GROUP DIFFERENTIAL GAMES FOR MULTIPARAMETER SINGULARLY PERTURBED SYSTEMS

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Abstract: In this paper, a group differential game problem is formulated using the system model of multiparameter singularly perturbed systems (MSPS). The case that there exist two groups of players with a conflict interest in the game is considered, and the players in each group must make their own decisions by taking into account the group interest. A method is proposed to find the approximate strategy for every player which will lead to an $O(\|\mu\|)$ near saddle–point equilibrium.

Keywords: Singular perturbations, Differential games, Equilibrium, Systems concepts, Approximate analysis, Matrix equations.

1. INTRODUCTION

In the real world, it is common that business competition takes place among a few major corporate groups. A corporate group is usually composed of a parent company and a number of subsidiary companies. For example, the headquarter of a financing corporation is located usually in a central city of business, but its branches are distributed everywhere in the country or in the world, an auto–manufacturer usually establish its factories all over the world, and a general electrical maker develops its business through numerous affiliated companies. All these groups have their own group interests. A decision maker in a parent company or a subsidiary company is of responsibility to make relatively independent decisions by taking into account the group interest. Moreover, comparing to the long–term decision–making in the parent company, the short–term decision making is often found in the subsidiary company. In other words, there exist the time–scale difference in decision–making between the parent company and the subsidiary company.

Taking these characteristics into account, we try to formulate a group differential game problem in this paper by using the system model of multiparameter singularly perturbed systems. We consider the case that there exist two groups of players with a different (conflict) group interest in competition. The multiparameter singularly perturbed system is composed of N lower level fast–subsystems (branches) and a higher level slow–subsystem (headquarter) (see e.g., Khalil and Kokotović, 1978; Coumarbatch and Gajić, 2000; Gajić, 1988; Mukaidani et al., 2003). The fact that fast–subsystems are interconnected through the slow–subsystem is used to describes the time–scale differences between the parent company and the subsidiary company.

We are now interested in the saddle–point equilibrium to the game. In order to solve the problem, the multiparameter algebraic Riccati equation (MARE), which is parameterized by the small positive same order parameters $\varepsilon_j,\ j=1,\ \ldots,N$ is studied. Various reliable approaches to the theory of the algebraic Riccati equation (ARE) have been reported in many literatures (see e.g., Laub, 1979). One of the approaches is the invariant subspace approach which is based on the Hamiltonian matrix (Laub, 1979). However, when the ARE is ill–conditioned, such an approach is not adequate to the MSPS due to the high dimension and the numerical stiffness (Coumarbatch and Gajić, 2000). Moreover, the exact slow–fast decomposition method for solving the MARE has been proposed recently in Coumarbatch and Gajić (2000). However, the limitation of this approach lies in that the small parameters are assumed to be known. In the real world, the small perturbation parameters ε_j are often not known. Thus, it is not applicable to a large class of problems where the parameters represent small perturbations whose values are not known exactly.

In this paper, the unique and bounded solution of the MARE is derived, and its asymptotic structure is also established. As the result, a method to find the perturbation–independent approximate strategies to the players is obtained in which the zero–order solution of the MARE is used. Since the approximate strategies of the player do not depend on the small parameters (perturbations), the decision making can be carried out even if the parameters are not known. Furthermore, it is proved that the perturbation–independent approximate strategies constitute an $O(\|\mu\|)$ near saddle–point equilibrium.

2. PROBLEM FORMULATION

It is assumed that the considered system is an actual business model with time–scale differences between the parent company and the subsidiary company. Therefore, we investigate the following MSPS which is composed of N lower level fast subsystems and a higher level slow subsystem as the ideal MSPS.

$$\dot{x}_0 = \sum_{j=0}^{N} A_{0j}x_j + \sum_{j=0}^{N} B_{0j}u_j + \sum_{j=0}^{N} D_{0j}v_j,\quad x_j(0)=x_j^0, \tag{1a}$$

$$\varepsilon_j \dot{x}_j = A_{j0}x_0 + A_{jj}x_j + B_{jj}u_j + D_{jj}v_j, \tag{1b}$$

$$y_0 = C_{00}x_0,\ y_j = C_{j0}x_0 + C_{jj}x_j, \tag{1c}$$

where $x_j \in \mathbf{R}^{n_j},\ j=0,\ 1,\ \ldots,N$ are the state vectors, $u_j \in \mathbf{R}^{m_{uj}},\ j=0,\ 1,\ \ldots,N$ are the controls of players in Group 1, and $v_j \in \mathbf{R}^{m_{vj}},\ j=0,\ 1,\ \ldots,N$ are the controls of players in Group 2, respectively. $y_j \in \mathbf{R}^{l_j},\ j=0,\ 1,\ \ldots,N$ are the outputs. It is assumed that the ratios of the small positive parameter $\varepsilon_j > 0, j=1,\ \ldots,N$ are bounded by some positive constants $\underline{k}_{ij},\ \bar{k}_{ij}$ (see e.g., Khalil and Kokotović, 1978, 1979; Mukaidani et al., 2003),

$$0 < \underline{k}_{ij} \le \alpha_{ij} \equiv \frac{\varepsilon_j}{\varepsilon_i} \le \bar{k}_{ij} < \infty. \tag{2}$$

Note that at least one of the fast state matrices $A_{jj},\ j=1,\ \ldots,N$, is singular. The performance criterion is given by

$$J(u,\ v) = \frac{1}{2}\int_0^\infty \left(y^T y + u^T R_u u - v^T R_v v\right) dt$$

$$= \frac{1}{2}\int_0^\infty \left(y^T y + \sum_{j=0}^{N} u_j^T R_{uj} u_j - \sum_{j=0}^{N} v_j^T R_{vj} v_j\right) dt, \tag{3}$$

where $y = \begin{bmatrix} y_0^T & \cdots & y_N^T \end{bmatrix}^T \in \mathbf{R}^{\bar{l}},\ \bar{l} = \sum_{j=0}^{N} l_j,\ R_{uj} > 0,\ R_{vj} > 0$.

The goal of players in Group 1 is to minimize the cost function J, while players in Group 2 would like to maximize it.

The decision makers in two groups are required to select the closed loop control laws u_j^* and $v_j^*,\ j=1,\ldots,N$, respectively, if they exist, such that

$$J(u^*,\ v) \le J(u^*,\ v^*) \le J(u,\ v^*), \tag{4}$$

where

$$u = \begin{bmatrix} u_0^T & u_1^T & \cdots & u_N^T \end{bmatrix}^T \in \mathbf{R}^{\bar{m}_u},\ \bar{m}_u = \sum_{j=0}^{N} m_{uj},$$

$$v = \begin{bmatrix} v_0^T & v_1^T & \cdots & v_N^T \end{bmatrix}^T \in \mathbf{R}^{\bar{m}_v},\ \bar{m}_v = \sum_{j=0}^{N} m_{vj}.$$

The strategy pair $(u^*,\ v^*)$ is called the saddle–point equilibrium.

The full-order strategy pair $(u^*,\ v^*)$ with the knowledge of the small perturbation parameters ε_j can be obtained as follow.

$$u^* = -R_u^{-1} B_e^T P_e x, \tag{5a}$$

$$v^* = R_v^{-1} D_e^T P_e x, \tag{5b}$$

where P_e satisfies the MARE

$$A_e^T P_e + P_e A_e - P_e S_e P_e + Q = 0, \tag{6}$$

with

$$A_e := \begin{bmatrix} A_{00} & A_{0f} \\ \Pi_e^{-1} A_{f0} & \Pi_e^{-1} A_f \end{bmatrix},$$

$$\Pi_e := \mathbf{block\ diag}\left(\varepsilon_1 I_{n_1}\ \cdots\ \varepsilon_N I_{n_N}\right),$$

$$A_{0f} := \left[\, A_{01} \ \cdots \ A_{0N} \,\right],$$
$$A_{f0} := \left[\, A_{10}^T \ \cdots \ A_{N0}^T \,\right]^T,$$
$$A_f := \textbf{block diag}\left(A_{11} \ \cdots \ A_{NN} \right),$$
$$S_e := B_e R_u^{-1} B_e^T - D_e R_v^{-1} D_e^T = \begin{bmatrix} S_{00} & S_{0f}\Pi_e^{-1} \\ \Pi_e^{-1} S_{0f}^T & \Pi_e^{-1} S_f \Pi_e^{-1} \end{bmatrix},$$
$$S_{00} := \sum_{j=0}^{N} \left[B_{0j} R_{uj}^{-1} B_{0j}^T - D_{0j} R_{vj}^{-1} D_{0j}^T \right],$$
$$S_{0f} := \left[\, S_{01} \ \cdots \ S_{0N} \,\right] = \left[\, B_{01} R_{u1}^{-1} B_{11}^T - D_{01} R_{v1}^{-1} D_{11}^T \ \cdots \ B_{0N} R_{uN}^{-1} B_{NN}^T - D_{0N} R_{vN}^{-1} D_{NN}^T \,\right],$$
$$S_f := \textbf{block diag}\left(S_{11} \ \cdots \ S_{NN} \right) = \textbf{block diag}\left(B_{11} R_{u1}^{-1} B_{11}^T - D_{11} R_{v1}^{-1} D_{11}^T \ \cdots \ B_{NN} R_{uN}^{-1} B_{NN}^T - D_{NN} R_{vN}^{-1} D_{NN}^T \right),$$
$$B_e := \begin{bmatrix} B_{00} & B_{0f} \\ 0 & \Pi_e^{-1} B_f \end{bmatrix},\ B_{0f} := \left[\, B_{01} \ \cdots \ B_{0N} \,\right],$$
$$B_f := \textbf{block diag}\left(B_{11} \ \cdots \ B_{NN} \right),$$
$$D_e := \begin{bmatrix} D_{00} & D_{0f} \\ 0 & \Pi_e^{-1} D_f \end{bmatrix},\ D_{0f} := \left[\, D_{01} \ \cdots \ D_{0N} \,\right],$$
$$D_f := \textbf{block diag}\left(D_{11} \ \cdots \ D_{NN} \right),$$
$$R_u := \textbf{block diag}\left(R_{u0} \ R_{u1} \ \cdots \ R_{uN} \right),$$
$$R_v := \textbf{block diag}\left(R_{v1} \ R_{v1} \ \cdots \ R_{vN} \right),$$
$$Q := \begin{bmatrix} Q_{00} & Q_{0f} \\ Q_{0f}^T & Q_f \end{bmatrix},\ Q_{00} := \sum_{j=0}^{N} C_{j0}^T C_{j0},$$
$$Q_{0f} := \left[\, Q_{01} \ \cdots \ Q_{0N} \,\right] = \left[\, C_{10}^T C_{11} \ \cdots \ C_{N0}^T C_{NN} \,\right],$$
$$Q_f := \textbf{block diag}\left(Q_{11} \ \cdots \ Q_{NN} \right) = \textbf{block diag}\left(C_{11}^T C_{11} \ \cdots \ C_{NN}^T C_{NN} \right).$$

However, the purpose of this paper is to find the approximate saddle–point equilibrium without the knowledge of the small perturbation parameters. In the later analysis, the following assumptions are made without loss of generality.

Assumption 1: The Hamiltonian matrices T_{jj} are nonsingular, where

$$T_{jj} := \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^T \end{bmatrix},\ j = 1,\ \dots,N.$$

Assumption 2:

$$\text{rank} \begin{bmatrix} sI_{n_0} - A_{00} & -A_{0f} & B_{00} & B_{0f} \\ -A_{f0} & -A_f & 0 & B_f \end{bmatrix} = \bar{n}, \quad (7a)$$
$$\text{rank} \begin{bmatrix} sI_{n_0} - A_{00} & -A_{0f} & D_{00} & D_{0f} \\ -A_{f0} & -A_f & 0 & D_f \end{bmatrix} = \bar{n}, \quad (7b)$$
$$\text{rank} \begin{bmatrix} sI_{n_0} - A_{00}^T & -A_{f0}^T & C_0^T \\ -A_{0f}^T & -A_f^T & C_f^T \end{bmatrix} = \bar{n}, \quad (7c)$$

where $\bar{n} := \sum_{j=0}^{N} n_j$,

$$C_0 := \begin{bmatrix} C_{00} \\ C_{10} \\ \vdots \\ C_{N0} \end{bmatrix},\ C_f := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ C_{11} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_{NN} \end{bmatrix},$$

with $\text{Re}[s] \geq 0$, $s \in \text{C}$.

In order to find the approximate saddle–point equilibrium, the asymptotic structure of the MARE (6) will be derived. Let us introduce a scaling matrix

$$\Phi_e := \begin{bmatrix} I_{n_0} & 0 \\ 0 & \Pi_e \end{bmatrix}.$$

In order to avoid the ill–conditionedness caused by a large value ε_j^{-1} which is contained in the MARE (6), the following useful lemma is introduced (Mukaidani et al., 2003).

Lemma 1: The MARE (6) is equivalent to the following generalized multiparameter algebraic Riccati equation (GMARE) (8)

$$\mathcal{G}(P) = A^T P + P^T A - P^T S P + Q = 0, \quad (8)$$

where $A := \Phi_e A_e$, $S := \Phi_e S_e \Phi_e$ and

$$P := \begin{bmatrix} P_{00} & P_{0f} \\ P_{f0} & P_f \end{bmatrix},\ P_{f0} := \begin{bmatrix} P_{10} \\ \vdots \\ P_{N0} \end{bmatrix},$$
$$P_{0f} = P_{f0}^T \Pi_e := \left[\, \varepsilon_1 P_{10}^T \ \cdots \ \varepsilon_N P_{N0}^T \,\right],$$
$$P_f := \begin{bmatrix} P_{11} & \alpha_{12} P_{21}^T & \cdots & \alpha_{1N} P_{N1}^T \\ P_{21} & P_{22}^T & \cdots & \alpha_{2N} P_{N2}^T \\ \vdots & \vdots & \ddots & \vdots \\ P_{N-11} & P_{N-12} & \cdots & \alpha_{N-1N} P_{NN-1}^T \\ P_{N1} & P_{N2} & \cdots & P_{NN} \end{bmatrix}.$$

The GMARE (8) can be partitioned into

$$f_1 = P_{00}^T A_{00} + A_{00}^T P_{00} + P_{f0}^T A_{f0} + A_{f0}^T P_{f0} - P_{00}^T S_{00} P_{00} - P_{f0}^T S_f P_{f0} - P_{00}^T S_{0f} P_{f0} - P_{f0}^T S_{0f}^T P_{00} + Q_{00} = 0, \quad (9a)$$
$$f_2 = A_{00}^T P_{f0}^T \Pi_e + A_{f0}^T P_f + P_{00}^T A_{0f} + P_{f0}^T A_f - P_{00}^T S_{00} P_{f0}^T \Pi_e - P_{f0}^T S_{0f}^T P_{f0}^T \Pi_e - P_{00}^T S_{0f} P_f - P_{f0}^T S_f P_f + Q_{0f} = 0, \quad (9b)$$
$$f_3 = P_f^T A_f + A_f^T P_f + \Pi_e P_{f0} A_{0f} + A_{0f}^T P_{f0}^T \Pi_e - P_f^T S_f P_f - P_f^T S_{0f}^T P_{f0}^T \Pi_e - \Pi_e P_{f0} S_{0f} P_f - \Pi_e P_{f0} S_{00} P_{f0}^T \Pi_e + Q_f = 0. \quad (9c)$$

It is assumed that the limitation of α_{ij} exists as ε_i and ε_j tend to zero (see e.g., Khalil and Kokotović, 1978, 1979), that is

$$\bar{\alpha}_{ij} = \lim_{\substack{\varepsilon_j \to +0 \\ \varepsilon_i \to +0}} \alpha_{ij}. \quad (10)$$

Let $\bar{P}_{00}$, $\bar{P}_{f0}$ and $\bar{P}_f$ be the limiting solutions of the above equation (8) as $\varepsilon_j \to +0$, $j = 1, \ldots, N$. In this case, the following equations are obtained.

$$\begin{aligned}
&\bar{P}_{00}^T A_{00} + A_{00}^T \bar{P}_{00} + \bar{P}_{f0}^T A_{f0} + A_{f0}^T \bar{P}_{f0} \\
&\quad - \bar{P}_{00}^T S_{00} \bar{P}_{00} - \bar{P}_{f0}^T S_f \bar{P}_{f0} - \bar{P}_{00}^T S_{0f} \bar{P}_{f0} \\
&\quad - \bar{P}_{f0}^T S_{0f}^T \bar{P}_{00} + Q_{00} = 0, \quad (11a) \\
&A_{f0}^T \bar{P}_f + \bar{P}_{00}^T A_{0f} + \bar{P}_{f0}^T A_f - \bar{P}_{00}^T S_{0f} \bar{P}_f \\
&\quad - \bar{P}_{f0}^T S_f \bar{P}_f + Q_{0f} = 0, \quad (11b) \\
&\bar{P}_f^T A_f + A_f^T \bar{P}_f - \bar{P}_f^T S_f \bar{P}_f + Q_f = 0, \quad (11c)
\end{aligned}$$

where

$$\bar{P}_f := \begin{bmatrix} \bar{P}_{11} & \bar{\alpha}_{12} \bar{P}_{21}^T & \cdots & \bar{\alpha}_{1N} \bar{P}_{N1}^T \\ \bar{P}_{21} & \bar{P}_{22} & \cdots & \bar{\alpha}_{2N} \bar{P}_{N2}^T \\ \vdots & \vdots & \vdots & \vdots \\ \bar{P}_{N-11} & \bar{P}_{N-12} & \cdots & \bar{\alpha}_{N-1N} \bar{P}_{NN-1}^T \\ \bar{P}_{N1} & \bar{P}_{N2} & \cdots & \bar{P}_{NN} \end{bmatrix},$$

$$\bar{P}_{jj} := \bar{P}_{jj}^T, \ j = 0, 1, \ldots, N.$$

The ARE (11c) appears nonsymmetric in the form. However, we will show that the ARE (11c) admits at least a symmetric positive semidefinite stabilizing solution.

Lemma 2: Assume that the following AREs admit a unique symmetric positive semidefinite stabilizing solution $\bar{P}_{jj}^*$, $j = 1, \ldots, N$, respectively

$$A_{jj}^T \bar{P}_{jj}^* + \bar{P}_{jj}^* A_{jj} - \bar{P}_{jj}^* S_{jj} \bar{P}_{jj}^* + Q_{jj} = 0.$$

Then there exists a symmetric positive semidefinite stabilizing solution $\bar{P}_f$ to the ARE (11c). It can be written as

$$\bar{P}_f^* := \textbf{block diag} \left(\bar{P}_{11}^* \ \cdots \ \bar{P}_{NN}^* \right). \quad (12)$$

Finally, 0–order equations (13) are given.

$$\begin{aligned}
&\bar{P}_{00}^* \mathcal{A} + \mathcal{A}^T \bar{P}_{00}^* + \bar{P}_{00}^* \mathcal{S} \bar{P}_{00}^* + \mathcal{Q} = 0, \quad (13a) \\
&\bar{P}_{j0}^* = \left[\ \bar{P}_{jj}^* \ -I_{n_j} \ \right] T_{jj}^{-1} T_{j0} \begin{bmatrix} I_{n_0} \\ \bar{P}_{00}^* \end{bmatrix}, \quad (13b) \\
&\bar{P}_{jj}^* \Xi_{jj} + \Xi_{jj}^T \bar{P}_{jj}^* + \bar{P}_{jj}^* S_{jj} \bar{P}_{jj}^* + W_{jj} = 0, \quad (13c)
\end{aligned}$$

where

$$\begin{bmatrix} \mathcal{A} & -\mathcal{S} \\ -\mathcal{Q} & -\mathcal{A}^T \end{bmatrix} := T_{00} - \sum_{j=1}^{N} T_{0j} T_{jj}^{-1} T_{j0},$$

$$T_{00} := \begin{bmatrix} \Xi_{00} & S_{00} \\ -W_{00} & -\Xi_{00}^T \end{bmatrix}, \ T_{0j} := \begin{bmatrix} \Xi_{0j} & S_{0j} \\ -W_{0j} & -\Xi_{j0}^T \end{bmatrix},$$

$$T_{j0} := \begin{bmatrix} \Xi_{j0} & S_{0j}^T \\ -W_{0j}^T & -\Xi_{0j}^T \end{bmatrix}, \ T_{jj} := \begin{bmatrix} \Xi_{jj} & S_{jj} \\ -W_{jj} & -\Xi_{jj}^T \end{bmatrix},$$

$$j = 1, \ \ldots, N.$$

The limiting behavior of P_e as the parameter $\|\mu\| := \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_N^2} \to +0$ is described by the following theorem.

Theorem 1: Under Assumptions 1 and 2, and the condition that (A_{jj}, C_{jj}) is observable, there exists a small σ^* such that for all $\|\mu\| \in (0, \sigma^*)$ the MARE (6) admits a symmetric positive semidefinite stabilizing solution P_e if the following two conditions are satisfied.

(i) The ARE (13c) admits a symmetric positive semidefinite stabilizing solution $\bar{P}_{jj}^*$.

(ii) The ARE (13a) admits a symmetric positive semidefinite stabilizing solution $\bar{P}_{00}^*$.

Moreover, the solution P_e can be written as

$$\Phi_e^{-1} P_e = \begin{bmatrix} \bar{P}_{00}^* + O(\|\mu\|) & [\bar{P}_{f0}^* + O(\|\mu\|)]^T \Pi_e \\ \bar{P}_{f0}^* + O(\|\mu\|) & \bar{P}_f^* + O(\|\mu\|) \end{bmatrix}$$

$$= \bar{P} + O(\|\mu\|) = \begin{bmatrix} \bar{P}_{00}^* & 0 \\ \bar{P}_{f0}^* & \bar{P}_f^* \end{bmatrix} + O(\|\mu\|). \quad (14)$$

Proof: The proof of the existence of P_e is obtained by the implicit function theorem (Gajić, 1988). To do so, it is sufficient to show that the corresponding Jacobian is nonsingular at $\|\mu\| = 0$. It can be shown, after some algebra, that the Jacobian of (8) in the limitation as $\|\mu\| \to 0$ is given by

$$\begin{aligned}
\mathbf{J} = \nabla \mathbf{F} &= \left. \frac{\partial(\text{vec} f_1, \ \text{vec} f_2, \ \text{vec} f_3)}{\partial(\text{vec} P_{00}, \ \text{vec} P_{f0}, \ \text{vec} P_f)^T} \right|_{\|\mu\|=0} \\
&= \begin{bmatrix} \mathbf{J}_{00} & \mathbf{J}_{01} & 0 \\ \mathbf{J}_{10} & \mathbf{J}_{11} & \mathbf{J}_{12} \\ 0 & 0 & \mathbf{J}_{22} \end{bmatrix}, \quad (15)
\end{aligned}$$

where vec denotes an ordered stack of the columns of its matrix and

$$\begin{aligned}
&\mathbf{J}_{00} = I_{n_0} \otimes \bar{A}_{00}^T + \bar{A}_{00}^T \otimes I_{n_0}, \\
&\mathbf{J}_{01} = (I_{n_0} \otimes \bar{A}_{f0}^T) U_{n_0 \hat{n}} + \bar{A}_{f0}^T \otimes I_{n_0}, \\
&\mathbf{J}_{10} = \bar{A}_{0f}^T \otimes I_{n_0}, \ \mathbf{J}_{11} = \bar{A}_f^T \otimes I_{n_0}, \\
&\mathbf{J}_{22} = I_{\hat{n}} \otimes \bar{A}_f^T + \bar{A}_f^T \otimes I_{\hat{n}}, \\
&\bar{A}_{00} = A_{00} - S_{00} \bar{P}_{00}^* - S_{0f} \bar{P}_{f0}^*, \\
&\bar{A}_{f0} = A_{f0} - S_{0f}^T \bar{P}_{00}^* - S_f \bar{P}_{f0}^*, \\
&\bar{A}_{0f} = A_{0f} - S_{0f} \bar{P}_f^*, \\
&\bar{A}_f = A_f - S_f \bar{P}_f^*, \ \hat{n} = \sum_{j=1}^{N} n_j,
\end{aligned}$$

where $\otimes$ denotes Kronecker products and $U_{n_0 n_0}$ is the permutation matrix in the Kronecker matrix sense.

The Jacobian (15) can be expressed as

$$\det\mathbf{J} = \det\mathbf{J}_{22} \cdot \det\mathbf{J}_{11} \cdot \det[I_{n_0} \otimes \bar{A}_0^T + \bar{A}_0^T \otimes I_{n_0}], \quad (16)$$

where $\bar{A}_0 \equiv \bar{A}_{00} - \bar{A}_{0f}\bar{A}_f^{-1}\bar{A}_{f0}$. Obviously, $\mathbf{J}_{jj}$, $j = 1,\ 2$ are nonsingular because the matrices $\bar{A}_f = A_f - S_f\bar{P}_f^*$ is stable. After some straightforward but tedious algebra, it is easy to verify that $\mathcal{A} - \mathcal{S}\bar{P}_{00}^* = \bar{A}_{00} - \bar{A}_{0f}\bar{A}_f^{-1}\bar{A}_{f0} = \bar{A}_0$. Therefore, the matrix $\bar{A}_0$ is also stable if Assumption 2 holds. Thus, $\det\mathbf{J} \neq 0$, i.e., $\mathbf{J}$ is nonsingular at $\|\mu\| = 0$. The conclusion of Theorem 1 is obtained directly by using the implicit function theorem. The remainder of the proof is to show that P_e is the positive semidefinite stabilizing solution. Since it can be done by using the similar technique in (Mukaidani et al., 2003), the proof is omitted. □

3. APPROXIMATE SADDLE–POINT EQUILIBRIUM

The required solution of the MARE (6) exists under the condition of Theorem 1. Our attention is focused on the design of the approximate strategy which does not depend on the values of the small perturbation parameters. Such an approximate strategy is obtained by eliminating $O(\|\mu\|)$ item of the linear state feedback strategy (5). If $\|\mu\|$ is very small, it is obvious that the linear state feedback strategy (5) for each group is approximated to

$$\bar{u}^* = \begin{bmatrix} \bar{u}_0^{*T} & \bar{u}_1^{*T} & \cdots & \bar{u}_N^{*T} \end{bmatrix}^T = -R_u^{-1}B^T\bar{P}x = -R_u^{-1}B^T \begin{bmatrix} \bar{P}_{00}^* & 0 \\ \bar{P}_{f0}^* & \bar{P}_f^* \end{bmatrix} x, \quad (17a)$$

$$\bar{v}^* = \begin{bmatrix} \bar{v}_0^{*T} & \bar{v}_1^{*T} & \cdots & \bar{v}_N^{*T} \end{bmatrix}^T = R_v^{-1}D^T\bar{P}x = R_v^{-1}D^T \begin{bmatrix} \bar{P}_{00}^* & 0 \\ \bar{P}_{f0}^* & \bar{P}_f^* \end{bmatrix} x, \quad (17b)$$

where $B = \Phi_e B_e$ and $D = \Phi_e D_e$.

When $\|\mu\|$ is sufficiently small, it is easy to find from Theorem 1 that the resulting perturbation–independent strategy pair (17) is close to the full-order strategy pair (5). We are now interested in how the approximate strategy effects the original game. In other words, we want to know if the approximate strategy pair is still in a saddle–point equilibrium, and how the game value changes if the approximate strategy pair constitutes a saddle–point equilibrium.

Theorem 2: Under the conditions of Theorem 1, the use of the approximate strategies (17) results in $J(\bar{u}^*,\ \bar{v}^*)$ satisfying

$$J(\bar{u}^*,\ \bar{v}^*) = J(u^*,\ v^*) + O(\|\mu\|^2). \quad (18)$$

Moreover, $J(\bar{u}^*,\ \bar{v}^*) > J(u^*,\ v^*)$ for $S > 0$, and $J(\bar{u}^*,\ \bar{v}^*) < J(u^*,\ v^*)$ for $S < 0$.

Before proving this theorem, the following lemma is useful (Mukaidani et al., 2001).

Lemma 3: Consider the iterative algorithm which is based on the Kleinman algorithm

$$(A - SP^{(i)})^T P^{(i+1)} + P^{(i+1)T}(A - SP^{(i)}) + P^{(i)T}SP^{(i)} + Q = 0, \quad i = 0,\ 1,\ \dots\ , \quad (19a)$$

$$P^{(i)} = \begin{bmatrix} P_{00}^{(i)} & P_{f0}^{(i)T}\Pi_e \\ P_{f0}^{(i)} & P_f^{(i)} \end{bmatrix}, \quad (19b)$$

with the initial condition obtained from

$$P^{(0)} = \bar{P} = \begin{bmatrix} \bar{P}_{00}^* & 0 \\ \bar{P}_{f0}^* & \bar{P}_f^* \end{bmatrix}. \quad (20)$$

Under Assumptions 1 and 2, there exists a small $\bar{\sigma}$ such that for all $\|\mu\| \in (0,\ \bar{\sigma})$, $\bar{\sigma} \leq \sigma^*$ Kleinman algorithm (19) converges to the exact solution of $P_e = \Phi_e P = P^T\Phi_e$ with the rate of quadratic convergence, where $P_e^{(i)} = \Phi_e P^{(i)} = P^{(i)T}\Phi_e$ is positive semidefinite. That is, the following relation holds

$$\|P^{(i)} - P\| = O(\|\mu\|^{2^i}),\ i = 0,\ 1,\ \dots\ , \quad (21)$$

where

$$\gamma = 2\|S\| < \infty,\ \beta = \|[\nabla\mathcal{G}(P^{(0)})]^{-1}\|,$$

$$\eta = \beta \cdot \|\mathcal{G}(P^{(0)})\|,\ \theta = \beta\eta\gamma,\ \nabla\mathcal{G}(P) = \frac{\partial \mathrm{vec}\mathcal{G}(P)}{\partial(\mathrm{vec}P)^T}.$$

Proof: When $\bar{u}^*$ and $\bar{v}^*$ are used, the value of the performance index is given by

$$J(\bar{u}^*,\ \bar{v}^*) = \frac{1}{2}x(0)^T X_e x(0),$$

where

$$(A_e - S_e\bar{P}_e)^T X_e + X_e(A_e - S_e\bar{P}_e) + \bar{P}_e S_e \bar{P}_e + Q = 0,\ \bar{P}_e := \Phi_e\bar{P}. \quad (22)$$

Subtracting (6) from (22), it is easy to verify that $V_e = X_e - P_e$ satisfies the following multiparameter algebraic Lyapunov equation (MALE)

$$(A_e - S_e\bar{P}_e)^T V_e + V_e(A_e - S_e\bar{P}_e) + (P_e - \bar{P}_e)S_e(P_e - \bar{P}_e) = 0. \quad (23)$$

Hence, since $A_e - S_e\bar{P}_e$ is stable, using the standard Lyapunov theorem (Zhou, 1998), $V_e = X_e - P_e > 0$ for $S > 0$, $V_e = X_e - P_e < 0$ for $S < 0$ are satisfied. Taking $J(u^*,\ v^*) = \frac{1}{2}x(0)^T P_e x(0)$ into account, $J(\bar{u}^*,\ \bar{v}^*) > J(u^*,\ v^*)$ for $S > 0$, $J(\bar{u}^*,\ \bar{v}^*) < J(u^*,\ v^*)$ for $S < 0$ are obtained. On the other hand, subtracting (6) from (19a), the following MALE holds

$$
\begin{aligned}
&(A_e - S_e P_e^{(i)})^T (P_e^{(i+1)} - P_e) \\
&\quad +(P_e^{(i+1)} - P_e)(A_e - S_e P_e^{(i)}) \\
&\quad +(P_e - P_e^{(i)}) S_e (P_e - P_e^{(i)}) = 0, \qquad (24)
\end{aligned}
$$

where $P_e^{(i)} = \Phi_e P^{(i)}$. When $i = 0$, the following equality holds.

$$
\begin{aligned}
&(A_e - S_e P_e^{(0)})^T (P_e^{(1)} - P_e) \\
&\quad +(P_e^{(1)} - P_e)(A_e - S_e P_e^{(0)}) \\
&\quad +(P_e - P_e^{(0)}) S_e (P_e - P_e^{(0)}) \\
&= (A_e - S_e \bar{P}_e)^T (P_e^{(1)} - P_e) \\
&\quad +(P_e^{(1)} - P_e)(A_e - S_e \bar{P}_e) \\
&\quad +(P_e - \bar{P}_e) S_e (P_e - \bar{P}_e) = 0.
\end{aligned}
$$

Therefore, it is easy to verify that $V_e = P_e^{(1)} - P_e$ because $A_e - S_e \bar{P}_e$ is stable from Theorem 1 of (Khalil and Kokotović, 1979). Using Lemma 3, the following relation satisfies.

$$
\begin{aligned}
&\|V_e\| = \|X_e - P_e\| = \|P_e^{(1)} - P_e\| \\
&\leq \|\Phi_e\| \cdot \|P^{(1)} - P\| \leq \|P^{(1)} - P\| = O(\|\mu\|^2). \quad (25)
\end{aligned}
$$

Hence, $V_e = X_e - P_e = O(\|\mu\|^2)$ results in (18). □

Using the similar analysis we have the following result.

Theorem 3: Under the conditions of Theorem 1, the following result holds.

$$
J(\bar{u}^*,\ v) = J(u^*,\ v) + O(\|\mu\|), \qquad (26a)
$$
$$
J(u,\ \bar{v}^*) = J(u,\ v^*) + O(\|\mu\|). \qquad (26b)
$$

Proof: Since the proof can be done by using the above technique, it is omitted. □

Finally, the main result is easily derived as the major extension of the existing result (Xu and Mizukami, 1997).

Theorem 4: Under the conditions of Theorem 1, the approximate strategy pair (17) constitutes an $O(\|\mu\|)$ near saddle–point equilibrium, that is,

$$
J(\bar{u}^*,\ v) - O(\|\mu\|) \leq J(\bar{u}^*,\ \bar{v}^*), \qquad (27a)
$$
$$
J(\bar{u}^*,\ \bar{v}^*) \leq J(u,\ \bar{v}^*) + O(\|\mu\|). \qquad (27b)
$$

Proof: We will give the proof to the inequality (27b). A similar proof can be obtained for another inequality. Let us rewrite the inequality (27b) as

$$
\begin{aligned}
&J(\bar{u}^*,\ \bar{v}^*) - J(u,\ \bar{v}^*) \\
&= J(\bar{u}^*,\ \bar{v}^*) - J(u^*,\ v^*) + J(u^*,\ v^*) - J(u,\ v^*) \\
&\quad + J(u,\ v^*) - J(u,\ \bar{v}^*). \qquad (28)
\end{aligned}
$$

Using (18), (4) and (26b), we get (27b) readily. □

4. CONCLUSION

In this paper, the group differential game problem for the MSPS has been studied. A method to find the perturbation–independent approximate strategies to the players is developed, which use the zero–order solution of the MARE in the approximate strategy. Since each approximate strategy of the player does not depend on the small parameters (perturbations), the decision making can be carried out even if the player does not have the knowledge of the small parameters. It has been proved that the perturbation–independent approximate strategies constitute an $O(\|\mu\|)$ near saddle–point equilibrium.

Applicability of the theoretical results will be used as the application of the business model with decision support. This problem will be addressed in future investigations.

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