedge billiard map is the composition of $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{I}$.

We now examine how angular momentum control of the edges modifies the flight map and the impact rule. The angular momentum control $\dot{\mu}[k]$ has no effect on the wedge geometry. As a consequence, it leaves the flight map unchanged and only modifies the impact rule as

$$V_n^+(t[k]) = -V_n^-(t[k]) + \frac{2}{\alpha}R(t[k])\dot{\mu}(t[k]) \tag{4}$$

with $R(t[k]) = \frac{r(t[k])}{\cos\theta}$ obtained from equation (1).

Composing the flight map $\mathcal{F}_1$ and the impact rule (4), one obtains the discrete controlled billiard map $\mathcal{A}$

$$\begin{pmatrix} V_r[k+1] \\ V_n[k+1] \end{pmatrix} = \begin{pmatrix} V_r[k] - 2|V_n[k]| \\ V_n[k] \end{pmatrix} + \frac{2}{\alpha}\begin{pmatrix} 0 \\ R[k+1] \end{pmatrix} \dot{\mu}[k+1] \qquad (5)$$

for impacts $k$ and $k+1$ on the same edge.

The composition of the flight map $\mathcal{F}_2$ and the impact rule $\mathcal{I}$ gives rise to the discrete controlled billiard map $\mathcal{B}$

$$\begin{pmatrix} V_r[k+1] \\ V_n[k+1] \end{pmatrix} = \begin{pmatrix} |V_n[k]| - V_r[k] - f_1[k] \\ -f_1[k] sign(V_n[k]) \end{pmatrix} + \frac{2}{\alpha}\begin{pmatrix} 0 \\ R[k+1] \end{pmatrix} \dot{\mu}[k+1] \qquad (6)$$

for impacts $k$ and $k+1$ on different edges, with

$$f_1[k] = \sqrt{4E[k] + 2\frac{1-\alpha^2}{(1+\alpha^2)^2}\left(|V_n[k]| - V_r[k]\right)^2 - V_n^2[k]}$$

The energy update is

$$E[k+1] = E[k] + \frac{1}{2}\frac{\alpha^2}{1+\alpha^2}\left(V_n[k+1]^2 - V_n^-[k+1]^2\right) + \frac{1}{2}\frac{1}{1+\alpha^2}\left(V_r[k+1]^2 - V_r^-[k+1]^2\right)$$

The model $\mathcal{A} - \mathcal{B}$ is suitable for the analysis and design of stabilizing control laws of various periodic orbits of the uncontrolled billiard.

## 3. THE SQUARE WEDGE BILLIARD

The *square wedge billiard* ($\theta = 45°$) configuration is special in that, in the absence of control, the 2 DOF dynamics decouples into two 1 DOF dynamics : in the (fixed) frame $(\underline{e}_1, \underline{e}_2)$ attached to the wedge (as depicted in Figure 2), the dynamics of the mass point $\underline{x} = x_1\underline{e}_1 + x_2\underline{e}_2$ satisfy

$$\begin{cases} \ddot{x}_i = -\frac{\sqrt{2}}{2} g \\ x_i(t) = 0 \Rightarrow \dot{x}_i^+(t) = -\dot{x}_i^-(t),\ i = 1, 2 \end{cases}$$

which directly yields the discrete map

$$V_i[k_i + 1] = V_i[k_i] \qquad (7)$$

$$t[k_i + 1] = t[k_i] + \frac{2}{g} V_i[k_i] \qquad (8)$$

of an elastic bouncing ball or impact oscillator. Each solution of the impact oscillator is periodic of period $T_i = \frac{2\sqrt{2}}{g}\ v_n^i$ where $v_n^i$ denotes the (constant) impact velocity $|\dot{x}_i(t[k_i])|$ ($V_i[.] = \sqrt{2} v_n^i$).

The periodic orbits of the square wedge billiard satisfy $T_1 = qT_2,\ \ q \in \mathbb{N}$. In the rest of the paper, we only consider the case $q = 1$. Such periodic orbits correspond to alternating impacts on the two billiard edges. They are therefore fixed points of the map $\mathcal{B}^l,\ \ l \geq 1$, where $l$ is the total number of impacts during one period.

The simplification of the map $\mathcal{B}$ when $\alpha = 1$ comes from the property

$$|V_n[k+2]| = f_1[k+1] = \sqrt{4E[k+1] - V_n[k+1]^2} = |V_n[k]|$$

which renders the map $\mathcal{B}$ linear in the coordinates

$$Z[k] = \begin{pmatrix} V_r[k] & |V_n[k]| & |V_n[k-1]| \end{pmatrix}^T \qquad (9)$$

We have :

$$Z[k+1] = \tilde{B} Z[k]$$

with

$$\tilde{B} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The factor $\frac{|V_n| - V_r}{g}$ has the convenient interpretation of a phase shift $\phi[k] = t[k] - t[k-1]$ between the two impact oscillators defining the billiard motion. This is a consequence of the formula

$$V_r[k] - |V_n[k-1]| = -g(t[k] - t[k-1]) = -g\phi[k] \qquad (10)$$

In the sequel, we focus on the stabilization of period-one and period-two orbits.

Period-one orbits and period-two orbits of the uncontrolled square wedge billiard are respectively fixed points of the maps $\tilde{B}$ and $\tilde{B}^2$. Fixed points of $\tilde{B}$ are of the form

$$\bar{Z} = \left(\bar{V}_r, |\bar{V}_n|, |\bar{V}_n|\right) = \left(0, \sqrt{2\bar{E}}, \sqrt{2\bar{E}}\right)$$

They characterize a one-parameter family of periodic orbits, parametrized by their total energy $\bar{E}$. Fixed points $\tilde{B}^2$ are of the form

$$\bar{Z} = (\bar{V}_r, |\bar{V}_n|, |\bar{V}_n|) = (\bar{V}_r, \sqrt{2\bar{E}}, \sqrt{2\bar{E}})$$

with $-\sqrt{2E} \leq \bar{V}_r \leq \sqrt{2E}$

They characterize a two-parameter family of periodic orbits parametrized by their total energy $\bar{E}$ and the difference $|\bar{V}_n| - \bar{V}_r$.

Period-one and period-two orbits are unstable because $\tilde{B}$ and $\tilde{B}^2$ have an eigenvalue of (algebraic) multiplicity greater than 1 on the unit circle. An illustration of the traces of period-one and period-two orbits is given in Figure 2.

Substituting the variable $\phi$ to the variable $V_r$ in the map $\tilde{B}^2$ and using the coordinates

$$\mathcal{Z}[k] = (\phi[k_r], V_n[k_r], V_n[k_l])^T$$

where $k_l$ and $k_r$ denote respectively the $k^{th}$ impact on the left edge and the right edge (impacts are

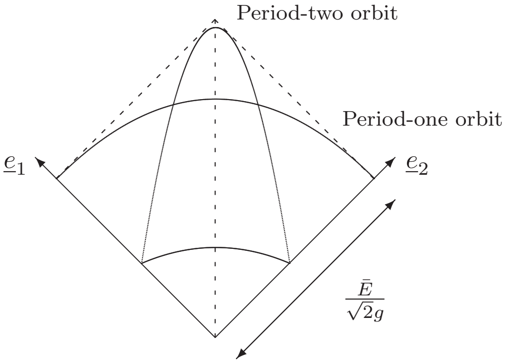


Fig. 2. Period-one and period-two orbits of the square wedge billiard.

numbered on each edge independently), the map $\tilde{B}^2$ expresses as :

$$\begin{aligned} \phi[k_r+1] &= \phi[k_r] + \tfrac{2}{g}\left(V_n[k_r] + V_n[k_l]\right) \\ V_n[k_r+1] &= V_n[k_r] \\ V_n[k_l+1] &= V_n[k_l] \end{aligned} \qquad (11)$$

where

$$\phi[k_r] = t[k_r] - t[k_l] > 0 \,^3 \qquad (12)$$

is the phase shift between both oscillators, which is computed with respect to the right oscillator.

A fixed point $\bar{\mathcal{Z}} = (\bar{\phi}, \bar{V}_n, -\bar{V}_n)$ of the system (11) is characterized by $\bar{V}_n = \sqrt{2\bar{E}}$, $0 < \bar{\phi} < \frac{2\bar{V}_n}{g}$. This fixed point corresponds to a period-one orbit of energy $\bar{E}$ when $\bar{\phi} = \frac{\bar{V}_n}{g}$ (the oscillators are in opposite phase configuration) and to a period-two orbit of energy $\bar{E}$ elsewhere.

## 4. STATE FEEDBACK STABILIZATION

In the rest of the paper, we only consider the square wedge configuration ($\theta = 45°$) and alternating impacts on the left and right edges. This situation leads to simple analytic derivations nevertheless illustrative of the general case, which will be analyzed in a forthcoming publication.

The simplest control strategy proposed in Sepulchre and Gerard (2003) for the square billiard is to maintain the edges at their uncontrolled angular position and to use angular momentum feedback control of each edge. Adding $\dot{\mu}$-control to the map $\tilde{B}^2$ (equations (11)) yields the model

$$\begin{aligned} \phi[k_r+1] &= \phi[k_r] + \tfrac{2}{g}\left(V_n[k_r] + V_n[k_l]\right) \\ V_n[k_r+1] &= V_n[k_r] + 2R[k_r+1]\dot{\mu}[k_r+1] \\ V_n[k_l+1] &= V_n[k_l] + 2R[k_l+1]\dot{\mu}[k_l+1] \end{aligned} \qquad (13)$$

with

$$R[k_r+1] = \phi[k_r+1](-V_n[k_l+1] - \frac{g}{2}\phi[k_r+1])$$

$$R[k_l+1] = -(\phi[k_r] + \frac{2}{g}V_n[k_l]) \times (V_n[k_r] + \frac{g}{2}(\phi[k_r] + \frac{2}{g}V_n[k_l]))$$

[3] The first impact is supposed to occur on the left edge, hence $t[k_l] < t[k_r]$.

The equilibrium point $\bar{\mathcal{Z}} = (\bar{\phi}, \bar{V}_n, -\bar{V}_n)$ is made asymptotically stable with the state feedback control law

$$\begin{aligned} \dot{\mu}[k_r+1] &= -\tfrac{k_P}{R}(V_n[k_r] - \bar{V}_n) - \tfrac{g k_I}{R}(\phi[k_r] - \bar{\phi}) \\ \dot{\mu}[k_l+1] &= -\tfrac{k_P}{R}(V_n[k_l] + \bar{V}_n) \end{aligned} \qquad (14)$$

Exponential stability of the Jacobian linearization is ensured with mild conditions $0 < k_P < 1$ and $0 < k_I < \frac{k_p}{2}$ on the (adimensional) design parameters $k_P$ and $k_I$.

The state feedback control (14) has the standard structure of a proportional-integral control. With the interpretation of the wedge billiard as two coupled impact oscillators, the proportional feedback assigns the energy of each oscillator to a common energy level $\bar{V}_n^2$ whereas the integral term regulates the phase difference between the two oscillators. Of particular interest are the rhythmic nature and the low-gain property of the control law (14).

The size of the basin of attraction and the gain margin of the controller are increased as the control parameters $k_P$ and $k_I$ are lowered. As a consequence, the basin of attraction of the desired equilibrium can be made large and an arbitrarily low bound can be imposed on the magnitude of the control $|\dot{\mu}|$.

## 5. RHYTHMIC FEEDBACK STABILIZATION

The rhythmic nature of the control law (14) is suggested by the time equation (8) showing that the control law (14) can be rewritten as a function that uses the sequence of impact times as sole feedback information.

More precisely, the controlled square wedge model (13) is uniformly completely observable (UCO, see Messina et al. (2003)) for the rhythmic output

$$y[k] = \left(\, t[k_l] \;\; t[k_r] \,\right)^T \qquad (15)$$

This UCO property is established as follows.

From the decoupling property of the square wedge configuration, we obtain

$$V_n[k_r] = \frac{g}{2}\left(t[k_r+1] - t[k_r]\right) \qquad (16)$$

$$V_n[k_l] = -\frac{g}{2}\left(t[k_l+1] - t[k_l]\right) \qquad (17)$$

From equations (16)-(17) and the definition of the phase shift (3), the state $\mathcal{Z}[k]$ of the system expresses as

$$\begin{aligned} \mathcal{Z}[k] &= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -\frac{g}{2} & 0 & \frac{g}{2} \\ \frac{g}{2} & 0 & -\frac{g}{2} & 0 \end{pmatrix} \begin{pmatrix} y[k] \\ y[k+1] \end{pmatrix} \\ &= \psi(y[k], y[k+1]), \end{aligned} \qquad (18)$$

Equation (18) proofs the uniform complete observability of the controlled square wedge model (13) with respect to the rhythmic output (15).

*Nonlinear deadbeat observer design*

Delaying equation (18) from one discrete instant of time and introducing it in the state equation (13) provides the expression of a nonlinear deadbeat observer

$$\begin{aligned}\xi[k+1] &= y[k] \\ \hat{\mathcal{Z}}[k] &= \underbrace{\begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & -\frac{g}{2} & 0 & \frac{g}{2} \\ \frac{g}{2} & 0 & -\frac{g}{2} & 0 \end{pmatrix}}_{M} \begin{pmatrix} \xi[k] \\ y[k] \end{pmatrix} \\ &\quad +2 \underbrace{\begin{pmatrix} 0 & 0 \\ \hat{R}[k_r] & 0 \\ 0 & \hat{R}[k_l] \end{pmatrix}}_{N[k]} \underbrace{\begin{pmatrix} \dot{\mu}[k_r] \\ \dot{\mu}[k_l] \end{pmatrix}}_{\dot{\mu}[k]} \end{aligned} \quad (19)$$

where

$$\hat{\mathcal{Z}}[k] = \left( \hat{\phi}[k_r] \ \hat{V}_n[k_r] \ \hat{V}_n[k_l] \right)^T \quad (20)$$

is the estimated state of the system at discrete time $k$ and

$$\begin{aligned}\hat{R}[k_r] &= \hat{\phi}[k_r](-\hat{V}_n[k_l] - \frac{g}{2}\hat{\phi}[k_r]) \\ \hat{R}[k_l] &= -(\hat{\phi}[k_r-1] + \frac{2}{g}\hat{V}_n[k_l-1]) \times \\ &\qquad (\hat{V}_n[k_r-1] + \frac{g}{2}(\hat{\phi}[k_r-1] + \frac{2}{g}\hat{V}_n[k_l-1]))\end{aligned}$$

A system state estimation hence requires the measurement of four successive impact times.

*Output feedback controller*

Combining the state feedback control law (14) with the observer dynamics (19) we derive the following output feedback controller :

$$\begin{aligned}\hat{\dot{\mu}}[k_i+1] &= -K_i \left( M \begin{pmatrix} \xi[k] \\ y[k] \end{pmatrix} + 2N[k] \begin{pmatrix} \hat{\dot{\mu}}[k_r] \\ \hat{\dot{\mu}}[k_l] \end{pmatrix} - \bar{\mathcal{Z}} \right) \\ &= F_{\hat{\dot{\mu}}}(y[k], y[k-1], \hat{\dot{\mu}}[k_l], \hat{\dot{\mu}}[k_r]) \qquad (21) \\ &\quad i \in \{l, r\}\end{aligned}$$

where $K_l = \left( 0 \ 0 \ \frac{k_P}{\bar{R}} \right)$, $K_r = \left( \frac{g k_I}{\bar{R}} \ \frac{k_P}{\bar{R}} \ 0 \right)$ The discrete-time control law stabilizes the desired period-two orbit using the sole measurement of impact times. In a robotic setup, these measurements can be obtained for instance from accelerometers mounted on each edge.

## 6. MIRROR LAW IMPLEMENTATION

To be implemented in a mechanical setup, the discrete-time control laws designed for the discrete-time impact model must be converted into continuous-time reference trajectories for the actuated edges. To this end, we employ the mirror-law strategy proposed by Buehler and Koditschek (Buehler et al. (1994)) : after impact $k_i$ has occurred at time $t[k_i]$, the impacted edge is given the reference trajectory

$$\mu_i(t) = F_{\dot{\mu}}[k_i]\beta_i(t), \ t[k_i] < t \le t[k_i+1], \ i \in \{l, r\} \quad (22)$$

where $\beta_i(t)$ is the angular deviation of the ball at time $t$ with respect to the equilibrium edge angle. By definition, the impact will occur when $\mu_i(t) = \beta_i(t)$, producing the discrete-time control law

$$\dot{\mu}[k_i+1] = F_{\dot{\mu}}[k_i]\dot{\beta}_i^-(t[k_i+1])$$

The mirror-law continuous-time implementation thus approximates the discrete-time nominal design $\dot{\mu}[k_i+1] = F_{\dot{\mu}}[k_i]$. The limited amplitude of $F_{\dot{\mu}}$ and the gain margin of the (low-gain) nominal design make it robust to the proposed approximate implementation.

The mirror law (22) is not purely rhythmic in that it requires the continuous-time measurement of the angular position $\beta(t)$. This additional feedback information is nevertheless necessary only in the vicinity of the impact position ($\theta = 45°$) and can be obtained for instance from a proximity sensor mounted on the edge. The pure rhythmic nature of the control is recovered in the limit of a fixed wedge with impulsive control.

Using an anthropomorphic analogy, the mirror law control (22) uses aural information (rhythmic sensor) and tactile information (proximity sensor) but no visual information (continuous-time monitoring of the ball position).

## 7. SIMULATION RESULTS

The mirror law implementation of the output feedback control (21) is illustrated by a simulation result. We choose to stabilize the periodic orbit characterized by $\bar{V}_n = \sqrt{11} m/s$, $\bar{\phi} = 1.42\frac{\bar{V}_n}{g}$. In the coordinates $(V_r, V_n, E)$, the initial condition is chosen as $V_r[0] = V_n[0] = -3.25 m/s$, $E[0] = \bar{E}$, which roughly corresponds to an initial drop of the ball above the left edge with the correct energy level. The system is left uncontrolled during the first four impacts in order to initialize the state estimator. Next the output feedback controller is switched on. Figure 3 illustrates the trace of the trajectories.

## 8. CONCLUSION

This paper has presented output feedback stabilization results for periodic orbits of the controlled square wedge billiard, a model we view as an interesting benchmark for impact control stabilization problems. The model of the square wedge configuration is simpler than the general wedge

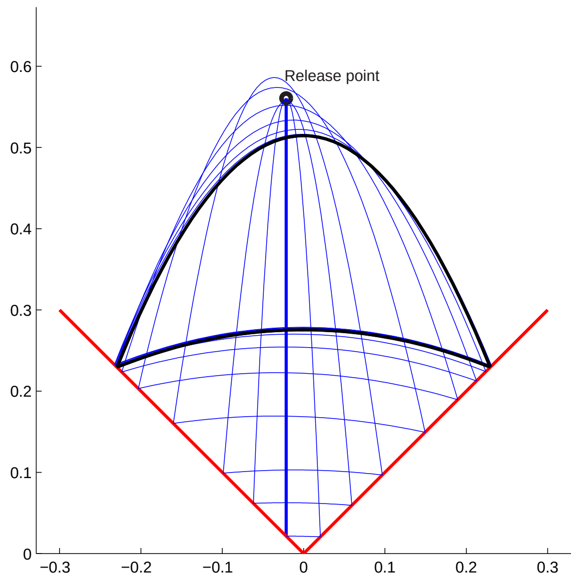


Fig. 3. Trace of the trajectories

model while displaying most of the relevant issues of the problem. Exploiting the uniform complete observability property of the controlled system w.r.t. a rhythmic output, we derived a nonlinear deadbeat state estimator. Combining the stabilizing state feedback control law with the state estimator dynamics gives rise to an output feedback strategy. The robustness of both the control law and the observer leads to a simple rhythmic controller achieving a large basin of attraction.

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