INPUT-TO-STATE STABILITY OF SWITCHED NONLINEAR SYSTEMS [1]

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Abstract: The input-to-state stability (ISS) problem is studied for switched systems with infinite subsystems. By the method of multiple Lyapunov function, a sufficient ISS condition is given based on a quantitative relation of the input and the values of the Lyapunov functions of the subsystems before and after the switching instants. In terms of the average dwell-time of the switching laws, some sufficient ISS conditions are obtained for switched nonlinear systems and switched linear systems, respectively.

Keywords: Switched system, input-to-state stability, Lyapunov function, dwell-time

1. INTRODUCTION

Since the performance of a real control system is affected more or less by uncertainties such as unmodelled dynamics, parameter perturbations, exogenous disturbances, measurement errors etc., the research on robustness of control systems do always have a vital status in the development of control theory and technology. Aiming at robustness analysis of nonlinear control systems, a new method from the point of view of input-to-state stability (ISS), input-to-output stability (IOS) and integral input-to-state stability (iISS) are developed and a series of fundamental results centralizing on the theory of ISS–, IOS–Lyapunov functions are obtained by many scholars (Sontag *et al.*, 1989, 1995, 1996, 2001; Lin *et al.*, 1996; Praly *et al.*, 1996; Angeli *et al.*, 2000, 2003). Recently, Mancilla-Aguilar and García applied the idea to studying the robustness of switched nonlinear (SNL) systems of the form $\dot{x}(t) = f_i(x(t), u(t))$ ($i \in \mathcal{I}$, where $\mathcal{I}$ is the index set) (Mancilla-Aguilar *et al.*, 2001).

For switched systems, although lots of results have been presented, they mainly focus on the problems of stability, controllability, observability and stabilization control (Hespanha *et al.*, 1999; Liberzon *et al.*, 1999; Sun *et al.*, 2002). For robustness study of such systems, the relevant literature is not rich, and (Mancilla-Aguilar *et al.*, 2001) seems the only one on the ISS of SNL systems, to our knowledge.

In this paper, we will investigate the ISS of general SNL systems (including the case where there is no common Lyapunov function). Unlike the existing results, which mainly focus on establishing ISS converse theorems for nonlinear systems (Sontag *et al.*, 1989, 1995, 1996), by opening out the characteristic of SNL systems, we aim at presenting some sufficient ISS conditions for SNL systems, including for instance, the relation of the ISS and the average dwell-time of the switching law. Precisely, we will investigate SNL time-varying systems, which may involve in infinite subsystems. In this case, switching among different subsystems

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may lead to discontinuity of the system function, and dissatisfies the continuity assumption required by (Sontag *et al.*, 1989, 1995, 1996). Thus, the results in (Sontag *et al.*, 1989, 1995, 1996) cannot be generalized to general SNL systems directly. In some special cases, for instance, where there exists a common ISS–Lyapunov function (CISSLF), a sufficient and necessary ISS condition of SNL systems with arbitrary switching laws is given (Mancilla-Aguilar *et al.*, 2001) under the assumption that $f_i(x, u)$ is uniformly (with respect to i) locally Lipschitz continuous on x, u. Here, by using the methods of multiple Lyapunov function and average dwell-time, some sufficient ISS conditions are given for general SNL systems, which may have no CISSLF. The ISS–Lyapunov functions of the subsystems are allowed to be different from each other rather than simply assuming the existence of a CISSLF. Besides, the uniformity assumption on the local Lipschitz continuity of $f_i(x, u)$ with respect to i is not required.

The remainder of this paper is organized as follows. Section 2 describes the problem to be investigated, and introduces some notations and definitions. In Section 3, by using the method of multiple Lyapunov function, a sufficient ISS condition is given for general SNL systems. In section 4, by using the method of average dwell-time, some sufficient ISS conditions are presented for SNL systems and switched linear (SL) systems, respectively. In section 5, some concluding remarks are given.

2. NOTATIONS AND PROBLEM FORMULATION

Consider the following SNL system

$$\dot{x}(t) = f_{\sigma(t,x(t))}(t, x(t), u(t)), \quad x(t_0) = x_0, \quad (1)$$

where $x(\cdot) \in \mathbb{R}^n$ and $u(\cdot) \in \mathbb{R}^m$ are the system state and input, respectively; and $\sigma(\cdot,\cdot) : [t_0, \infty) \times \mathbb{R}^n \to \mathcal{I}$ ($\mathcal{I}$ is the index set, maybe infinite) is the switching law and is righthand continuous and piecewise constant on t; for every $i \in \mathcal{I}$, function $f_i : [t_0, \infty) \times \mathbb{R}^{n+m} \to \mathbb{R}^n$ is continuous with respect to t, x, u, uniformly locally Lipschitz continuous with respect to x, u, and satisfies $f_i(\cdot, 0, 0) \equiv 0$.

Here, it is different from (Mancilla-Aguilar *et al.*, 2001), $f_i(t, x, u)$ is time-varying, and the uniformity of the local Lipschitz continuity of $f_i(t, x, u)$ is with respect to t rather than i.

Remark 1 By using two arguments t and x in the switching function $\sigma(t, x)$ we would like to emphasize that the switching can depend on both time and events determined by the status of the system state $x(t)$.

Throughout the paper, $\mathbb{R}^+$ denotes the real number set $[0, \infty)$; for a function $\gamma(t) : \mathbb{R}^+ \to \mathbb{R}^+$, $\gamma \in \mathcal{K}$ means that γ is continuous and strictly increasing, and satisfies $\gamma(0) = 0$; $\gamma \in \mathcal{K}_\infty$ means that $\gamma \in \mathcal{K}$ and γ is unbounded; for a function $\beta(t, s) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $\beta \in \mathcal{KL}$ means that for any fixed s, $\beta(t, s) \in \mathcal{K}$, and for any fixed t, $\beta(t, s)$ is continuous and decreases to zero as $s \to \infty$; for two functions $\varphi(\cdot)$ and $\chi(\cdot)$, symbol $\varphi \circ \chi(\cdot)$ denotes the composite function $\varphi(\chi(\cdot))$; ∇ is the gradient operator as usual; $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ and the corresponding induced matrix norm, and for a nonempty subset $\mathcal{M} \subset \mathbb{R}^n$, $|x|_\mathcal{M} \triangleq \inf_{\eta \in \mathcal{M}} |x - \eta|$ (obviously, it holds $|x|_{\{0\}} = |x|$ when $\mathcal{M} = \{0\}$); L_∞^m denotes the set of all the measurable and locally essentially bounded input $u(\cdot) \in \mathbb{R}^m$ on $[t_0, \infty)$ with norm

$$\|u\| = \sup_{t \geq t_0} |u(t)| < \infty. \quad (2)$$

For any given switching law $\sigma(\cdot,\cdot)$, initial condition $x_0 \in \mathbb{R}^n$, $u(\cdot) \in L_\infty^m$, $x(t) \triangleq x_\sigma(t; t_0, x_0, u)$ denotes the state trajectory of System (1) with the maximal existing interval $[t_0, T_\sigma)$, where the constant $T_\sigma \triangleq T_\sigma(t_0, x_0, u) \leq \infty$.

Definition 1 *Consider the following general nonlinear system*

$$\dot{\omega}(t) = g(t, \omega(t), v(t)), \quad \omega(t_0) = \omega_0, \quad (3)$$

where function $g : [t_0, \infty) \times \mathbb{R}^{n+m} \to \mathbb{R}^n$ satisfies $g(\cdot, 0, 0) \equiv 0$. If the trajectory $\omega(t) \triangleq \omega(t; t_0, \omega_0, v)$ of (3) is defined well on $[t_0, \infty)$ for every $\omega_0 \in \mathbb{R}^n$ and $v \in L_\infty^m$, then the system is called forward complete. For a closed set $\mathcal{M} \subset \mathbb{R}^n$, if System (3) is forward complete for every $\omega_0 \in \mathcal{M}$, $v \in L_\infty^m$, and $\omega(t) \in \mathcal{M}$, $\forall t \geq t_0$, then $\mathcal{M}$ is called a closed invariant set of System (3).

Remark 2 By Definition 1, if System (1) is forward complete for any $\sigma(t, x)$, then all of the subsystems are forward complete.

Remark 3 Obviously, if $\mathcal{M}$ is a closed invariant set of all subsystems of System (1), then it is a closed invariant set of system (1), too.

Definition 2 (Sontag, 1989) *For the forward complete system (3) and its closed invariant set $\mathcal{M} \subset \mathbb{R}^n$, the system (3) is called (globally) input-to-state stable (ISS) with respect to $\mathcal{M}$, if there exist two functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $\omega_0 \in \mathbb{R}^n \backslash \mathcal{M}$, $v \in L_\infty^m$ and $t \geq t_0$,*

$$|\omega(t; t_0, \omega_0, v)|_\mathcal{M} \leq \beta(|\omega_0|_\mathcal{M}, t - t_0) + \gamma(\|v\|). \quad (4)$$

Definition 3 (Sontag *et al.*, 1995) *For the forward complete system (3) and its closed invariant set $\mathcal{M} \subset \mathbb{R}^n$, a smooth function $V_g(t, \xi) : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^+$ is called an ISS-Lyapunov function of the system (3) with respect to $\mathcal{M} \subset \mathbb{R}^n$, if there* *exist functions* $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, α, $\chi \in \mathcal{K}$ *such that for all* $\xi \in \mathbb{R}^n \backslash \mathcal{M}$, $\mu \in \mathbb{R}^m$ *and* $t \geq t_0$,

$$\underline{\alpha}(|\xi|_\mathcal{M}) \leq V_g(t,\xi) \leq \overline{\alpha}(|\xi|_\mathcal{M}), \quad (5)$$

$$|\xi|_\mathcal{M} \geq \chi(|\mu|) \Rightarrow \frac{\partial V_g(t,\xi)}{\partial t} + \nabla V_g(t,\xi) \cdot g(t,\xi,\mu) \leq -\alpha(|\xi|_\mathcal{M}). \quad (6)$$

For short, they will be denoted as $(V_g; \underline{\alpha}, \overline{\alpha}, \alpha, \chi)$ *in the sequel.*

3. ISS CONDITIONS BASED ON MULTIPLE LYAPUNOV FUNCTIONS

In this section, by using the multiple Lyapunov function method, sufficient ISS conditions are explored for SNL systems. For simplicity of expression, we denote the switching instants of switching law $\sigma(t,x)$ by $t_1<t_2<\cdots<t_k<\cdots$, and let $\sigma(t_l, x(t_l))=i_l$.

Lemma 1 *For the forward complete system (3) and its closed invariant set* $\mathcal{M} \subset \mathbb{R}^n$, *if the system (3) has an ISS-Lyapunov function* $(V_g; \underline{\alpha}, \overline{\alpha}, \alpha, \chi)$ *such that (5)-(6) hold for all* $\xi \in \mathbb{R}^n \backslash \mathcal{M}$, $\mu \in \mathbb{R}^m$ *and* $t \geq t_0$, *then there exists a* C^1 *function* $\rho \in \mathcal{K}_\infty$ *depending only on* $\overline{\alpha}$ *and* α *such that*

$$\frac{\partial W_g(t,\xi)}{\partial t} + \nabla W_g(t,\xi) \cdot g(t,\xi,\mu) \leq -W_g(t,\xi)$$

for $W_g(t,\xi) \geq \overline{\chi}(|\mu|)$, *where* $W_g(t,\xi) = \rho \circ V_g(t,\xi)$ *and* $\overline{\chi}(\cdot) = \rho \circ \overline{\alpha} \circ \chi(\cdot) \in \mathcal{K}$.

This lemma can be regarded as a corollary of the Remark 2.2 of (Sontag and Wang, 2001).

Lemma 2 *For the forward complete system (1), suppose that* $\mathcal{M} \subset \mathbb{R}^n$ *is a closed invariant set of System (1). If for each* $i \in \mathcal{I}$, *subsystem* $f_i(t,x,u)$ *has an ISS-Lyapunov function* $(V_i; \underline{\alpha}_i, \overline{\alpha}_i, \alpha_i, \chi_i)$ *such that* $\overline{\alpha}(\cdot) \triangleq \sup_{i\in\Lambda} \overline{\alpha}_i(\cdot) \in \mathcal{K}_\infty$ *and*

$$\max\left\{V_{i_{l-1}}(t_l, x(t_l)),\ \overline{\alpha} \circ \chi(\|u\|)\right\} \geq V_{i_l}(t_l, x(t_l)), \quad (7)$$

then there exists a common time-instant $t^*_\sigma \triangleq t^*_\sigma(t_0, x_0, u(t))$ *such that*

$$(t, x(t)) \notin S_{\sigma(t,x(t))}, \forall t \in [t_0, t^*_\sigma), \quad (8)$$

$$(t, x(t)) \in S_{\sigma(t,x(t))}, \forall t \in [t^*_\sigma, \infty), \quad (9)$$

where $S_i=\{(t,\xi)\colon V_i(t,\xi) \leq \overline{\alpha}\circ\chi(\|u\|)\}$, $i\in\mathcal{I}$, *and* l^* *is the largest integer* l *such that* $t_l \leq t^*_\sigma(t_0, x_0, u)$.

Proof By the claim in Lemma 2.14 of (Sontag *et al.*, 1995), for the subsystem $f_{i_l}(t,x,u)$ on the interval $[t_l, t_{l+1})$, there exists

$$t'_{i_l} \triangleq t'_{i_l}(t_l, x(t_l), u) \geq t_l \quad (10)$$

such that $(t, x(t)) \in S_{i_l}$ for all $t \geq t'_{i_l}$, and $(t, x(t)) \notin S_{i_l}$ for all $t < t'_{i_l}$.

If $t_{l+1} < t'_{i_l}$ for $l = 0, 1, 2, \cdots$, then $(t, x(t)) \notin S_{i_l} (l = 0, 1, 2, \cdots)$ for all $t \geq t_0$. In this case, set $t^*_\sigma = \infty$. Otherwise, there exists a nonnegative integer l_0 such that $t'_{i_{l_0}} \leq t_{l_0+1}$. Let

$$l^* = \min_{0\leq l\leq l_0}\{l : t'_{i_l} \leq t_{l+1}\}, \quad t^*_\sigma = t'_{i_{l^*}}.$$

Then we have (8), and $(t, x(t)) \in S_{i_{l^*}}$ for all $t \in [t^*_\sigma, t_{l^*+1})$. Particularly, by the continuity of the state trajectory $x(t)$ and the function $V_i(t,x)$ $(i \in \mathcal{I})$ we have $(t_{l^*+1}, x(t_{l^*+1})) \in S_{i_{l^*}}$. Thus from (7) it follows that

$$V_{i_{l^*+1}}(t_{l^*+1}, x(t_{l^*+1})) \leq \overline{\alpha} \circ \chi(\|u\|).$$

This together with (10) implies that $t'_{i_{l^*+1}} = t_{l^*+1}$. Therefore,

$$(t, x(t)) \in S_{i_{l^*+1}},\ \forall t \in [t_{l^*+1}, t_{l^*+2}).$$

Repeating the above process for $l = l^* + 2, l^* + 3, \cdots$, one can get (9). □

Theorem 1 *Consider the forward complete system (1). Suppose that* $\mathcal{M} \subset \mathbb{R}^n$ *is its closed invariant set, and the switching instants of the switching law* $\sigma(t,x)$ *are* $t_1 < t_2 < \cdots < t_k < \cdots$. *If there exists ISS-Lyapunov function* $(V_i; \underline{\alpha}_i, \overline{\alpha}_i, \alpha_i, \chi_i)$ *of subsystem* $f_i(t,x,u)$, $i \in \mathcal{I}$, *such that*

(i) $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ *and* $\alpha, \chi \in \mathcal{K}$, *where* $\underline{\alpha}(\cdot) \triangleq \inf_{i\in\mathcal{I}} \underline{\alpha}_i(\cdot)$, $\overline{\alpha}(\cdot) \triangleq \sup_{i\in\mathcal{I}} \overline{\alpha}_i(\cdot)$, $\alpha(\cdot) \triangleq \inf_{i\in\mathcal{I}} \alpha_i(\cdot)$ *and* $\chi(\cdot) \triangleq \sup_{i\in\mathcal{I}} \chi_i(\cdot)$;
(ii) *(7) holds at each switching instant* t_l $(l = 0, 1, 2, \cdots)$,

then System (1) is input-to-state stable.

Proof First, by Definition 3 and condition (i), for all $\xi \in \mathbb{R}^n \backslash \mathcal{M}$, $\mu \in \mathbb{R}^m$ and $t \geq t_0$, we have

$$\underline{\alpha}(|\xi|_\mathcal{M}) \leq V_i(t,\xi) \leq \overline{\alpha}(|\xi|_\mathcal{M}),\ \forall i \in \mathcal{I},$$

$$|\xi|_\mathcal{M} \geq \chi(|\mu|) \Rightarrow \frac{\partial V_i(t,\xi)}{\partial t} + DV_i(t,\xi) \cdot f_i(t,\xi,\mu) \leq -\alpha(|\xi|_\mathcal{M}),\ \forall i \in \mathcal{I}; \quad (11)$$

and by (Praly & Wang, 1996), we know that there exists a C^1 function $\rho \in \mathcal{K}_\infty$ depending only on $\overline{\alpha}$ and α such that $\dot{\rho}(r)\alpha\circ\overline{\alpha}^{-1}(r) \geq \rho(r), \forall r \geq 0$. Let $W_i(t,\xi) = \rho \circ V_i(t,\xi)$ and $\overline{\chi}(\|u\|) = \rho \circ \overline{\alpha} \circ \chi(\|u\|)$. Then, by Lemma 1 we have for all $i \in \mathcal{I}$,

$$\rho \circ \underline{\alpha}(|\xi|_\mathcal{M}) \leq W_i(t,\xi) \leq \rho \circ \overline{\alpha}(|\xi|_\mathcal{M}), \quad (12)$$

$$W_i(t,\xi) \geq \overline{\chi}(\|u\|) \Rightarrow \frac{dW_i(t,\xi)}{dt} \leq -W_i(t,\xi). \quad (13)$$

By Lemma 2, there exists t^*_σ such that (8)-(9) hold. Then from (8)- (9) and the definition of $W_i(t,\xi)$ it follows that

$$W_{\sigma(t,x(t))}(t, x(t)) > \overline{\chi}(\|u\|),\ \forall t \in [t_0, t^*_\sigma), \quad (14)$$

$$W_{\sigma(t,x(t))}(t, x(t)) \leq \overline{\chi}(\|u\|),\ \forall t \in [t^*_\sigma, \infty). \quad (15)$$

This together with (13)-(14) gives

$$\frac{dW_{\sigma(t,x(t))}(t,x(t))}{dt} < -W_{\sigma(t,x(t))}(t,x(t)) \quad (16)$$

for $t \in [t_0, t_\sigma^*)$. Hence, for $t \in [t_{l^*}, t_\sigma^*)$ we get

$$W_{i_{l^*}}(t,x(t)) \leq W_{i_{l^*}}(t_{l^*}, x(t_{l^*}))e^{-(t-t_{l^*})}, \quad (17)$$

and for $l = 0, 1, \cdots, l^* - 1$,

$$W_{i_l}(t_{l+1}, x(t_{l+1})) \leq W_{i_l}(x(t_l), t_l)e^{-(t_{l+1}-t_l)}. \quad (18)$$

From (7)-(8), the definition of $W_i(t,\xi)$, (14) and (13) it follows that for $l = 1, 2, \cdots, l^*$,

$$W_{i_l}(t_l, x(t_l)) \leq W_{i_{l-1}}(t_l, x(t_l)). \quad (19)$$

Thus, by (17), (18)-(19) we have

$$W_{\sigma(t,x(t))}(t,x(t)) \leq \cdots$$
$$\leq \max\left\{W_{i_0}(t_0, x(t_0))e^{-(t-t_0)}, \overline{\chi}(\|u\|)\right\}.$$

This together with (12) and (15) leads to

$$\rho\circ\underline{\alpha}(|x(t)|_{\mathcal{M}}) \leq \max\{\rho\circ\overline{\alpha}(|x_0|_{\mathcal{M}})e^{-(t-t_0)}, \overline{\chi}(\|u\|)\}$$

Let $\beta(r,s) = \underline{\alpha}^{-1} \circ \rho^{-1}\big(\rho(\overline{\alpha}(r))e^{-s}\big)$ and $\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(r)$. Then $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$ and

$$|x(t)|_{\mathcal{M}} \leq \beta(|x_0|_{\mathcal{M}}, t-t_0) + \gamma(\|u\|), \ \forall t \geq t_0.$$

Thus, the system (1) is input-to-state stable. □

Remark 4 Condition (ii) of Theorem 1 says that the energy of the system should not be increasing at switching instants. This is because that the ISS is a global property holding for all $t \geq t_0$ with respect to $x(t_0) = x_0$ and $u(t)$, rather than a limit-sup property. Otherwise, for instance, if $\limsup_{t\to\infty} |x(t)|_{\mathcal{M}}$ is considered, then the condition can be relaxed to that: (7) holds after finite switching instants.

Remark 5 From the proof of Theorem 1 we see that $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ are independent of the concrete choice of $\sigma(\cdot,\cdot)$. In other words, the switched nonlinear system (1) is ISS for all $\sigma(\cdot,\cdot)$ satisfying (7).

Corollary 1 *For the forward complete system (1), suppose that $\mathcal{M} \subset \mathbb{R}^n$ is its closed invariant set, and the switching instants of switching law $\sigma(t,x)$ are $t_1 < t_2 < \cdots < t_k < \cdots$. Under the conditions and notations of Theorem 1, if there are positive constants $k_1 < k_2$, k_3, p such that*

$$\underline{\alpha}(r) = k_1 r^p, \ \overline{\alpha}(r) = k_2 r^p, \ \alpha(r) = k_3 r^p, \ \forall r \geq 0,$$

then system (1) is input-to-state stable.

Corollary 2 *Consider the forward complete system (1) and suppose that $\mathcal{M} \subset \mathbb{R}^n$ is its closed invariant set and the index set $\mathcal{I} = \{1, \cdots, N\}$ with $N < \infty$. If there exist ISS-Lyapunov function $(V_i, \underline{\alpha}_i, \overline{\alpha}_i, \alpha_i, \chi_i)$, $i \in \mathcal{I}$, such that*

$$\max\big\{V_{i_{l-1}}(t_l, x(t_l)), \ \overline{\alpha}_1 \circ \chi(\|u\|), \cdots,$$
$$\overline{\alpha}_N \circ \chi(\|u\|)\big\} \geq V_{i_l}(t_l, x(t_l)), \ l = 0, 1, \cdots$$

then System (1) is input-to-state stable.

Proof Let

$$\underline{\alpha}(\cdot) = \min_{1\leq i\leq N} \underline{\alpha}_i(\cdot), \ \overline{\alpha}(\cdot) = \max_{1\leq i\leq N} \overline{\alpha}_i(\cdot),$$
$$\alpha(\cdot) = \min_{1\leq i\leq N} \alpha_i(\cdot), \ \chi(\cdot) = \max_{1\leq i\leq N} \chi_i(\cdot).$$

Then by the definition of ISS-Lyapunov function we have $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, and α, $\chi \in \mathcal{K}$. This means Conditions (i)-(ii) of Theorem 1 hold. Thus, system (1) is input-to-state stable. □

4. ISS CONDITIONS BASED ON THE AVERAGE DWELL-TIME

In this section, we will use the concept of average dwell-time to get some sufficient ISS conditions for both SNL systems and switched linear (SL) systems.

Definition 4 (Hespanha *et al.*, 1999) *For any given constants $\tau^* > 0$ and N_0, let $N_\sigma(s,t)$ denote the switch number of $\sigma(t,x)$ in $[s,t)$, $\forall t > s \geq t_0$, and let*

$$\mathcal{S}[\tau^*, N_0] = \Big\{\sigma(\cdot,\cdot) : N_\sigma(s,t) \leq N_0 + \frac{t-s}{\tau^*},$$
$$\forall t > t_0, \forall s \in [t_0, t)\Big\}.$$

Then τ^* is called the average dwell-time of $\mathcal{S}[\tau^*, N_0]$, and $\tau_\sigma \triangleq \sup_{t\geq t_0}\sup_{t>s\geq t_0} \frac{t-s}{N_\sigma(s,t)-N_0}$ is called the average dwell-time of $\sigma(\cdot,\cdot)$.

4.1 ISS analysis of SNL systems

Theorem 2 *For the forward complete system (1), suppose that $\mathcal{M} \subset \mathbb{R}^n$ is its closed invariant set, and switching instants of switching law $\sigma(t,x)$ are $t_1 < t_2 < \cdots$. If there are ISS-Lyapunov function $(V_i; \underline{\alpha}_i, \overline{\alpha}_i, cV_i, \chi_i)$ of subsystem $f_i(t,x,u)(i \in \mathcal{I})$ and constants $c > 0$, $\eta_0 \geq 1$, such that for all $\xi \in \mathbb{R}^n\backslash\mathcal{M}$, $\mu \in \mathbb{R}^m$ and $t \geq t_0$,*

$$\underline{\alpha}(|\xi|_{\mathcal{M}}) \leq V_i(t,\xi) \leq \overline{\alpha}(|\xi|_{\mathcal{M}}); \quad (20)$$

$$|\xi|_{\mathcal{M}} \geq \chi(|\mu|) \Rightarrow \frac{\partial V_i(t,\xi)}{\partial t} + \nabla V_i(t,\xi)\cdot f_i(t,\xi,\mu)$$
$$\leq -c\ V_i(t,\xi), \quad (21)$$

and for $l = 0, 1, 2, \cdots$,

$$\max\Big\{\eta_0 V_{\sigma(t_{l-1}, x(t_{l-1}))}(t_l, x(t_l)), \ \overline{\alpha} \circ \chi(\|u\|)\Big\}$$
$$\geq V_{\sigma(t_l, x(t_l))}(t_l, x(t_l)), \quad (22)$$

then System (1) is input-to-state stable for all $\sigma \in \mathcal{S}_0\left[\frac{\ln \eta_0}{c}, N_0\right] \triangleq \Big\{\sigma(\cdot,\cdot) \in \mathcal{S}[\tau^*, N_0] : \tau^* > \frac{\ln \eta_0}{c}\Big\}$.

The proof is omitted.

Remark 6 For $\eta_0 \geq 1$, the condition (22) in Theorem 2 is obviously weaker than the condition (7) in Theorem 1. This includes the case where the energy of the subsystem after a switching instant is greater than that of the subsystem before the switching instant.

4.2 ISS analysis of SL systems

In this subsection, we will investigate SL systems of the form

$$\begin{cases} \dot{x}(t) = A_{\sigma(t,x(t))}x(t) + B_{\sigma(t,x(t))}u(t), \\ x(t_0) = x_0, \end{cases} \tag{23}$$

where $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m}$ are constant matrices for each $i \in \Lambda$, respectively.

In the sequel, for any $n \times n$ matrix A, J_A denotes the Jordanian normal form of A, $\eta(A)$ is the largest real part of the eigenvalues of A, and $\triangle_M(A)$ and $\triangle_m(A)$ are the largest and smallest singular values of matrix A, respectively.

For a given matrix set $\mathcal{A} \subset \mathbb{R}^{n\times n}$, $\mathcal{A}_1$ denotes the set of all stable matrices of $\mathcal{A}$, $\mathcal{A}_2$ denotes $\mathcal{A} \setminus \mathcal{A}_1$. Noticing that $\eta(A)$ depends continuously upon the parameter of A, when $\mathcal{A}_1$ is compact, we have $\max_{A\in\mathcal{A}_1} \eta(A) < 0$ and

$$\Big|\max_{A\in\mathcal{A}_1} \eta(A)\Big| = \min_{A\in\mathcal{A}_1} \Big|\eta(A)\Big|.$$

In particular, when the number of all stable matrices in $\mathcal{A}$ is finite and greater than zero, we have $\max_{A\in\mathcal{A}_1} \eta(A) < 0$.

Proposition 1 *For any given matrix set $\mathcal{A} \subset \mathbb{R}^{n\times n}$, if $\mathcal{A}$ and $\mathcal{A}_1$ are compact, and $\mathcal{A}_1$ is nonempty, then for any $\varepsilon \in (0, \min_{A\in\mathcal{A}_1} |\eta(A)|)$, there exists a constant $M(\varepsilon) > 0$ such that*

$$|e^{At}| \leq \lambda_\varepsilon(A)e^{a_\varepsilon(A)t}, \quad \forall A \in \mathcal{A},\ \forall t \geq 0, \tag{24}$$

$$\lambda_\varepsilon(\mathcal{A}) \triangleq \max_{A\in\mathcal{A}} \lambda_\varepsilon(A) < \infty \tag{25}$$

where $\lambda_\varepsilon(A) \triangleq \sqrt{M(\varepsilon)} \frac{\triangle_M(P(A))}{\triangle_m(P(A))} > 0$, $a_\varepsilon(A) = \eta(A) + \varepsilon$, $P(A)$ is $n \times n$ nonsingular matrix satisfying

$$A = P(A)J_AP(A)^{-1}.$$

The proof is omitted.

Now, we study the ISS property of the SL system (23).

For system (23), let $\mathcal{A} = \{A_i, i \in \mathcal{I}\} \subset \mathbb{R}^{n\times n}$ and $\mathcal{B} = \{B_i, i \in \mathcal{I}\} \subset \mathbb{R}^{n\times m}$, assume that $\mathcal{A}$ and $\mathcal{B}$ are compact, and the subset $\mathcal{A}_1$ consisting of all the stable matrices of $\mathcal{A}$ is nonempty and compact. For any given $\varepsilon \in (0, \min_{A\in\mathcal{A}_1} |\eta(A)|)$, define $a_\varepsilon(A)$ and $\lambda_\varepsilon(A)$ as in Proposition 1, and set

$$\begin{cases} a_\varepsilon^-(\mathcal{A}) = \min_{A\in\mathcal{A}_1} |a_\varepsilon(A)|, \\ a_\varepsilon^+(\mathcal{A}) = \max\big\{0,\ \max_{A\in\mathcal{A}} a_\varepsilon(A)\big\}, \end{cases} \tag{26}$$

$$b_0(\mathcal{B}) = \max_{B_i\in\mathcal{B}} |B_i|, \quad M_\varepsilon = e^{(1+N_0)\ln\lambda_\varepsilon(\mathcal{A})}. \tag{27}$$

For a given switching law $\sigma(\cdot,\cdot)$ and a time interval $[s,t)$, let $T_\sigma^+(s,t)$ and $T_\sigma^-(s,t)$ be the total time of system (23) running on stable subsystems and unstable subsystems in $[s,t)$, respectively; and for any $a^* \in (0, a_\varepsilon^-]$ and $\tau^* > 0$, define

$$\begin{aligned} &\mathcal{S}[a^*, \tau^*; \mathcal{A}] \\ &= \Big\{\sigma :\ \sigma \in \mathcal{S}[\tau^*, N_0],\ \tau_\sigma > \tau^* \quad \text{and} \\ &\qquad \sup_{t\geq u\geq t_0} \frac{T_\sigma^+(s,t)}{T_\sigma^-(s,t)} \leq \frac{a_\varepsilon^-(\mathcal{A}) - a^*}{a_\varepsilon^+(\mathcal{A}) + a^*}\Big\}, \end{aligned}$$

where the average dwell-time τ_σ of $\sigma(\cdot,\cdot)$ is given by Definition 4.

In the sequel, for simplicity of expression, we will drop the arguments of $\lambda_\varepsilon(\mathcal{A})$, $a_\varepsilon^-(\mathcal{A})$, $a_\varepsilon^+(\mathcal{A})$ and $b_0(\mathcal{B})$, and denote them by $\lambda_\varepsilon, a_\varepsilon^-, a_\varepsilon^+$ and b_0, respectively.

Theorem 3 *For SL system (23), assume that $\mathcal{A}$ and $\mathcal{B}$ are compact, and the subset $\mathcal{A}_1$ consisting of all the stable matrices of $\mathcal{A}$ is nonempty and compact. Then for any given $\varepsilon \in (0, \min_{A\in\mathcal{A}_1} |\eta(A)|)$ and $a^* \in (0, a_\varepsilon^-]$, there exists $\tau^* \geq \frac{1}{a^*}\ln\lambda_\varepsilon$ such that*

(i) for all $\sigma(\cdot,\cdot) \in \mathcal{S}[a^*, \tau^*; \mathcal{A}]$, System (23) is forward complete, and

(ii) System (23) is ISS if and only if the control-free system $\dot{x}(t) = A_{\sigma(t,x(t))}x(t)$ is asymptotically stable.

Proof Part (i) and the necessity of Part (ii) are obvious. So, below we need only to show the sufficiency.

For any given $t \geq t_0$, assume that in $[t_0, t)$, System (23) has switching instants $t_1 < t_2 < \cdots < t_j$. Let $\sigma(t_l, x(t_l)) = i_l$ $(l = 0, 1, \cdots, j)$. Then the solution of system (23) can be expressed as

$$x_\sigma(t) = \Phi(t,t_0)x_0 + \int_{t_0}^t \Phi(t,s)B_{\sigma(s,x(s))}u(s)ds, \tag{28}$$

where for $s \in [t_l, t_{l+1})$,

$$\Phi(t,s) = e^{A_{i_j}(t-t_j)}e^{A_{i_{j-1}}(t_j-t_{j-1})}\cdots e^{A_{i_l}(t_{l+1}-s)}.$$

We first show that for any given constant $a^* \in (0, a_\varepsilon^-]$, $\tau^* \geq \frac{\ln\lambda_\varepsilon}{a^*}$, and $\sigma(\cdot,\cdot) \in \mathcal{S}[a^*, \tau^*; \mathcal{A}]$, there are $a > 0$ and $M > 0$ such that

$$|\Phi(t,s)| \leq Me^{-a(t-s)}, \quad \forall t \geq s \geq t_0. \tag{29}$$

Noticing that

$$N_\sigma(s,t) = \begin{cases} j-l,\ t_l \leq s < t_{l+1},\ l = 0, \cdots, j-1; \\ 0, \qquad t_j \leq s < t, \end{cases}$$

we have $\lambda_\varepsilon = e^{[1+N_\sigma(s,t)]\ln\lambda_\varepsilon}$ for $s \in [t_j, t)$; and $\lambda_\varepsilon^{j-l+1} = e^{[1+N_\sigma(s,t)]\ln\lambda_\varepsilon}$ for $s \in [t_l, t_{l+1})$

$(l = 0, 1, \cdots, j-1)$. In particular, $N_\sigma(t_0, t) = j$ and $\lambda_\varepsilon^{j+1} = e^{[1+N_\sigma(t_0,t)]\ln\lambda_\varepsilon}$. Thus,

$$|\Phi(t,s)| \le \lambda_\varepsilon^{j-l+1} e^{a_\varepsilon(A_{i_j})(t-t_j)} \cdots e^{a_\varepsilon(A_{i_l})(t_{l+1}-s)}$$
$$\le e^{[1+N_\sigma(s,t)]\ln\lambda_\varepsilon} e^{a_\varepsilon^+ T_\sigma^+(s,t) - a_\varepsilon^- T_\sigma^-(s,t)}.$$

By the definition of $\mathcal{S}[a^*, \tau^*; \mathcal{A}]$, for any given $\sigma \in \mathcal{S}[a^*, \tau^*; \mathcal{A}]$, we have $a_\varepsilon^+ T_\sigma^+(s,t) - a_\varepsilon^- T_\sigma^-(s,t) \le -a^*(t-s)$. This together with $N_\sigma(s,t) \le N_0 + \frac{t-s}{\tau_\sigma}$ and $\tau_\sigma > \tau^* \ge \frac{\ln\lambda_\varepsilon}{a^*}$ implies that $a \triangleq a^* - \frac{\ln\lambda_\varepsilon}{\tau^*} > 0$. Then, by some straightforward calculations we have

$$|\Phi(t,s)| \le M e^{-a(t-s)}, \quad \forall s \in [t_0, t),$$

where $M = e^{(1+N_0)\ln\lambda_\varepsilon}$. Thus, (29) is true, which together with (28) gives

$$|x(t)| \le M e^{-a(t-t_0)}|x_0| + \frac{Mb_0}{a}\|u\|.$$

Let $\beta(r,s) = Me^{-as}r$ and $\gamma(r) = \frac{Mb_0}{a}r$. Then $\beta(\cdot,\cdot) \in \mathcal{KL}$, $\gamma(\cdot) \in \mathcal{K}_\infty$, and

$$|x(t)| \le \beta(|x_0|, t-t_0) + \gamma(\|u\|),$$

i.e. system (23) is input-to-state stable. □

Remark 7 Comparing Theorem 3 with Theorem 2, one can see that for SL system case, some of the subsystems of system (23) are allowed to be unstable. But for SNL systems, all of its subsystems are required to be stable, since the degree of instability of nonlinear systems is hard to be characterized.

Remark 8 By Theorem 3, the ISS of SL system (23) is independent of the concrete choice of $\sigma(\cdot,\cdot)$ in $\mathcal{S}[a^*, \tau^*; \mathcal{A}]$.

5. CONCLUSIONS

In this paper, the ISS of SNL system and SL system are investigated, respectively. The main results can roughly be divided into two classes. One is based on multiple Lyapunov function method, and the other is based on (average) dwell-time method. Firstly, by using the method of multiple Lyapunov function, a sufficient ISS condition is given for general SNL systems based on a quantitative relation of the control and the values of the Lyapunov functions of the subsystems before and after the switching instants. Here, the ISS–Lyapunov functions of the subsystems are allowed to be different from each other rather than simply assuming the existence of a CISSLF. So, the condition is sufficient not only for the switched systems possessing a CISSLF, but also sufficient for the switched systems without any CISSLF. Secondly, by employing the method of the average dwell-time, some ISS sufficient conditions are given for SNL systems and SL systems, respectively. Among others, the condition on SNL systems is characterized by the size of the dwell-time, and that on switched linear systems is characterized by the average dwell-time and the ratio of the total time that the system runs on unstable subsystems to the total time that the system runs on stable subsystems.

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