ROBUST STABILITY ANALYSIS OF LINEAR SYSTEMS WITH TIME-VARYING DELAYS

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Abstract: Robust stability of linear systems in presence of bounded uncertain time-varying time delays is studied. The time delay robustness problem is treated in the Integral Quadratic Constraint (IQC) framework. The stability criterion is formulated as frequency dependent linear matrix inequalities. The criterion can be equivalently formulated as a Semi-Definite Program (SDP) by applying Kalman-Yakubovich-Popov (KYP) lemma. Therefore, checking the criterion can be done efficiently by using various SDP solvers.

Keywords: Time delay, Robust stability, Stability analysis, Linear system

1. INTRODUCTION

Consider the following linear time delay system

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau(t)) \tag{1}$$

where $\tau(t)$ is a unknown time-varying parameter which satisfies

$$0 \leq \tau(t) \leq \tau_0, \quad |\dot{\tau}(t)| \leq d, \quad \forall\, t \geq 0, \tag{2}$$

$x \in \mathbf{R}^n$ is the state, A and $A_d \in \mathbf{R}^{n\times n}$ are constant matrices. The initial condition $x(\theta)$ is a continuous function defined on $[-\tau_0, 0]$. In this paper, delay-dependent conditions for robust stability of time-delay system (1) is developed. More specifically, given a pair of scalars (τ_0, d), our objective is to derive conditions under which the delay system (1) is stabile for all $\tau(t)$ that satisfy condition (2).

If the delay parameter τ is unknown but constant, then the energy of $x(t-\tau)$ is the same as the energy of $x(t)$. Hence, a simple but conservative delay-independent stability criterion for the system, $\sup_\omega \|(j\omega I_n - A)^{-1}A_d\| < 1$, immediately follows the small gain theorem. In the case of constant time delay, the exact condition for delay-independent stability was derived in (Chen and Latchman, 1995) using structured singular value. For delay-dependent stability, the robustness conditions can either be derived using frequency-domain analysis (by μ or IQC analysis) (Megretski and Rantzer, 1997; Huang and Zhou, 2000; Zhang *et al.*, 2001; Jun and Safonov, 2001) or time-domain analysis (Park, 1999; Kolmanovskii and Richard, 1999; Li and de Souza, 1997*b*; Han and Gu, 2001). See also (Kolmanovskii *et al.*, 1999) and (Niculescu, 2001) for the recent development on stability analysis of time delay systems.

When the delay parameter is time-varying, stability analysis is more involved. One of the difficulties, for instance, is that the delay operator is no longer energy-preserving. In fact, if there is no restriction on the variation $\dot{\tau}(t)$, the delay operator is not even a bounded operator on the L_2 space no matter how small the length of the delay is. To see this, let $v(t)$ and $\tau(t)$ be

$$v(t) = \begin{cases} 1 & t = [0, \epsilon] \\ 0 & \text{otherwise} \end{cases} \qquad \tau(t) = \begin{cases} t & t \in [0, \tau_0] \\ \tau_0 & \text{otherwise} \end{cases},$$

Then $v(t-\tau(t))$ is equal to 1 for $t \in [0, \tau_0+\epsilon]$ and 0 otherwise. The energy of $v(t-\tau(t))$ is equal to $\tau_0+\epsilon$ while the energy of $v(t)$ is equal to ϵ. Hence, the

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gain of the delay operator becomes unbounded as $\epsilon \to 0$. Intuitively, systems with energy generating components are easier to be rendered unstable, and it is not obvious that robust stability criteria for system with constant time delays can be easily generalized to verify robustness of time-varying delay systems. Over the past few years, researchers have been working on stability analysis of linear systems with time-varying delays. Most of the available results in the literature are developed in the time-domain framework, based on Lyapunov's second method using Lyapunov-Razumikhin functions (Cao *et al.*, 1998; Li and de Souza, 1997*a*), or various Lyapunov-Krasovskii functionals (Fridman and Shaked, 2002; Fridman and Shaked, 2003; Kim, 2001; Mehdi *et al.*, 2002; Kharitonov and Niculescu, 2003).

In this paper, we consider a robust stability problem where time-varying delays appear in a closed-loop continuous-time linear system. The frequency-domain approach is adopted to developed stability criteria. Specifically, the stability problem is treated in the Integral Quadratic Constraint (IQC) framework (Megretski and Rantzer, 1997). The main advantage of IQC analysis is that, the result obtained can be easily generalized to systems with delays and uncertainties or nonlinearities.

Notations: L_2^m is used to denote the space of $\mathbf{R}^m$ valued, square summable functions defined on time interval $(-\infty, \infty)$, and L_{2e}^m to denote the extension of the space L_2^m, which consists of those functions whose time truncations lie in L_2^m. Notation $\mathbf{RL}_\infty^{l\times m}$ is used to denote the space of proper rational transfer matrices of dimension $l \times m$ which have no pole on the imaginary axis, while $\mathbf{RH}_\infty^{l\times m}$ denotes the subspace of $\mathbf{RL}_\infty^{l\times m}$ consisting of functions which have no pole in the closed right half plane. Every $H \in \mathbf{RL}_\infty^{l\times m}$ defines a convolution operator on L_2: for any $u \in L_2$,

$$(Hu)(t) := \int_{-\infty}^{\infty} h(t-\theta)u(\theta)d\theta,$$

where $h(t)$ is the inverse Laplace transform of H. $\|\cdot\|_{L_2}$ is used to denote the L_2 norm of L_2 signals, or the L_2 induced norm of bounded operators on the L_2 space,

Let $\mathcal{D}_\tau$ denote the time-delay operator such that $\mathcal{D}_\tau(v) := v(t-\tau(t))$, and $\mathcal{S}_\tau$ be $(I - \mathcal{D}_\tau)$; i.e., $\mathcal{S}_\tau(v) := v(t) - v(t-\tau(t))$. For simplicity, in the rest of the paper the time dependency on $\tau(t)$ is suppressed and we simply write τ and $\dot{\tau}$.

2. STABILITY ANALYSIS VIA INTEGRAL QUADRATIC CONSTRAINTS

In this section, the Integral Quadratic Constraint (IQC) analysis, which is needed for the main result of this paper, is briefly introduced. The system under consideration is

$$v = Gw + e, \quad w = \Delta(v) \tag{3}$$

where $G(s) \in \mathbf{RH}_\infty^{l\times m}$ and Δ is a bounded and causal operator on $L_{2e}^{m\times l}$. *Well-posedness* and *stability* of such a system are defined as follows.

Definition 1. (Megretski and Rantzer, 1997) The feedback interconnection of G and Δ as defined in equation (3) is said to be *well-posed* if the map $(v, w) \mapsto e$ has a causal inverse on L_{2e}. That is, for any $e \in L_{2e}$, there exists a solution $(v, w) \in L_{2e}$ which depends causally on e. If, in addition, there exists a positive constant C such that

$$\int_{-\infty}^{T} \|v\|^2 + \|w\|^2 dt \le C \int_{-\infty}^{T} \|e\|^2 dt, \quad \forall\, T \ge 0$$

then the system is said to be *stable*.

Let Π be a bounded self-adjoint operator on L_2 space. Then Π defines a quadratic form on L_2

$$\begin{aligned}\sigma_\Pi(v,w) &:= \left\langle \begin{bmatrix} v \\ w \end{bmatrix}, \Pi \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle \\ &= \int_{-\infty}^{\infty} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}' \left(\Pi \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \right) dt \\ &= \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega\end{aligned}$$

where $\hat{v}$ and $\hat{w}$ are Fourier transforms of v and w, respectively. The operator Π is referred to as the multiplier of the quadratic form σ_Π.

Given an operator $\mathcal{H}$ and a quadratic form $\sigma_\Pi(v, w)$ defined on L_2 space, we said that $\mathcal{H}$ satisfies the integral quadratic constraint defined by σ_Π, or more often "$\mathcal{H}$ satisfies IQC defined by Π" to emphasize the multiplier involved, if $\sigma_\Pi(v, \mathcal{H}(v)) \ge 0$ for all $v \in L_2$.

The main stability criterion, the so-called IQC theorem, in (Megretski and Rantzer, 1997) is formulated as follows.

Theorem 1. (Megretski and Rantzer, 1997) Let $G(s) \in \mathbf{RH}_\infty^{l\times m}$ and let Δ be a bounded causal operator. Suppose

(1) for every $\rho \in [0, 1]$, the interconnection of G and $\rho\Delta$ is well-posed;
(2) for every $\rho \in [0, 1]$, the IQC defined by Π is satisfied by $\rho\Delta$;
(3) there exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -\epsilon I, \ \forall\, \omega \in [0, \infty]. \tag{4}$$

Then the feedback interconnection of G and Δ is stable.

Note that if $\rho\Delta$ satisfies IQCs defined by Π_i, $i = 1, \cdots, n$, then the conic combination $x_1\Pi_1 + \cdots + x_n\Pi_n$, $x_i \ge 0$ also defines an IQC for $\rho\Delta$. Hence, a sufficient condition for stability is the existence of $x_1, \cdots, x_n \ge 0$ such that (4) holds for $\Pi := x_1\Pi_1 + \cdots + x_n\Pi_n$.

Furthermore, assume that the overall Δ is diagonally structured by n components, Δ_i, $i = 1, \cdots, n$; i.e., $\Delta = \mathrm{diag}(\Delta_1, \cdots, \Delta_n)$. Suppose that each Δ_i satisfies IQC defined by Π_i, respectively. Then an IQC for Δ can be easily defined by assembling Π_i appropriately.

Condition (4) is a frequency dependent, infinite dimensional Linear Matrix Inequality (LMI). Suppose that $\Pi \in \mathbf{RL}_\infty$. Then this matrix inequality can be converted into a frequency independent finite dimensional LMI using the Kalman-Yakubovich- Popov (KYP) Lemma. See (Megretski and Rantzer, 1997) for details.

3. INTEGRAL QUADRATIC CONSTRAINTS FOR OPERATORS $\mathcal{D}_\tau$ AND $\mathcal{S}_\tau$

In this section, conically parameterized integral quadratic constraints for operators $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$ are derived. These IQCs will be used in the next section to derive stability criteria for linear systems with time-varying delays.

3.1 IQC's with frequency independent multipliers

Lemma 1. Let $\psi_1 \in \mathbf{R}^{n\times n}$ be $\sqrt{1-d} \cdot I_n$ and ψ_2, be the following rational transfer matrix

$$\psi_2 := \frac{1}{\tau_0 s} \cdot I_n.$$

Suppose that $\tau \in [0, \tau_0]$ and $|\dot{\tau}| \le d \le 1$. Then $\mathcal{D}_\tau \circ \psi_1$ and $\mathcal{S}_\tau \circ \psi_2$ are bounded operators on the L_2 space. Furthermore, the L_2-gains of these operators are upper bounded by 1.

Proof 1. See (Kao and Rantzer, 2003).

Remark 1. It can be shown that the bounds on $\|\mathcal{D}_\tau\|_{L_2}$ and $\|\mathcal{S}_\tau \circ \frac{1}{s}\|_{L_2}$ are tight. See (Kao and Rantzer, 2003) for details. Furthermore, as long as $\tau_0 > 0$, $\|\mathcal{D}_\tau\|_{L_2}$ is a function of d but **not dependent** on τ_0. In contrast, $\|\mathcal{S}_\tau \circ \frac{1}{s}\|_{L_2}$ is a function of τ_0 but **not dependent** on d.

The norm estimation in Lemma 1 gives rise the following IQC for $\mathcal{D}_\tau \circ \psi_1$ and $\mathcal{S}_\tau \circ \psi_2$.

Lemma 2. Suppose $\tau \in [0, \tau_0]$ and $|\dot{\tau}| \le d \le 1$. Then operators $\mathcal{D}_\tau \circ \psi_1$ and $\mathcal{S}_\tau \circ \psi_2$ satisfy integral quadratic constraints defined by

$$\Pi = \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix} \tag{5}$$

where $X = X' \ge 0$ is any positive semi-definite matrix.

Proof 2. Let $w_1 = \mathcal{D}_\tau \circ \psi_1(v)$ and $w_2 = \mathcal{S}_\tau \circ \psi_2(v)$. Then Lemma 2 immediately follows Lemma 1 and the fact that, given a positive definite matrix X,

$$X^{\frac{1}{2}} w_1 = X^{\frac{1}{2}} \mathcal{D}_\tau \circ \psi_1(v) = \mathcal{D}_\tau \circ \psi_1(X^{\frac{1}{2}} v)$$
$$X^{\frac{1}{2}} w_2 = X^{\frac{1}{2}} \mathcal{S}_\tau \circ \psi_2(v) = \mathcal{S}_\tau \circ \psi_2(X^{\frac{1}{2}} v).$$

3.2 IQC's with frequency-dependent multipliers

In this section, integral quadratic constraints with frequency dependent multipliers for $\mathcal{D}_\tau \circ \psi_1$ and $\mathcal{S}_\tau \circ \psi_2$ are derived. The following lemma, often referred to as the swapping lemma, plays a key role in deriving those IQCs.

Lemma 3. (Swapping lemma). Let $H(s) := C(sI-A)^{-1}B+D$, $H_L(s) := C(sI-A)^{-1}$, and $H_R(s) := (sI-A)^{-1}B$ be proper rational transfer matrices from $\mathbf{RL}_\infty^{n\times n}$. Furthermore, let T denote the operator of multiplying $\dot{\tau}$; i.e., $T(v(t)) := \dot{\tau} \cdot v(t)$. Then

$$\begin{aligned} \mathcal{D}_\tau \circ H(s) &= H(s) \circ \mathcal{D}_\tau \\ &\quad - H_L(s) \circ T \circ \mathcal{D}_\tau \circ sH_R(s) \end{aligned} \tag{6}$$

Proof 3. Let $y = \mathcal{D}_\tau \circ Hv$, and $z = H \circ \mathcal{D}_\tau(v)$. Then $y(t)$ is equal to $Cx_1(t-\tau)+Dv(t-\tau)$, where x_1 is the state of the system $\dot{x_1}(t) = Ax_1(t) + Bv(t)$, $x_1(0) = 0$. Signal $z(t)$ is equal to $Cx_2(t) + Dv(t-\tau)$, where x_2 satisfies $\dot{x}_2(t) = Ax_2(t) + Bv(t-\tau)$, $x_2(0) = 0$. Let $w = y-z = C(x_1(t-\tau)-x_2(t)) = Cx_3(t)$. It is observed by differentiating x_3 w.r.t. time that

$$\begin{aligned} \dot{x}_3(t) &= \dot{x}_1(t-\tau) - \dot{x}_2(t) - \dot{\tau} \cdot \dot{x}_1(t-\tau) \\ &= Ax_3(t) - \dot{\tau} \cdot \dot{x}_1(t-\tau) \end{aligned}$$

Hence,

$$\begin{aligned} w :&= (\mathcal{D}_\tau \circ H - H \circ \mathcal{D}_\tau)v = H_L(s)(-\dot{\tau} \cdot \mathcal{D}_\tau(\dot{x}_1)) \\ &= -H_L(s) \circ T \circ \mathcal{D}_\tau \circ sH_R(s)v \end{aligned}$$

This concludes the proof.

Remark 2. Using (6), the following equalities can be readily verified

$$\begin{aligned} &(\mathcal{D}_\tau \circ \psi_1) \circ H(s) - H(s) \circ (\mathcal{D}_\tau \circ \psi_1) \\ &\qquad = -H_L(s) \circ T \circ \mathcal{D}_\tau \circ H_{C,1}(s) \\ &(\mathcal{S}_\tau \circ \psi_2) \circ H(s) - H(s) \circ (\mathcal{S}_\tau \circ \psi_2) \\ &\qquad = H_L(s) \circ T \circ \mathcal{D}_\tau \circ H_{C,2}(s) \end{aligned}$$

where $H_{C,i}(s) = sH_R(s)\psi_i$, $i = 1, 2$.

Using these swapping formulas, the following frequency dependent integral quadratic constraints for $\mathcal{D}_\tau \circ \psi_1$ and $\mathcal{S}_\tau \circ \psi_2$ can be derived.

Lemma 4. Let $H(s) := h(s) \cdot I_n$ where $h(s) \in \mathbf{RL}_\infty^{1\times 1}$. Let the corresponding $H_L(s)$, $H_R(s)$, and $H_{C,i}(s)$, $i = 1, 2$ be defined similarly as those in Lemma 4 and Remark 2. Suppose $\tau \in [0, \tau_0]$ and $|\dot{\tau}| \le d < 1$. Then operators $\mathcal{D}_\tau \circ \psi_1$ and $\mathcal{S}_\tau \circ \psi_2$ satisfy integral quadratic constraints defined by the following multipliers, respectively

$$\Pi_i(j\omega) = \begin{bmatrix} \Pi_{i,1} & 0 \\ 0 & \Pi_{i,2} \end{bmatrix}, \quad i = 1, 2 \tag{7}$$
$$\Pi_{i,1}(j\omega) = H^*XH + \frac{d}{1-d} \cdot H_{C,i}^* X H_{C,i}$$
$$\Pi_{i,2}(j\omega) = -H^*XH + d \cdot H^* H_L X H_L^* H$$

where $X = X' \ge 0$ is any positive semi-definite matrix.

Proof 4. It is shown here that operator $\mathcal{D}_\tau \circ \psi_1$ satisfies the integral quadratic constraint defined by $\Pi_1(j\omega)$. The other IQC can be derived in a similar fashion. Given a L_2 signal v, let $w = \mathcal{D}_\tau \circ \psi_1(v)$. Then

$$\begin{aligned}
2\langle Hw, Hw\rangle &= 2\langle Hw, (\mathcal{D}_\tau \circ \psi_1) \circ Hv + \\
&\qquad\qquad H_L \circ T \circ \mathcal{D}_\tau \circ H_{C,1}v\rangle \\
&= 2\langle Hw, (\mathcal{D}_\tau \circ \psi) \circ Hv\rangle + \\
&\qquad 2\langle \sqrt{d}H_L^* Hw, \frac{1}{\sqrt{d}}T \circ \mathcal{D}_\tau \circ H_{C,1}v\rangle \\
&\le \langle Hw, Hw\rangle + \langle Hv, Hv\rangle + d\langle H_L^* Hw, H_L^* Hw\rangle \\
&\qquad \frac{d}{1-d}\langle H_{C,1}v, H_{C,1}v\rangle
\end{aligned}$$

which implies

$$\left\langle \begin{bmatrix} v \\ w \end{bmatrix}, \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle \ge 0$$

where

$$\begin{aligned}
\Pi_1 &= H^* H + \frac{d}{1-d} \cdot H_{C,1}^* H_{C,1} \\
\Pi_2 &= -H^* H + d \cdot H^* H_L H_L^* H
\end{aligned}$$

Finally, to see the conic parametrization of Π_1, note that given any positive semi-definite matrix X, matrix $X^{\frac{1}{2}}$ commutes with $\mathcal{D}_\tau \circ \psi_1$, $H(s)$, $H_L(s)$, $H_R(s)$, and $H_{C,1}(s)$. This concludes the proof.

Remark 3. Note that the IQC stated in Lemma 2 is a special case of those stated in Lemma 4. Indeed, by choosing $h(s) \equiv 1$, the Π_i matrices in (8) reduce to the Π matrix in (5).

Lemma 5. Consider the following operator

$$\mathcal{C}_\tau(v) := \begin{bmatrix} \mathcal{D}_\tau(v) \\ \mathcal{S}_\tau(v) \end{bmatrix}$$

Then $\mathcal{C}_\tau$ satisfies IQC defined by the multiplier

$$\Pi_5(j\omega) := \begin{bmatrix} X_1(j\omega) & X_2(j\omega) & X_2(j\omega) \\ X_2(j\omega)^* & X_3(j\omega) & X_3(j\omega) \\ X_2(j\omega)^* & X_3(j\omega) & X_3(j\omega) \end{bmatrix} \tag{8}$$

where $X_i \in \mathbf{RL}_\infty^{n\times n}$ satisfying $X_1 = X_1^* \ge 0$, $X_3 = X_3^* \le 0$, and $X_1 + X_2 + X_2^* + X_3 \ge 0$, $\forall\ \omega$.

Proof 5. Let $w_1 = \mathcal{D}_\tau(v)$ and $w_2 = \mathcal{S}_\tau(v)$ where v is a L_2 signal. Note that $w_1 + w_2 = v$. Hence, given a $\Pi_5(j\omega)$ matrix as described above,

$$\begin{aligned}
&\left\langle \begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix}, \Pi_5 \begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} v \\ w_1 + w_2 \end{bmatrix}, \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} \begin{bmatrix} v \\ w_1 + w_2 \end{bmatrix} \right\rangle \\
&= \langle v, (X_1 + X_2 + X_2^* + X_3)v\rangle \ge 0
\end{aligned}$$

This concludes the proof. The restriction that X_1 is positive semi-definite and X_3 is negative semi-definite is to ensure that the integral quadratic constraint is satisfied by all $\rho \cdot \mathcal{C}_\tau$, $\rho \in [0, 1]$, which is required for applying the IQC theorem.

4. STABILITY CRITERIA FOR LINEAR TIME-VARYING DELAY SYSTEMS

Consider now the linear time-varying delay system (1). Let $Q \in \mathbf{R}^{n\times n}$ be a constant matrix. Then the time-delay system (1) can be equivalently expressed as

$$\begin{aligned}
\dot{x}(t) &= (A + Q)x + (A_d - Q)x(t - \tau) \\
&\quad - Q\int_{t-\tau}^{t} \dot{x}(\theta)d\theta \\
&= (A + Q)x + (A_d - Q)\mathcal{D}_\tau(x) \\
&\quad - Q(\mathcal{S}_\tau \circ \psi_2)(\tau_0 \dot{x})
\end{aligned} \tag{9}$$

Two different cases arise.

4.1 The case where d *is greater than or equal to* 1

In this case, the L_2 gain of $\mathcal{D}_\tau$ is infinite. Hence let $Q = A_d$ and consider the following system

$$\begin{aligned}
\dot{x}(t) &= (A + A_d)x(t) - A_d w \qquad (10) \\
w &:= \int_{t-\tau}^{t} (A + A_d)x(\theta) - A_d w(\theta)d\theta \\
&= (\mathcal{S}_\tau \circ \psi_2)(\tau_0 \cdot ((A + A_d)x(t) - A_d w))
\end{aligned}$$

It is well-known that *descriptor* model (10) is *not* equivalent to system (1) (Gu and Niculescu, 2000); however, stability of system (10) does imply stability of system (1). System (10) can be viewed as a feedback interconnection of $G_{eq1}(s)$ and $\Delta_{eq1} := \mathcal{S}_\tau \circ \psi_2$, where

$$G_{eq1}(s) := C_{eq1}(sI - (A + A_d))^{-1}A_d + D_{eq1},$$

and $C_{eq1} = -\tau_0(A + A_d)$, $D_{eq1} = -\tau_0 A_d$. Using the IQC defined in Lemma 2 for $\mathcal{S}_\tau \circ \psi_2$, the following sufficient condition for stability of system (1) follows immediately the IQC theorem.

Proposition 1. Suppose $A + A_d$ has no eigenvalue in the closed right half plane. Then system (10) is stable, which consequently implies stability of time delay system (1), if there exists a symmetric positive-definite matrix X of suitable dimensions such that for some $\epsilon > 0$,

$$\begin{aligned}
G_{eq1}(j\omega)^* X G_{eq1}(j\omega) - X \le -\epsilon I, \\
\forall\ \omega \in [0, \infty]
\end{aligned} \tag{11}$$

Remark 4. By the KYP lemma, stability criterion (11) can be expressed as linear matrix inequality

$$\begin{bmatrix} P(A + A_d) + (A + A_d)'P & PA_d \\ A_d'P & 0 \end{bmatrix} + \begin{bmatrix} \tau_0^2(A + A_d)'X(A + A_d) & \tau_0^2(A + A_d)'XA_d \\ \tau_0^2 A_d'X(A + A_d) & -X \end{bmatrix} < 0$$

where $P = P' > 0$. The condition becomes to find positive-definite matrices P and X such that the above linear matrix inequality holds.

4.2 The case where d is strictly less than 1

In this case, let $w_1 := x(t-\tau) = \mathcal{D}_\tau(x(t))$,

$$w_2 := \int_{t-\tau}^{t} (A+Q)x + (A_d - Q)w_1 - Qw_2 \; d\theta$$

and consider a feedback interconnection of $G_{eq2}(s)$ and Δ_{eq2}, where

$$\begin{aligned} G_{eq2} &:= C_{eq2}(sI - A_{eq2})^{-1}B_{eq2} + D_{eq2}, \\ \Delta_{eq2} &:= \begin{bmatrix} \mathcal{D}_\tau & 0 \\ 0 & \mathcal{S}_\tau \circ \frac{1}{s} \end{bmatrix} \end{aligned} \tag{12}$$

and $A_{eq2} = A + Q$, $B_{eq2} = [A_d - Q \;\; -Q]$,

$$C_{eq2} = \begin{bmatrix} I_n \\ A_{eq2} \end{bmatrix}, \quad D_{eq2} = \begin{bmatrix} 0_{n\times 2n} \\ B_{eq2} \end{bmatrix}$$

Again, stability of (12) implies stability of time delay system (1), and IQC theorem is applied to derive stability conditions for (12).

Let $H_{ij}(s) = h_{ij}(s)I_n$, where $h_{ij}(s) \in \mathbf{RL}_\infty^{1\times 1}$, $i = 1, 2$, $j = 1, \cdots, n$. Define the corresponding $H_{L,ij}(s)$, $H_{R,ij}(s)$, and $H_{C,ij}(s) := sH_{R,ij} \circ \psi_i$ as in Lemma 3 and Remark 2, and let

$$M_{1j,1} = \frac{1}{1-d} H_{1j}^* X_{1j} H_{1j} + \frac{d}{(1-d)^2} H_{C,1j}^* X_{1j} H_{C,1j} \qquad j = 1, \cdots, n$$

$$M_{2j,1} = \tau_0^2 H_{2j}^* X_{2j} H_{2j} + \frac{\tau_0^2 d}{1-d} H_{C,2j}^* X_{2j} H_{C,2j} \qquad j = 1, \cdots, n$$

$$M_{ij,2} = -H_{ij}^* X_{ij} H_{ij} + d \cdot H_{ij}^* H_{L,ij} X_{ij} H_{L,ij}^* H_{ij} \qquad i = 1, 2 \quad j = 1, \cdots, n$$

where $X_{ij} = X_{ij}' > 0$. Moreover, let $P_i(s) \in \mathbf{RH}_\infty^{n\times n}$, $i = 1, \cdots, 4$ and defined

$$N_1 = P_1(j\omega)^* Y_1 P_1(j\omega), \quad N_2 = Y_2 P_2(j\omega),$$
$$N_3 = P_3(j\omega)^* Y_3 P_3(j\omega),$$

where Y_i are such that

$$\begin{aligned} &N_1 = N_1^* > 0, \; N_3 = N_3^* < 0, \\ &\quad \text{and } N_1 + N_2 + N_2^* + N_3 > 0, \; \forall \omega. \end{aligned} \tag{13}$$

Consider now the matrix

$$\Pi_{cmb} := \begin{bmatrix} \mathcal{M}_1 + N_1 & 0 & N_2 & N_2 \\ 0 & \mathcal{M}_2 & 0 & 0 \\ N_2^* & 0 & \mathcal{M}_3 + N_3 & N_3 \\ N_2^* & 0 & N_3 & \mathcal{M}_4 + N_3 \end{bmatrix} \tag{14}$$

where

$$\mathcal{M}_1 = \sum_{j=1}^{n} M_{1j,1}, \; \mathcal{M}_2 = \sum_{j=1}^{n} M_{2j,1},$$
$$\mathcal{M}_3 = \sum_{j=1}^{n} M_{1j,2}, \; \mathcal{M}_4 = \sum_{j=1}^{n} M_{2j,2}.$$

Following lemmas 4 and 5, it is easy to verify that Π_{cmb} is a bounded self-adjoint operator which defines a valid integral quadratic constraint for Δ_{eq2}. Applying the IQC theorem, we have the following stability criterion for system (12).

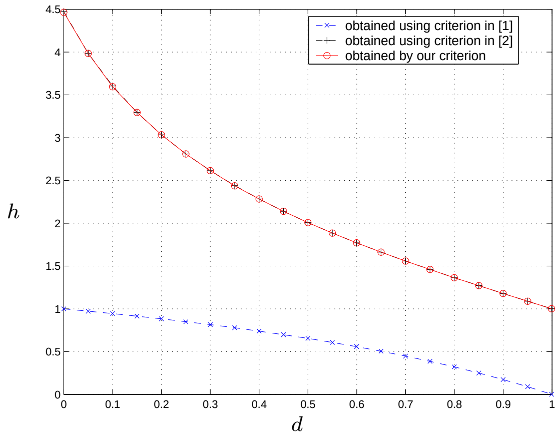

Fig. 1. Stability margins obtained using criteria given in [1] (Fridman and Shaked, 2002), [2] (Kim, 2001), and in this paper.

Proposition 2. Suppose $G_{eq2} \in \mathbf{RH}_\infty$. Then system (12) is stable if there exist symmetric positive-definite matrices X_{ij}, $i = 1, 2$, $j = 1, \cdots, n$, of suitable dimensions and Y_k, $k = 1, \cdots, 3$ satisfying condition (13), such that for some $\epsilon > 0$,

$$\begin{bmatrix} G_{eq2}(j\omega) \\ I \end{bmatrix}^* \Pi_{cmb}(j\omega) \begin{bmatrix} G_{eq2}(j\omega) \\ I \end{bmatrix} \le -\epsilon I, \qquad \forall\, \omega \in [0, \infty] \tag{15}$$

where Π_{cmb} is defined as in (14).

Remark 5. The descriptor model (12) is parameterized by Q which appears in the system matrices A_{eq2}, B_{eq2}, C_{eq2} and D_{eq2}. Hence the finite dimensional matrix inequality equivalent to (15) is in general bilinear (w.r.t the parameters Q and the 'P' matrix coming from the KYP lemma). To solve the bilinear matrix inequality (BMI), one could use an iterative procedure; namely, fix either matrix Q or the KYP matrix P iteratively. Alternatively, there is one way to make the BMI linear, at the expense of introducing conservatism. By letting w_2 (which is supposed to model the influence of $\mathcal{S}_\tau(x)$) to be

$$\int_{t-\tau}^{t} Ax(\theta) + A_d w_1(\theta) d\theta$$

and imposing a diagonal structure on the P matrix, it can be shown that the BMI becomes linear under certain re-parametrization. Here, due to the limitation of space, the details are omitted.

5. EXAMPLE AND COMPARISON

In this section, a numerical experiment is presented to demonstrate the effectiveness of stability criteria proposed in this paper.

Consider the following system (Example 4 of (Li and de Souza, 1997*a*)),

$$\begin{aligned} \dot{x}(t) &:= Ax(t) + A_d x(t-\tau) \\ &= \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-\tau) \end{aligned} \tag{16}$$

If τ is a constant, the Nyquist criterion shows that when $\tau = \frac{\arccos(-0.9)}{\sqrt{0.19}} \approx 6.1725$, the system has an eigenvalue on the imaginary axis.

Figure 1 shows the results obtained using the methodology proposed in this paper, and the criteria given in (Fridman and Shaked, 2002) and (Kim, 2001). It is observed that our method gives the same results as the criterion found in (Fridman and Shaked, 2002), both of which significantly outperform the criterion found in (Kim, 2001).

As a final remark, the IQC's used to obtain the above mentioned results are frequency-independent; namely, only non-ω-dependent multipliers were searched for applying condition (15) when stability analysis is performed. Less conservative results could be obtained by selecting appropriate frequency dependent multipliers, which is an ongoing research of ours in this topic.

6. CONCLUSIONS

Stability conditions for linear time delay systems were derived. The delay parameter is an unknown time-varying function for which the upper bounds on the magnitude and the variation are given. The influence of time-varying delay is modelled as perturbation caused by uncertainties in the system, and integral quadratic constraints were derived to characterize the effect of these uncertain operators. Conditions for stability were then derived based on IQC analysis. The advantage of this approach is that the results can be easily generalized for stability verification of systems with multiple delays, and extended to deal with systems with parametric uncertainties, unmodelled dynamics, and/or various simple non-linearities.

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