ROBUST CAPACITY FOR ADDITIVE COLORED GAUSSIAN UNCERTAIN CHANNELS

Charalambos D. Charalambous *,[1]
Stojan Z. Denic ** **Seddik M. Djouadi** ***

* *Department of Electrical and Computer Engineering, University of Cyprus, 75 Kallipoleos Avenue, P.O. Box 20537, Nicosia, 1678, Cyprus, E-mail: chadcha@ucy.ac.cy*
** *School of Information Technology and Engineering, University of Ottawa, Ottawa, 800 King Edward, Ontario K1N 6N5, Canada, E-mail: sdenic@site.uottawa.ca*
*** *Electrical and Computer Engineering Department, University of Tennessee, 414 Ferris Hall, Knoxville, TN 37996-210,USA, E-mail: djouadi@ece.utk.edu*

Abstract: This paper is concerned with the definition and computation of channel capacity of continuous time additive Gaussian channels, when the channel is subject to uncertainty, the noise power spectral density is known and the input signal is wide-sense stationary and constrained in power. The uncertainty description of the channel transfer function is described by the set of all channels which belong to a ball in a normed linear space, known as H^∞ space. Two uncertainty models are used that are borrowed from the control theory, additive, and multiplicative. The channel capacity, that we call robust capacity, is then defined as a maxi-min of mutual information rate in which the minimization is over the uncertainty set while the maximization is over all transmitted signals having finite power. An exact formulae for the robust capacity is derived. Part of the results include a modified version of the water-filling equation, describing how the optimal transmitter power depends on the channel uncertainty. The conditions are introduced under which the robust capacity is equivalent to operational capacity. Finally, an example is worked out to show the effect of uncertainty in case of the second order system.

Keywords: Communication channels, Robust transmission, Uncertainty.

1. INTRODUCTION

The definition of the channel capacity as introduced by Shannon (1948) is subject to the assumption that the communication channel, source signal, and disturbances are perfectly known to the transmitter and receiver. This is an underlying assumption that is present in the construction of most encoding/decoding schemes. The question that can be asked, is what kind of performance could one expect if the blocks of communication system are not perfectly known to the transmitter and/or receiver? Thus, if the true models are

[1] This work was supported by the European Commission under the project ICCCSYSTEMS and the NSERC under an operating grant T-810-289-01.

different from the models assumed in the capacity computation, then the computed capacity could lead to overly optimistic performance estimates of the maximum rate for reliable transmission.

Although, there exists a number of papers that consider the problem of communication under uncertainties (for a comprehensive review see (Biglieri et. al. , 1998), (Lapidoth , 1998), (Medard , 2003)), this topic did not get sufficient attention in the communication community. While, the majority of papers that consider communication under uncertainty gives the priority to the probabilistic description, we model the uncertainty of the communication channel through uncertainty of its frequency response. A paper that considers the problem similar to ours is (Baker, and Chao, 1996), which treats the continuous-time Gaussian channels where the noise is unknown, and where the transmitted signal, and noise energy constraints are introduced through their covariance functions. Our approach enables also consideration of noise uncertainty, but through power constraints on the power spectral densities of the transmitted signal, and noise that lead to a completely different solution technique. Most of the other papers discuss communication under channel uncertainties for random variables (McEliece, 1983), and vector random variables (Hughes, and Narayan, 1988). Usually, the channel uncertainty is defined by introducing additive disturbance with unknown statistics. Those channels are commonly called arbitrary varying channels (AVC). The special case are so-called Gaussian AVC (GAVC) (Csiszar, and Narayan, 1991). Our approach is different because it defines the uncertainty through the channel frequency response in a normed linear space, which is the right mathematical framework to quantify uncertainty as shown in robust control theory. On the other hand our work rely on the results of Gallager (1968), and Root, and Varaiya (1968). Namely, Gallager (1968) computed the channel capacity of the additive Gaussian channel when the frequency response of the channel, and power spectral density of the noise are known, while Root, and Varaiya (1968) proved the channel coding theorem for the class of unknown white Gaussian noise channels. Our goal is to unify these two approaches, to give the explicit formula for the channel capacity of uncertain colored additive Gaussian noise channels, to derive the new water filling formula (that has not been done in (Root, and Varaiya, 1968), and whose importance was pointed out in (Medard , 2003)) and to show that the channel coding theorem, and its converse found in (Root, and Varaiya, 1968) under certain conditions still can be applied in colored Gaussian noise case. Also, we want to emphasize the importance of uncertainty modeling in the frequency domain that is very practical, because for most communication systems, the designers have some ideas about the form of the frequency response which is based on previous experience or physical characteristics of the communication medium.

In this paper the case of continuous time colored Gaussian noise communication channel when the true frequency response of the channel is not perfectly known is put in appropriate mathematical setting that enables the derivation of explicit formula for channel capacity. Namely, two basic models for the description of the channel uncertainty in frequency domain are borrowed from control theory, and employed in the channel capacity computation. One is the additive, and the other is the multiplicative uncertainty model (Doyle, and Tannenbaum, 1992). The choice of uncertainty model depends on the communication channel at hand. Although, the capacity formulas are derived for two specific types of uncertainties, similar derivations may be used for other types of uncertainties found in (Doyle, and Tannenbaum, 1992). Thus, the uncertainty is modeled in the following way. It is assumed that so-called nominal frequency response is known. The nominal frequency response is based on the previous experience or belief, while the deviation from the nominal frequency response represents uncertainty, and it is described by a ball in the frequency domain that belongs to H^∞ (the space of bounded, and analytic transfer functions in the open right-half plane). The channel capacity, called robust capacity, is defined as a max-min of mutual information rate, where the maximum is over all power spectral densities of transmitted signal with constrained power, and minimum is over all frequency responses from the uncertainty set. The formula for mutual information rate follows from the fact that the distribution of transmitted signal that maximizes mutual information for additive Gaussian channels is also Gaussian. The explicit formula for robust capacity is obtained accompanied with a modified water-filling formula. The formula for robust capacity shows how the capacity decreases when the size of uncertainty set increases. The water-filling formula contains a factor that describes how the uncertainty affects the optimal transmitted power. It is concluded that under certain conditions, the robust capacity is equal to the operational capacity, i.e., the maximal attainable rate over the set of communication channels. In other words, the channel coding theorem, and its converse hold for so the robust capacity.

The rest of the paper is organized in the following way. In the Section 2, the model of the communication system is given accompanied by additive and multiplicative uncertainty description. In the Section 3, robust capacity formulas are derived for the case of both uncertainty description. Section 4 gives the conditions under which robust capacity is equal to the operational capacity for the colored Gaussian channels. Section 5 gives an example to illustrate the derived formulas for the capacity, and power spectral density of the transmitted signal when the channel can be modeled as a second order system with uncertain damping ratio.

2. COMMUNICATION SYSTEM MODEL

The communication system is modeled by $y(t) = x(t) * \tilde{h}(t) + n(t)$, where $*$ represents convolution, and $\tilde{h}(t)$ is a channel impulse response with the frequency response $\tilde{H}(f)$. The assumptions are the following. The transmitted signal $\mathbf{x} = \{x(t); -\infty < t < +\infty\}$, noise signal $\mathbf{n} = \{n(t); -\infty < t < +\infty\}$, and received signal $\mathbf{y} = \{y(t); -\infty < t < +\infty\}$ are wide-sense stationary processes with power spectral densities $S_{\mathbf{x}}(f)$, $S_{\mathbf{n}}(f)$, $S_{\mathbf{y}}(f)$, respectively. In addition, the transmitted signal $\mathbf{x}$ is constrained in power, and the noise $\mathbf{n}$ is Gaussian noise whose power spectral density is known. The frequency response $\tilde{H}(f)$ of the channel is unknown. The uncertainty of $\tilde{H}(f)$ will be described by a set in the H^{∞} space, which when endowed with $\|.\|_{\infty}$ norm defined by $\|\tilde{H}(f)\|_{\infty} := \sup_f |H(f)|$, is a Banach space (see Fig. 1, and Fig. 2).

2.1 Additive Uncertainty

The additive uncertainty model of the frequency response $\tilde{H}(f) = H_{nom}(f) + \Delta(f)W_1(f)$ is the sum of two terms. One is the so-called nominal frequency response $H_{nom}(f)$ that represents the known part of $\tilde{H}(f)$, and the other represents perturbation $\Delta(f)W_1(f)$. The nominal frequency response $H_{nom}(f)$ can be obtained, for instance, by measurement or by applying physical laws that govern the channel behavior. The transfer functions $H_{nom}(f)$, $W_1(f)$, and $\Delta(f)$ belong to the H^{∞} space. $W_1(f)$ is a known stable transfer function, $\Delta(f)$ is a variable stable transfer function with $\|\Delta(f)\|_{\infty} \le 1$, and $\tilde{H}(f)$, and $H(f)$ have the same number of unstable poles. From $\tilde{H}(f) = H_{nom}(f) + \Delta(f)W_1(f)$ it follows that $|\tilde{H}(f) - H_{nom}(f)| \le |W_1(f)|$, i.e., the uncertainty set is the set of all $\tilde{H}(f)$ that belong to the ball centered at $H_{nom}(f)$ with radius determined by the magnitude of a fixed known $W_1(f)$. Thus, the size of uncertainty set depends on the frequency, and it is determined by $|W_1(f)|$. The smaller the magnitude $|W_1(f)|$, the smaller the uncertainty set.

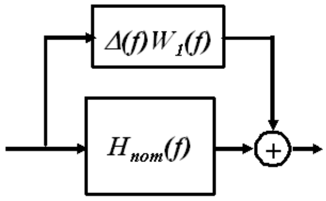

Fig. 1. Additive uncertainty description

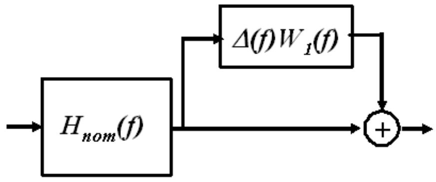

Fig. 2. Multiplicative uncertainty description

2.2 Multiplicative Uncertainty

The multiplicative uncertainty model is described by equation $\tilde{H}(f) = H_{nom}(f)(1 + \Delta(f)W_1(f))$, where the transfer functions satisfy all conditions as for the additive uncertainty model. The equation implies that $|\frac{\tilde{H}(f)}{H_{nom}(f)} - 1| \le |W_1(f)|$ meaning that the ratio $\frac{\tilde{H}(f)}{H_{nom}(f)}$ at each frequency belongs to the ball centered at 1, with radius $|W_1(f)|$. The size of uncertainty set is again determined by $|W_1(f)|$.

3. ROBUST CAPACITY

3.1 Additive Uncertainty

Define the two following sets

$$A_1 := \Big\{ S_{\mathbf{x}}(f)\,; \int S_{\mathbf{x}}(f)df \le P \Big\}$$
$$A_2 := \Big\{ \tilde{H}(f) \in H^{\infty}; \tilde{H} = H_{nom} + \Delta W_1; \; H_{nom} \in H^{\infty}, W_1 \in H^{\infty}, \Delta \in H^{\infty}, \; \|\Delta\|_{\infty} \le 1 \Big\}.$$

The set A_1 is the set of all possible power spectral densities of transmitted signal, and A_2 is the uncertainty set. The size of uncertainty set will be determined by using the fact that $\|\Delta\|_{\infty} \le 1$.

Definition 3.1. The robust capacity of a continuous time Gaussian channel, when a transmitted signal $\mathbf{x}$ is subject to the power constraint $\int S_{\mathbf{x}}(f)df \le P$, where $S_{\mathbf{n}}(f)$ is the power spectral density of the noise, and with channel uncertainty defined through set A_2, is defined by

$$C_R := \sup_{S_{\mathbf{x}} \in A_1} \inf_{\tilde{H} \in A_2} \frac{1}{2} \int \log\left(1 + \frac{S_{\mathbf{x}}|\tilde{H}|^2}{S_{\mathbf{n}}}\right) df. \quad (1)$$

The region of integration in (1) will be clear from Theorem 3.2. This definition is a version of the well-known formula for continuous time Gaussian channels that was first introduced by Shannon (1949) for strictly band-limited white noise channels, and afterward re-derived by Gallager (1968) for colored noise, and not necessarily strictly band-limited channels. In (1), the channel capacity is defined as the worst case value of mutual information rate. Although this may seem as a conservative approach, it turns out to be the maximum theoretical transmission rate, i.e., the operational capacity for uncertain channels which is proven in (Root, and Varaiya, 1968).

Theorem 3.2. Consider an additive uncertainty description of $\tilde{H}(f)$, and supposed that

$$\frac{(|H_{nom}(f)| + |W_1(f)|)^2}{S_{\mathbf{n}}(f)}$$

is bounded, integrable, and that $|H_{nom}(f)| \geq |W_1(f)|$. Then the following holds.

(1) The robust capacity of a continuous time Gaussian channel with additive channel uncertainty shown in Fig. 1 is given parametrically by

$$C_R = \frac{1}{2}\int \log\left(\frac{\nu^*(|H_{nom}| - |W_1|)^2}{S_{\mathbf{n}}}\right) df, \quad (2)$$

where ν^* is a Lagrange multiplier found via

$$\int \left(\nu^* - \frac{S_{\mathbf{n}}}{(|H_{nom}| - |W_1|)^2}\right) df = P, \quad (3)$$

subject to the condition

$$\nu^*(|H_{nom}| - |W_1|)^2 - S_{\mathbf{n}} > 0, \nu^* > 0, \quad (4)$$

in which the integrations in (2), and (3) are over the frequency interval over which the condition (4) holds.

(2) The infimum over the channel uncertainty in (1) is achieved for $\Delta^*(f)$ equals

$$\exp[-j\arg(W_1) + j\arg(H_{nom}) + j\pi],$$

when $\|\Delta^*(f)\|_\infty = 1$. The resulting mutual information rate after minimization is given by

$$\inf \int \log\left(1 + \frac{S_{\mathbf{x}}|H_{nom} + \Delta W_1|^2}{S_{\mathbf{n}}}\right) df = \int \log\left(1 + \frac{S_{\mathbf{x}}(|H_{nom}| - |W_1|)^2}{S_{\mathbf{n}}}\right) df,$$

where the infimum is over $\|\Delta\|_\infty \leq 1$. The supremum of previous equation over A_1 yields the water-filling equation

$$S_{\mathbf{x}}^* + \frac{S_{\mathbf{n}}}{(|H_{nom}| - |W_1|)^2} = \nu^*. \quad (5)$$

Proof. The proof is omitted due to the space constraint, but is available from the authors upon request.

The main conclusions that can be drawn from the previous theorem are obtained from (2), (5). The first equation shows how the robust capacity depends on the channel uncertainty, while the second illustrates the effect of channel uncertainty on the optimal transmitted power. It can be conjectured that robust capacity decreases with the magnitude of transfer function $W_1(f)$, that is, it decreases as the size of uncertainty set increases. If the channel is perfectly known then $|W_1(f)| = 0$ giving a known formula derived in (Gallager, 1968). This is an intuitive result because the channel capacity should be determined by the worst case channel. This follows from the definition of the channel capacity which determines a single code that should be good for each channel from uncertainty set. But, the channel capacity also depends on the water-filling level ν^*, which increases when uncertainty increases for constant power P. To get better insight into the dependence of robust capacity upon the uncertainty, an example is considered in the next section. The example will also show the impact of uncertainty on the optimal transmitted power through a water-filling formula (5).

Here, we make some comments on the relations between $|H_{nom}|$, and $|W_1(f)|$. It is reasonable to assume that in practical cases, $|H_{nom}| \geq |W_1(f)|$, because the uncertainty could represent the errors in channel estimation. Thus, the second term in logarithm could go to zero, implying zero capacity, just if the channel estimation is very poor.

3.2 Multiplicative Uncertainty

Define the following set

$$A_3 := \Big\{\tilde{H}(f) \in H^\infty; \tilde{H} = H_{nom}(1 + \Delta W_1); H_{nom} \in H^\infty, W_1 \in H^\infty, \Delta \in H^\infty, \|\Delta\|_\infty \leq 1\Big\}.$$

The definition 3.1 still applies in this case with the difference that the infimum is taken over all frequency responses that belong to A_3, instead. Therefore the following theorem holds.

Theorem 3.3. Consider an multiplicative uncertainty description of $\tilde{H}(f)$, and supposed that $\frac{[|H_{nom}(f)|(1+|W_1(f)|)]^2}{S_{\mathbf{n}}(f)}$ is bounded, integrable, and that $|W_1(f)| \leq 1$. Then the following hold.

(1) The robust capacity of a continuous time Gaussian channel with multiplicative channel

uncertainty shown in Fig. 2 is given parametrically by

$$\frac{1}{2}\int \log\left(\frac{\nu^*[|H_{nom}|(1-|W_1|)]^2}{S_{\mathbf{n}}}\right)df, \quad (6)$$

where ν^* is a Lagrange multiplier found via

$$\int\left(\nu^* - \frac{S_{\mathbf{n}}}{[|H_{nom}|(1-|W_1|)]^2}\right)df = P, \quad (7)$$

subject to the condition

$$\nu^*[|H_{nom}|(1-|W_1|)]^2 - S_{\mathbf{n}} > 0, \nu^* > 0, \quad (8)$$

in which the integrations in (6), and (7) are over the frequency interval over which the condition (8) holds.

(2) The infimum over the channel uncertainty in (1) is achieved at

$$\Delta^* = \exp[-j\arg(W_1) + j\pi]$$
$$\|\Delta^*\|_\infty = 1,$$

and the resulting mutual information rate after minimization is given by

$$\inf\int \log\left(1 + \frac{S_{\mathbf{x}}|H_{nom}(1+\Delta W_1|^2}{S_{\mathbf{n}}}\right)df$$
$$= \int \log\left(1 + \frac{S_{\mathbf{x}}[|H_{nom}|(1-|W_1|)]^2}{S_{\mathbf{n}}}\right)df,$$

where the infimum is over $\|\Delta\|_\infty \leq 1$. The supremum of previous equation over A_1 yields the water-filling equation

$$S_{\mathbf{x}}^* + \frac{S_{\mathbf{n}}}{[|H_{nom}|(1-|W_1|)]^2} = \nu^*. \quad (9)$$

Proof. The proof is omitted due to the space constraint, but is available from the authors upon request.

Again, similar conclusions can be drawn. It can be seen that the robust capacity is determined by the size of uncertainty set $|W_1(f)|$. Thus, if the perturbation is reasonable enough ($|W_1(f)| \leq 1$), robust capacity decreases when the size of uncertainty set increases. When the channel is perfectly known ($|W_1(f)| = 0$) the robust capacity is equal to the classical case. It should be noted that preceding examples imply that the robust capacity formula can be applied on other types of uncertainties, such as $\tilde{H} = H_{nom}/(1 + \Delta W_1 H_{nom})$.

4. CHANNEL CODING AND CONVERSE TO CHANNEL CODING THEOREM

In this section, it is shown that under certain conditions the coding theorem, and its converse hold for the set of communication channels with uncertainties defined by sets A_2, and A_3. It means that there exists a code, whose code rate R is less than the robust capacity C_R given by (2), or (6) for which the error probability is arbitrary small. This result is obtained in (Denic et. al., preprint), by combining two approaches found in (Gallager, 1968), and (Root, and Varaiya, 1968).

First define the frequency response of the equivalent communication channel by

$$G(f) = \left(\frac{S_{\mathbf{x}}(f)|\tilde{H}(f)|^2}{S_{\mathbf{n}}(f)}\right)^{1/2},$$

and denote its inverse Fourier transform by $g(t)$. Further define two sets, $\mathcal{A} := \{G(f); \tilde{H}(f) \in A_2 \text{ or } A_3\}$, and $\mathcal{B} := \{g(t); G(f) \in \mathcal{A}, g(t) \text{ satisfies } 1), 2), 3)\}$, where

(1) $g(t)$ has finite duration δ,
(2) $g(t)$ is square integrable ($g(t) \in L_2$),
(3) $\int_{-\infty}^{-\alpha} |G(f)|^2 df + \int_{\alpha}^{+\infty} |G(f)|^2 df \to 0$ when $\alpha \to +\infty$.

The set of all $g(t)$ that satisfy these conditions is conditionally compact set in L_2 (see (Root, and Varaiya, 1968)), and this enables the proof of coding theorem, and its converse. Note that the condition 1) can be relaxed (see Lemma 4 in (Forys, and Varaiya, 1969)).

Theorem 4.1. The operational capacity C (supremum of all attainable rates (Root, and Varaiya, 1968)) for the set of communication channels $\mathcal{B}$ is given by (2), or (6) and is equal to robust capacity C_R.

Proof. The proof follows from (Gallager, 1968), and (Root, and Varaiya, 1968), and details can be seen in Denic et. al. (preprint).

5. EXAMPLE AND CONCLUSION

To illustrate the effect of the channel uncertainty on the capacity, we consider the following example. The channel is modeled by a second order transfer function $H(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$, $s = j\omega = j2\pi f$. It is assumed that the damping ratio ξ is unknown (ξ can take values between 0, and 1), whose value is within certain interval, $\xi_{low} \leq \xi \leq \xi_{up}$. This set will be roughly approximated by using the following procedure. We choose the natural frequency to be $\omega_n = 900$ rad/s, nominal damping ratio $\xi_{nom} = 0.3$, and $0.2 \leq \xi \leq 0.5$ (see Fig. 3). Further, the size of uncertainty set is defined by $|W_1| = |H_{nom}| - |H_{low}|$, where $H_{low}(s) = \frac{\omega_n^2}{s^2+2\xi_{up}\omega_n s+\omega_n^2}$, $H_{nom}(s) = \frac{\omega_n^2}{s^2+2\xi_{nom}\omega_n s+\omega_n^2}$. The values of $\xi_{low} = 0.2$, and $\xi_{up} = 0.5$ are deliberately chosen such that $|H_{ub}| = |H_{nom}| + |W_1|$ is a good approximation of $H_{up}(s) = \frac{\omega_n^2}{s^2+2\xi_{low}\omega_n s+\omega_n^2}$. Thus,

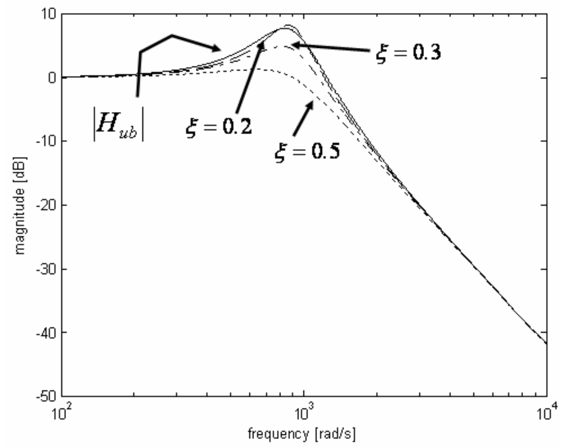

Fig. 3. Approximation of channel uncertainty set

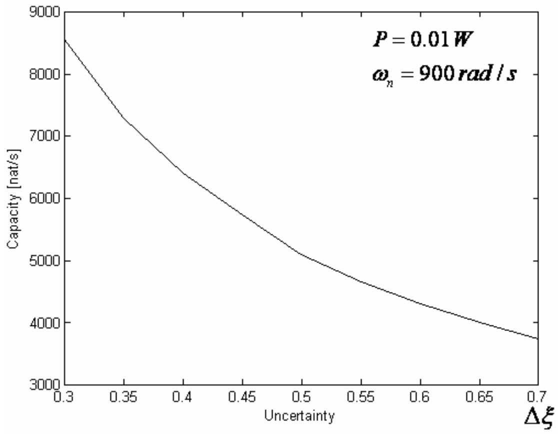

Fig. 4. Robust capacity vs. uncertainty

the frequency response uncertainty set is roughly described by $|H_{nom}| \pm |W_1|$. But, $|W_1| = |H_{nom}| - |H_{low}|$ implies $|H_{low}| = |H_{nom}| - |W_1|$. That means that the robust capacity is determined by the transfer function H_{low}. The uncertainty set is identified by the range of the damping ratio, $\Delta\xi = \xi_{up} - \xi_{low}$, $\Delta\xi = 0.30$ in this particular case. Notation $\Delta\xi = 0$ stands for the nominal channel model. The power spectral density of the noise is given by first order transfer function $S_{\mathbf{n}}(f) = |W(s)|^2 = |\frac{\alpha}{s+\beta}|^2$, $s = j\omega = j2\pi f$, where $\alpha = 1$, $\beta = 1000$ rad/s. The power of transmitted signal is limited to P =0.01 W. Fig. 4 depicts the robust capacity for different sizes of channel frequency response uncertainty sets. It can be seen that robust capacity indeed decreases with channel uncertainty. Fig. 5 shows the effect of channel uncertainty on optimal psd of the transmitter. The change of optimal bandwidth is negligible, which can be expected because the uncertainty in damping ratio does not affect channel bandwidth.

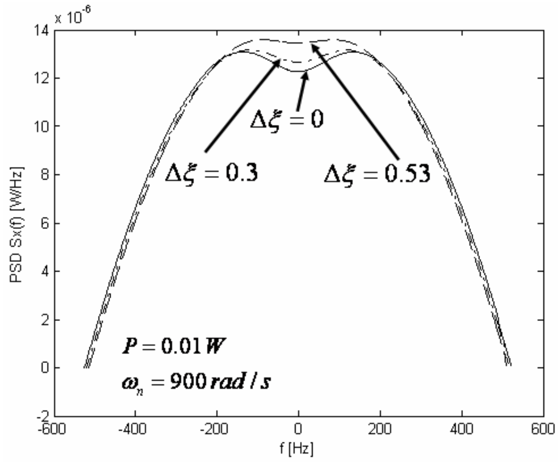

Fig. 5. Optimal power spectral density

REFERENCES

Baker, C. R., and I.-F. Chao (1996). Information capacity of channels with partially unknown noise. I. Finite dimensional channels, *SIAM J. Appl. Math.*, **56**(3), 946-963.

Biglieri, E., J. Proakis, S. Shamai (1998). Fading Channels - Information-Theoretic and Communications Aspects, *IEEE Trans. Info. Theo.*, **44**(6), 2619-2692.

Csiszar, I., and P. Narayan (1991). Capacity of the Gaussian arbitrary varying channels, *IEEE Trans. Info. Theo.*, **37**(1), 18-26.

Denic, S.Z., C. D. Charalambous, S. M. Djouadi, Robust Capacity for Additive Colored Gaussian Uncertain Channels," *preprint, http://www.uottawa.ca/˜ sdenic/colored.*

Doyle, J. C., Francis, B. A., Tannenbaum, A. R. (1992). *Feedback Control Theory.* New York: Mcmillan Publishing Company.

Forys, L.J., and P. P. Varaiya (1969). The ϵ-Capacity of Classes of unknown channels, *Information and control*, **44**, 376-406.

Gallager, R. (1968). *Information Theory and Reliable Communication*, New York: Wiley.

Hughes, B., and P. Narayan (1988). The capacity of vector Gaussian arbitrary varying channel, *IEEE Transactions on Information Theory*, **34**(5), 995-1003.

Lapidoth, A., and P. Narayan (1998). Reliable Communication Under Channel Uncertainty, *IEEE Trans. Info. Theo.*, **44**(6), 2148-2177.

McElice, R. J. (1983). Communications in the presence of jamming An information theoretic approach, in *Secure Digital Communications*, G. Longo, ed., Springer-Verlang, New York, 127-166.

Medard, M. (2003). Channel Uncertainty in Communications, *IEEE Information Society Newsletter*, **53**(2), 1, 10-12.

Root, W. L., and P.P. Varaiya (1968). Capacity of Classes of Gaussian Channels, *SIAM J. Appl. Math.*, **16**(6), 1350-1393.

Shannon, C.E. (1948). A Mathematical Theory of Communication, *Bell Syst. Tech. J.*, **27**, 379-423, 623-656.

Shannon, C. E. (1949). Communication in the Presence of Noise, *Proceedings of the IRE*, **37**(1), 10-21.